# Computer algebra independent integration tests

Summer 2022 edition

## 1-Algebraic-functions/1.2-Trinomial-products/1.2.1-Quadratic/37-1.2.1.6-g+h-x-^m-a+b-x+c-x^2-^p-d+e-x+f-x^2-^q

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# Contents

# Chapter 1

# Introduction

## Local contents

This report gives the result of running the computer algebra independent integration test.

The download section in the appendix contains links to download the problems in plain text format used for all CAS systems.

The number of integrals in this report is [ 143 ]. This is test number [ 37 ].

## 1.1 Listing of CAS systems tested

The following are the CAS systems tested:

1. Mathematica 13.1 (June 29, 2022) on windows 10.
2. Rubi 4.16.1 (Dec 19, 2018) on Mathematica 13.0.1 on windows 10.
3. Maple 2022.1 (June 1, 2022) on windows 10.
4. Maxima 5.46 (April 13, 2022) using Lisp SBCL 2.1.11.debian on Linux via sagemath 9.6.
5. Fricas 1.3.8 (June 21, 2022) based on sbcl 2.1.11.debian on Linux via sagemath 9.6.
6. Giac/Xcas 1.9.0-13 (July 3, 2022) on Linux via sagemath 9.6.
7. Sympy 1.10.1 (March 20, 2022) Using Python 3.10.4 on Linux.
8. Mupad using Matlab 2021a with Symbolic Math Toolbox Version 8.7 on windows 10.

Maxima and Fricas and Giac are called using Sagemath. This was done using Sagemath `integrate` command by changing the name of the algorithm to use the different CAS systems.

Sympy was called directly from Python.

## 1.2 Results

Important note: A number of problems in this test suite have no antiderivative in closed form. This means the antiderivative of these integrals can not be expressed in terms of elementary, special functions or `Hypergeometric2F1` functions. `RootSum` and `RootOf` are not allowed.

If a CAS returns the above integral unevaluated within the time limit, then the result is counted as passed and assigned an A grade.

However, if CAS times out, then it is assigned an F grade even if the integral is not integrable, as this implies CAS could not determine that the integral is not integrable in the time limit.

If a CAS returns an antiderivative to such an integral, it is assigned an A grade automatically and this special result is listed in the introduction section of each individual test report to make it easy to identify as this can be important result to investigate.

The results given in in the table below reflects the above.

| System | % solved | % Failed |
|---|---|---|
| Rubi | 100.00 ( 143 ) | 0.00 ( 0 ) |
| Mathematica | 100.00 ( 143 ) | 0.00 ( 0 ) |
| Maple | 98.60 ( 141 ) | 1.40 ( 2 ) |
| Fricas | 58.04 ( 83 ) | 41.96 ( 60 ) |
| Giac | 34.97 ( 50 ) | 65.03 ( 93 ) |
| Mupad | 13.29 ( 19 ) | 86.71 ( 124 ) |
| Maxima | 10.49 ( 15 ) | 89.51 ( 128 ) |
| Sympy | 6.99 ( 10 ) | 93.01 ( 133 ) |

Table 1.1: Percentage solved for each CAS

The table below gives additional break down of the grading of quality of the antiderivatives generated by each CAS. The grading is given using the letters A,B,C and F with A being the best quality. The grading is accomplished by comparing the antiderivative generated with the optimal antiderivatives included in the test suite. The following table describes the meaning of these grades.

| grade | description |
|---|---|
| A | Integral was solved and antiderivative is optimal in quality and leaf size. |
| B | Integral was solved and antiderivative is optimal in quality but leaf size is larger than twice the optimal antiderivatives leaf size. |
| C | Integral was solved and antiderivative is non-optimal in quality. This can be due to one or more of the following reasons<br>1. antiderivative contains a hypergeometric function and the optimal antiderivative does not.<br>2. antiderivative contains a special function and the optimal antiderivative does not.<br>3. antiderivative contains the imaginary unit and the optimal antiderivative does not. |
| F | Integral was not solved. Either the integral was returned unevaluated within the time limit, or it timed out, or CAS hanged or crashed or an exception was raised. |

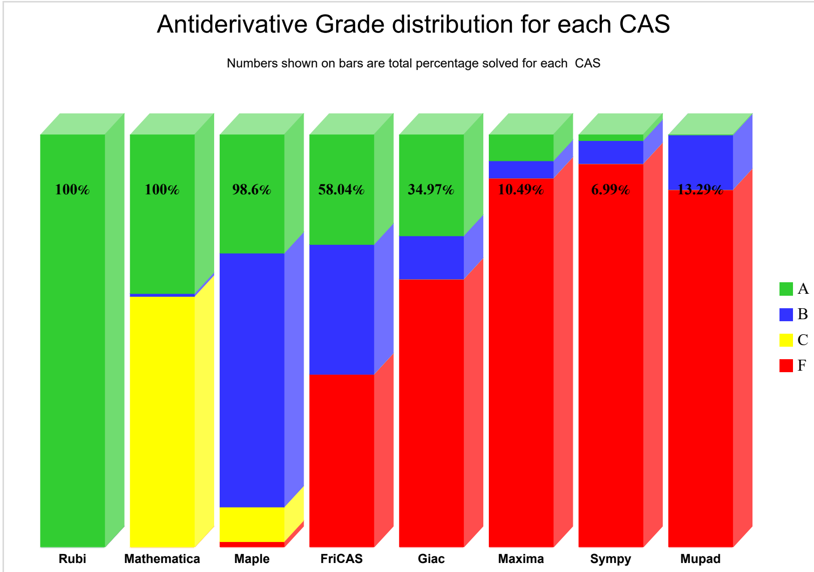
Table 1.2: Description of grading applied to integration result

Grading is implemented for all CAS systems. Based on the above, the following table summarizes the grading for this test suite.

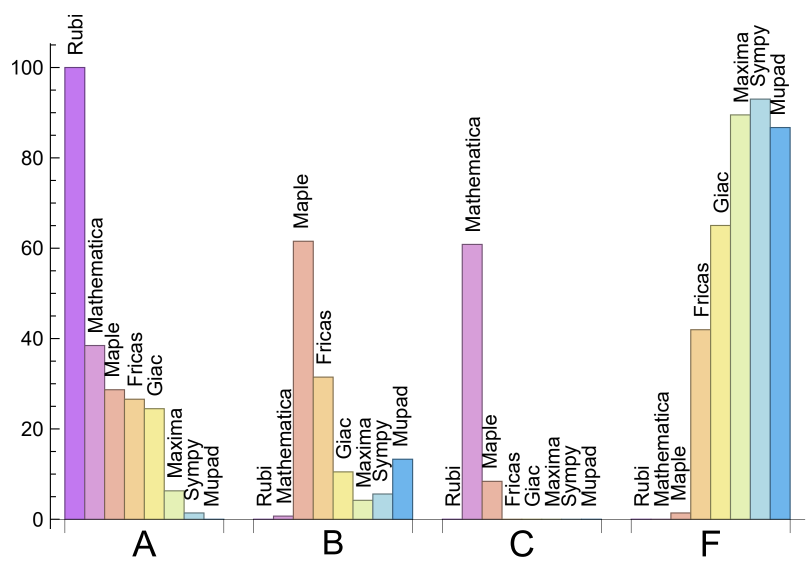
| System | % A grade | % B grade | % C grade | % F grade |
|---|---|---|---|---|
| Rubi | 100.00 | 0.00 | 0.00 | 0.00 |
| Mathematica | 38.46 | 0.70 | 60.84 | 0.00 |
| Maple | 28.67 | 61.54 | 8.39 | 1.40 |
| Fricas | 26.57 | 31.47 | 0.00 | 41.96 |
| Giac | 24.48 | 10.49 | 0.00 | 65.03 |
| Maxima | 6.29 | 4.20 | 0.00 | 89.51 |
| Sympy | 1.40 | 5.59 | 0.00 | 93.01 |
| Mupad | N/A | 13.29 | 0.00 | 86.71 |

Table 1.3: Antiderivative Grade distribution of each CAS

The following is a Bar chart illustration of the data in the above table.




The figure below compares the CAS systems for each grade level.




The following table shows the distribution of the different types of failure for each CAS. There are 3 types of reasons why it can fail. The first is when CAS returns back the input within the time limit, which means it could not solve it. This the typical normal failure **F** .

The second is due to time out. CAS could not solve the integral within the 3 minutes time limit which is assigned **F(-1)**.

The third is due to an exception generated. Assigned **F(-2)**. This most likely indicates an interface problem between sagemath and the CAS (applicable only to FriCAS, Maxima and

Giac) or it could be an indication of an internal error in CAS. This type of error requires more investigations to determine the cause.

| System | Number failed | Percentage normal failure | Percentage timeout failure | Percentage exception failure |
|---|---|---|---|---|
| Rubi | 0 | 0.00 % | 0.00 % | 0.00 % |
| Mathematica | 0 | 0.00 % | 0.00 % | 0.00 % |
| Maple | 2 | 100.00 % | 0.00 % | 0.00 % |
| Fricas | 60 | 1.67 % | 98.33 % | 0.00 % |
| Giac | 93 | 8.60 % | 43.01 % | 48.39 % |
| Maxima | 128 | 52.34 % | 0.00 % | 47.66 % |
| Sympy | 133 | 90.98 % | 9.02 % | 0.00 % |
| Mupad | 124 | 99.19 % | 0.81 % | 0.00 % |

Table 1.4: Failure statistics for each CAS

## 1.3 Time and leaf size Performance

The table below summarizes the performance of each CAS system in terms of time used and leaf size of results.

Mean size is the average leaf size produced by the CAS (before any normalization). The Normalized mean is relative to the mean size of the optimal anti-derivative given in the input files.

For example, if CAS has **Normalized mean** of 3, then the mean size of its leaf size is 3 times as large as the mean size of the optimal leaf size.

Median size is value of leaf size where half the values are larger than this and half are smaller (before any normalization). i.e. The Middle value.

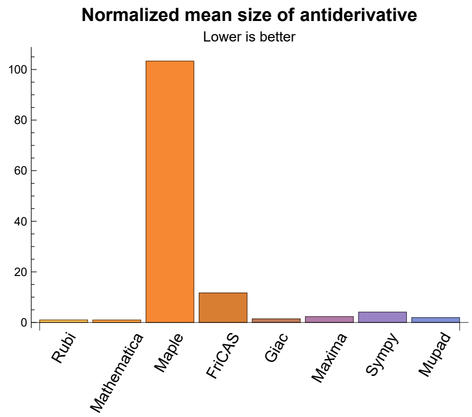
Similarly the **Normalized median** is relative to the median leaf size of the optimal.

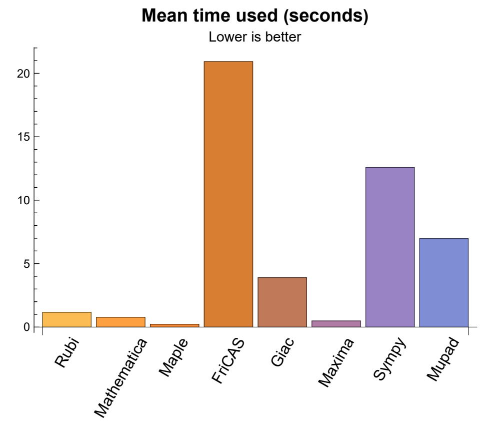
For example, if a CAS has Normalized median of 1.2, then its median is 1.2 as large as the median leaf size of the optimal.

| System | Mean time (sec) | Mean size | Normalized mean | Median size | Normalized median |
|---|---|---|---|---|---|
| Rubi | 1.16 | 333.90 | 1.00 | 302.00 | 1.00 |
| Mathematica | 0.77 | 333.82 | 0.97 | 264.00 | 0.88 |
| Maple | 0.22 | 49918.60 | 103.31 | 761.00 | 2.12 |
| Maxima | 0.49 | 388.27 | 2.32 | 220.00 | 1.11 |
| Fricas | 20.92 | 3794.67 | 11.68 | 516.00 | 3.24 |
| Sympy | 12.58 | 872.30 | 4.09 | 506.50 | 3.82 |
| Giac | 3.90 | 259.92 | 1.44 | 146.00 | 1.21 |
| Mupad | 6.98 | 568.32 | 1.93 | 187.00 | 1.25 |

Table 1.5: Time and leaf size performance for each CAS

The following are bar charts for the normalized leafsize and time used from the above table.


Normalized mean size of antiderivative
Lower is better
100
80
60
40
20
0
Rubi
Mathematica
Maple
FriCAS
Giac
Maxima
Sympy
Mupad


Mean time used (seconds)
Lower is better
20
15
10
5
0
Rubi
Mathematica
Maple
FriCAS
Giac
Maxima
Sympy
Mupad

## 1.4 list of integrals that has no closed form antiderivative

{}

## 1.5 List of integrals solved by CAS but has no known antiderivative

**Rubi** {}

**Mathematica** {}

**Maple** {}

**Maxima** {}

**Fricas** {}

**Sympy** {}

**Giac** {}

**Mupad** {}

## 1.6 list of integrals solved by CAS but failed verification

The following are integrals solved by CAS but the verification phase failed to verify the anti-derivative produced is correct. This does not mean necessarily that the anti-derivative is wrong, as additional methods of verification might be needed, or more time is needed (3 minutes time limit was used). These integrals are listed here to make it easier to do further investigation to determine why it was not possible to verify the result produced.

**Rubi** $\{19, 20\}$

**Mathematica** $\{\}$

**Maple** Verification phase not implemented yet.

**Maxima** Verification phase not implemented yet.

**Fricas** Verification phase not implemented yet.

**Sympy** Verification phase not implemented yet.

**Giac** Verification phase not implemented yet.

**Mupad** Verification phase not implemented yet.

## 1.7 Timing

The command `AbsoluteTiming[]` was used in Mathematica to obtain the elapsed time for each integrate call. In Maple, the command `Usage` was used as in the following example

```
cpu_time := Usage(assign ('result_of_int',int(expr,x)),output='realtime'
```

For all other CAS systems, the elapsed time to complete each integral was found by taking the difference between the time after the call completed from the time before the call was made. This was done using Python's `time.time()` call.

All elapsed times shown are in seconds. A time limit of 3 CPU minutes was used for each integral. If the integrate command did not complete within this time limit, the integral was aborted and considered to have failed and assigned an F grade. The time used by failed integrals due to time out was not counted in the final statistics.

## 1.8 Verification

A verification phase was applied on the result of integration for `Rubi` and `Mathematica`.

Future version of this report will implement verification for the other CAS systems. For the integrals whose result was not run through a verification phase, it is assumed that the antiderivative was correct.

Verification phase also had 3 minutes time out. An integral whose result was not verified could still be correct, but further investigation is needed on those integrals. These integrals were marked in the summary table below and also in each integral separate section so they are easy to identify and locate.

## 1.9 Important notes about some of the results

### 1.9.1 Important note about Maxima results

Since tests were run in a batch mode, and using an automated script, then any integral where Maxima needed an interactive response from the user to answer a question during the evaluation of the integral will fail.

The exception raised is `ValueError`. Therefore Maxima results is lower than what would result if Maxima was run directly and each question was answered correctly.

The percentage of such failures were not counted for each test file, but for an example, for the `Timofeev` test file, there were about 14 such integrals out of total 705, or about 2 percent. This percentage can be higher or lower depending on the specific input test file.

Such integrals can be identified by looking at the output of the integration in each section for Maxima. The exception message will indicate the cause of error.

Maxima integrate was run using SageMath with the following settings set by default

```
'besselexpand : true'
'display2d : false'
'domain : complex'
'keepfloat : true'
'load(to_poly_solve)'
'load(simplify_sum)'
'load(abs_integrate)' 'load(diag)'
```

SageMath automatic loading of Maxima `abs_integrate` was found to cause some problems. So the following code was added to disable this effect.

```
from sage.interfaces.maxima_lib import maxima_lib
maxima_lib.set('extra_definite_integration_methods', '[]')
```

```
maxima_lib.set('extra_integration_methods', '[]')
```

See https://ask.sagemath.org/question/43088/integrate-results-that-are-different-from-using-maxima/ for reference.

### 1.9.2 Important note about FriCAS result

There were few integrals which failed due to SageMath interface and not because FriCAS system could not do the integration.

These will fail With error `Exception raised: NotImplementedError`.

The number of such cases seems to be very small. About 1 or 2 percent of all integrals. These can be identified by looking at the exception message given in the result.

### 1.9.3 Important note about finding leaf size of antiderivative

For Mathematica, Rubi, and Maple, the builtin system function `LeafSize` was used to find the leaf size of each antiderivative.

The other CAS systems (SageMath and Sympy) do not have special builtin function for this purpose at this time. Therefore the leaf size for Fricas and Sympy antiderivative was determined using the following function, thanks to user `slelievre` at https://ask.sagemath.org/question/57123/could-we-have-a-leaf_count-function-in-base-sagemath/

```
def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)
```

For Sympy, which was called directly from Python, the following code was used to obtain the leafsize of its result

```
try:
  # 1.7 is a fudge factor since it is low side from actual leaf count
  leafCount = round(1.7*count_ops(anti))

  except Exception as ee:
        leafCount =1
```

### 1.9.4 Important note about Mupad results

Matlab's symbolic toolbox does not have a leaf count function to measure the size of the antiderivative. Maple was used to determine the leaf size of Mupad output by post processing Mupad result.

Currently no grading of the antiderivative for Mupad is implemented. If it can integrate the problem, it was assigned a B grade automatically as a placeholder. In the future, when grading function is implemented for Mupad, the tests will be rerun again.

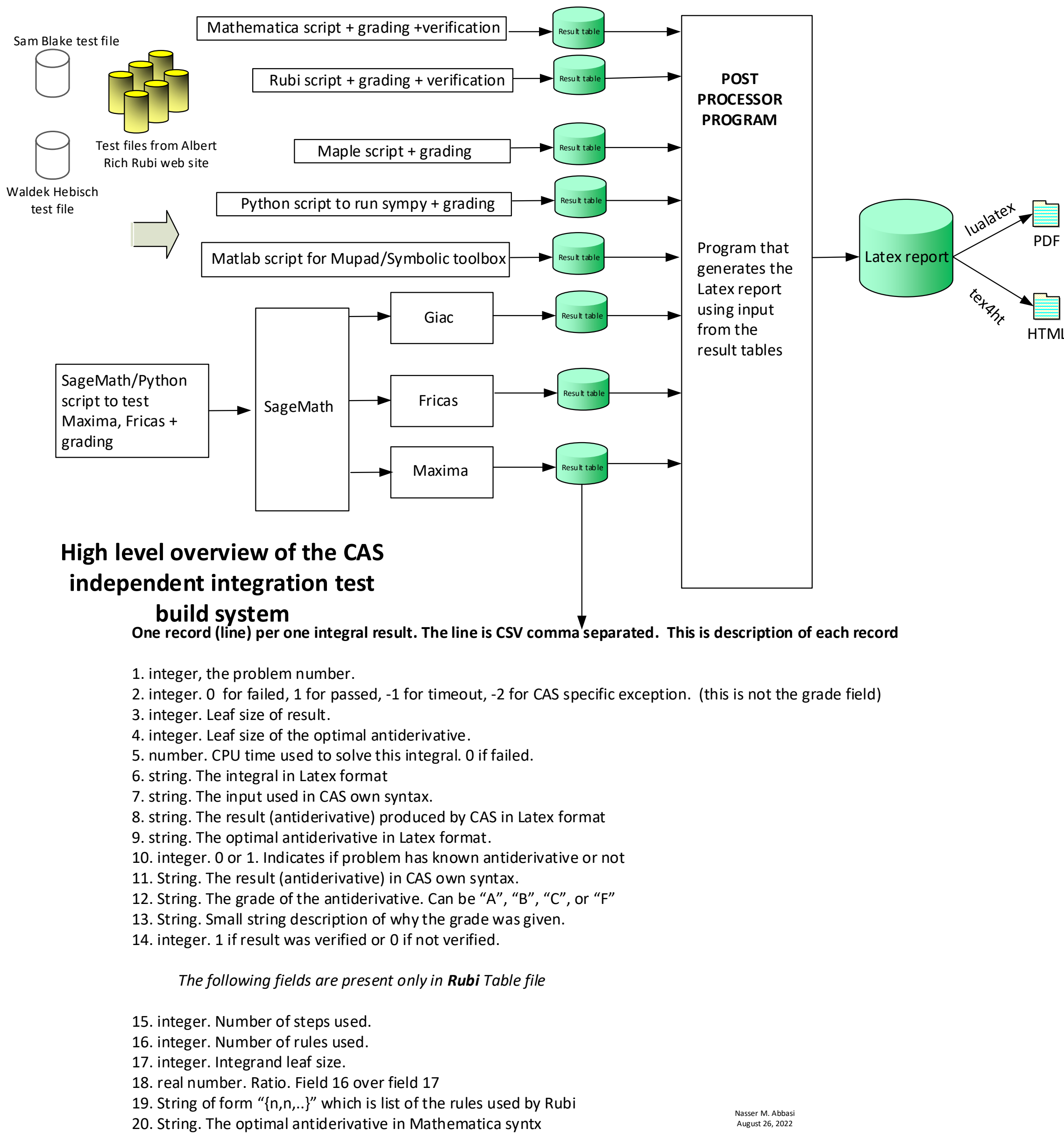
The following is an example of using Matlab's symbolic toolbox (Mupad) to solve an integral

```
integrand = evalin(symengine,'cos(x)*sin(x)')
the_variable = evalin(symengine,'x')
anti = int(integrand,the_variable)
```

Which gives `sin(x)^2/2`

# 1.10 Design of the test system

The following diagram gives a high level view of the current test build system.



# Chapter 2

# detailed summary tables of results

## Local contents

## 2.1 List of integrals sorted by grade for each CAS

### Local contents

### 2.1.1 Rubi

A grade: { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143 }

B grade: { }

C grade: { }

F grade: { }

### 2.1.2 Mathematica

A grade: { 1, 2, 3, 4, 5, 9, 10, 13, 14, 15, 16, 17, 18, 28, 29, 30, 31, 32, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 91, 92, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143 }

B grade: { 24 }

C grade: { 6, 7, 8, 11, 12, 19, 20, 21, 22, 23, 25, 26, 27, 34, 35, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125 }

F grade: { }

### 2.1.3 Maple

A grade: { 1, 2, 3, 4, 10, 13, 14, 15, 17, 18, 22, 25, 28, 31, 32, 33, 34, 38, 39, 92, 94, 95, 96, 99, 100, 101, 114, 120, 126, 127, 128, 129, 130, 131, 132, 133, 134, 138, 139, 140, 141 }

B grade: { 5, 6, 7, 8, 9, 11, 12, 16, 19, 20, 21, 23, 24, 26, 27, 29, 30, 35, 36, 37, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 93, 97, 98, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 115, 116, 117, 118, 119, 121, 122, 123, 124, 125, 135, 136, 137 }

C grade: { 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51 }

F grade: { 142, 143 }

### 2.1.4 Maxima

A grade: { 1, 2, 3, 10, 92, 133, 134, 138, 139 }

B grade: { 25, 26, 27, 28, 29, 30 }

C grade: { }

F grade: { 4, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 135, 136, 137, 140, 141, 142, 143 }

### 2.1.5 FriCAS

A grade: { 1, 2, 3, 10, 13, 14, 33, 36, 38, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 91, 92, 126, 127, 128, 129, 130, 131, 132, 133, 134, 136, 137, 138, 139, 140, 141 }

B grade: { 7, 11, 17, 18, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 37, 54, 55, 56, 66, 67, 68, 71, 72, 73, 74, 79, 80, 81, 82, 83, 93, 97, 98, 99, 100, 103, 104, 105, 106, 116, 117, 135 }

C grade: { }

F grade: { 4, 5, 6, 8, 9, 12, 15, 16, 19, 20, 35, 39, 52, 53, 57, 58, 59, 60, 61, 62, 63, 64, 65, 69, 70, 75, 76, 77, 78, 84, 85, 86, 87, 88, 89, 90, 94, 95, 96, 101, 102, 107, 108, 109, 110, 111, 112, 113, 114, 115, 118, 119, 120, 121, 122, 123, 124, 125, 142, 143 }

### 2.1.6 Sympy

A grade: { 33, 139 }

B grade: { 1, 2, 13, 14, 17, 18, 31, 138 }

C grade: { }

F grade: { 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 15, 16, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 133, 134, 135, 136, 137, 140, 141, 142, 143 }

### 2.1.7 Giac

A grade: { 1, 2, 3, 4, 10, 13, 14, 15, 17, 18, 28, 29, 30, 36, 40, 41, 42, 43, 44, 45, 46, 47, 48, 92, 126, 127, 129, 131, 133, 134, 137, 138, 139, 140, 141 }

B grade: { 5, 11, 16, 31, 32, 33, 34, 37, 38, 51, 128, 130, 132, 135, 136 }

C grade: { }

F grade: { 6, 7, 8, 9, 12, 19, 20, 21, 22, 23, 24, 25, 26, 27, 35, 39, 49, 50, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 142, 143 }

### 2.1.8 Mupad

A grade: { }

B grade: { 1, 2, 3, 4, 5, 13, 14, 16, 17, 18, 31, 33, 36, 133, 134, 138, 139, 140, 141 }

C grade: { }

F grade: { 6, 7, 8, 9, 10, 11, 12, 15, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 32, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100, 101, 102, 103, 104, 105, 106, 107, 108, 109, 110, 111, 112, 113, 114, 115, 116, 117, 118, 119, 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131, 132, 135, 136, 137, 142, 143 }

## 2.2 Detailed conclusion table per each integral for all CAS systems

Detailed conclusion table per each integral is given by table below. The elapsed time is in seconds. For failed result it is given as F(-1) if the failure was due to timeout. It is given as F(-2) if the failure was due to an exception being raised, which could indicate a bug in the system. If the failure was due to integral not being evaluated within the time limit, then it is given just an F.

In this table,the column `N.S.` in the table below, which stands for **normalized size** is defined as $\frac{\text{antiderivative leaf size}}{\text{optimal antiderivative leaf size}}$. To help make the table fit, `Mathematica` was abbreviated to `MMA`.

| Problem 1 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 94 | 94 | 86 | 84 | 84 | 200 | 333 | 87 | 97 |
| N.S. | 1 | 1.00 | 0.91 | 0.89 | 0.89 | 2.13 | 3.54 | 0.93 | 1.03 |
| time (sec) | N/A | 0.066 | 0.053 | 0.108 | 0.511 | 0.782 | 0.838 | 4.090 | 3.436 |

| Problem 2 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 228 | 228 | 204 | 235 | 220 | 500 | 933 | 263 | 253 |
| N.S. | 1 | 1.00 | 0.89 | 1.03 | 0.96 | 2.19 | 4.09 | 1.15 | 1.11 |
| time (sec) | N/A | 0.194 | 0.149 | 0.157 | 0.498 | 0.328 | 15.740 | 3.922 | 0.249 |

| Problem 3 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F(-1) | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 441 | 441 | 422 | 592 | 471 | 1014 | 0 | 623 | 552 |
| N.S. | 1 | 1.00 | 0.96 | 1.34 | 1.07 | 2.30 | 0.00 | 1.41 | 1.25 |
| time (sec) | N/A | 0.381 | 0.333 | 0.197 | 0.518 | 0.332 | 0.000 | 4.018 | 3.792 |

| Problem 4 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | F(-1) | F(-1) | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 274 | 274 | 212 | 239 | 0 | 0 | 0 | 266 | 2500 |
| N.S. | 1 | 1.00 | 0.77 | 0.87 | 0.00 | 0.00 | 0.00 | 0.97 | 9.12 |
| time (sec) | N/A | 0.176 | 0.255 | 0.306 | 0.000 | 0.000 | 0.000 | 5.052 | 38.323 |

| Problem 5 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | F(-1) | F(-1) | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 596 | 596 | 523 | 1254 | 0 | 0 | 0 | 1313 | 2500 |
| N.S. | 1 | 1.00 | 0.88 | 2.10 | 0.00 | 0.00 | 0.00 | 2.20 | 4.19 |
| time (sec) | N/A | 1.030 | 1.214 | 0.946 | 0.000 | 0.000 | 0.000 | 4.236 | 7.534 |

| Problem 6 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 331 | 331 | 511 | 803 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.54 | 2.43 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.379 | 0.678 | 0.168 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 7 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 249 | 249 | 219 | 389 | 0 | 6113 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.88 | 1.56 | 0.00 | 24.55 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.128 | 0.387 | 0.137 | 0.000 | 30.081 | 0.000 | 0.000 | 0.000 |

| Problem 8 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F(-1) | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 381 | 380 | 641 | 934 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.68 | 2.45 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.487 | 1.146 | 0.139 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 9 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | F(-1) | F(-1) | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 797 | 796 | 674 | 1768 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.85 | 2.22 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.137 | 12.757 | 0.136 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 10 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 47 | 47 | 37 | 46 | 65 | 46 | 0 | 48 | -1 |
| N.S. | 1 | 1.00 | 0.79 | 0.98 | 1.38 | 0.98 | 0.00 | 1.02 | -0.02 |
| time (sec) | N/A | 0.021 | 0.108 | 0.167 | 0.546 | 0.361 | 0.000 | 4.310 | 0.000 |

| Problem 11 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 117 | 117 | 106 | 637 | 0 | 758 | 0 | 457 | -1 |
| N.S. | 1 | 1.00 | 0.91 | 5.44 | 0.00 | 6.48 | 0.00 | 3.91 | -0.01 |
| time (sec) | N/A | 0.108 | 0.118 | 0.548 | 0.000 | 0.368 | 0.000 | 5.176 | 0.000 |

| Problem 12 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 484 | 484 | 210 | 6871419 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.43 | 14197.15 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 22.897 | 0.377 | 4.340 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 13 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 184 | 182 | 175 | 190 | 0 | 584 | 1260 | 191 | 273 |
| N.S. | 1 | 0.99 | 0.95 | 1.03 | 0.00 | 3.17 | 6.85 | 1.04 | 1.48 |
| time (sec) | N/A | 0.216 | 0.149 | 0.210 | 0.000 | 0.428 | 7.623 | 2.482 | 3.852 |

| Problem 14 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | A | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 542 | 542 | 535 | 800 | 0 | 1921 | 4663 | 738 | 893 |
| N.S. | 1 | 1.00 | 0.99 | 1.48 | 0.00 | 3.54 | 8.60 | 1.36 | 1.65 |
| time (sec) | N/A | 0.678 | 0.412 | 0.272 | 0.000 | 0.512 | 91.258 | 3.078 | 4.854 |

| Problem 15 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | F(-1) | F(-1) | A | F(-1) |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 406 | 398 | 267 | 384 | 0 | 0 | 0 | 416 | -1 |
| N.S. | 1 | 0.98 | 0.66 | 0.95 | 0.00 | 0.00 | 0.00 | 1.02 | -0.00 |
| time (sec) | N/A | 0.294 | 0.309 | 0.837 | 0.000 | 0.000 | 0.000 | 2.059 | 0.000 |

| Problem 16 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | F(-1) | F(-1) | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 1075 | 1067 | 952 | 51470 | 0 | 0 | 0 | 3226 | 2500 |
| N.S. | 1 | 0.99 | 0.89 | 47.88 | 0.00 | 0.00 | 0.00 | 3.00 | 2.33 |
| time (sec) | N/A | 2.533 | 4.414 | 0.038 | 0.000 | 0.000 | 0.000 | 2.296 | 30.314 |

| Problem 17 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 140 | 140 | 131 | 141 | 0 | 1150 | 709 | 219 | 395 |
| N.S. | 1 | 1.00 | 0.94 | 1.01 | 0.00 | 8.21 | 5.06 | 1.56 | 2.82 |
| time (sec) | N/A | 0.091 | 0.102 | 0.185 | 0.000 | 0.530 | 1.212 | 3.475 | 0.422 |

| Problem 18 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | B | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 140 | 140 | 131 | 141 | 0 | 1130 | 680 | 207 | 375 |
| N.S. | 1 | 1.00 | 0.94 | 1.01 | 0.00 | 8.07 | 4.86 | 1.48 | 2.68 |
| time (sec) | N/A | 0.070 | 0.020 | 0.182 | 0.000 | 0.381 | 1.177 | 5.025 | 4.020 |

| Problem 19 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | NO | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 617 | 615 | 889 | 1595 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.44 | 2.59 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 5.928 | 0.927 | 0.169 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 20 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F(-1) | F(-2) | F |
| verified | N/A | NO | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 1092 | 1092 | 2733 | 2908 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 2.50 | 2.66 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 16.593 | 4.640 | 0.184 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 21 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 416 | 416 | 278 | 805 | 0 | 21959 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.67 | 1.94 | 0.00 | 52.79 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.787 | 0.455 | 0.195 | 0.000 | 169.420 | 0.000 | 0.000 | 0.000 |

| Problem 22 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 780 | 780 | 218 | 425 | 0 | 6737 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.28 | 0.54 | 0.00 | 8.64 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 3.386 | 0.385 | 0.142 | 0.000 | 25.988 | 0.000 | 0.000 | 0.000 |

| Problem 23 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 302 | 302 | 195 | 639 | 0 | 8977 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.65 | 2.12 | 0.00 | 29.73 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.552 | 0.388 | 0.157 | 0.000 | 33.476 | 0.000 | 0.000 | 0.000 |

| Problem 24 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | B | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 101 | 101 | 349 | 337 | 0 | 1515 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 3.46 | 3.34 | 0.00 | 15.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.079 | 1.770 | 0.084 | 0.000 | 0.419 | 0.000 | 0.000 | 0.000 |

| Problem 25 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | B | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 139 | 139 | 149 | 176 | 361 | 322 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.07 | 1.27 | 2.60 | 2.32 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.138 | 0.199 | 0.697 | 0.513 | 0.369 | 0.000 | 0.000 | 0.000 |

| Problem 26 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | B | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 166 | 166 | 137 | 455 | 678 | 344 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.83 | 2.74 | 4.08 | 2.07 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.144 | 0.319 | 0.566 | 0.519 | 0.399 | 0.000 | 0.000 | 0.000 |

| Problem 27 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | B | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 193 | 193 | 183 | 868 | 1276 | 439 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.95 | 4.50 | 6.61 | 2.27 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.167 | 0.462 | 0.612 | 0.558 | 0.435 | 0.000 | 0.000 | 0.000 |

| Problem 28 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | B | B | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 151 | 151 | 109 | 186 | 363 | 245 | 0 | 93 | -1 |
| N.S. | 1 | 1.00 | 0.72 | 1.23 | 2.40 | 1.62 | 0.00 | 0.62 | -0.01 |
| time (sec) | N/A | 0.140 | 0.517 | 0.624 | 0.515 | 0.356 | 0.000 | 7.022 | 0.000 |

| Problem 29 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 174 | 174 | 130 | 466 | 668 | 365 | 0 | 112 | -1 |
| N.S. | 1 | 1.00 | 0.75 | 2.68 | 3.84 | 2.10 | 0.00 | 0.64 | -0.01 |
| time (sec) | N/A | 0.157 | 0.803 | 0.620 | 0.531 | 0.356 | 0.000 | 4.423 | 0.000 |

| Problem 30 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | B | B | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 197 | 197 | 154 | 878 | 1276 | 435 | 0 | 121 | -1 |
| N.S. | 1 | 1.00 | 0.78 | 4.46 | 6.48 | 2.21 | 0.00 | 0.61 | -0.01 |
| time (sec) | N/A | 0.184 | 0.891 | 0.604 | 0.539 | 0.420 | 0.000 | 3.773 | 0.000 |

| Problem 31 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | B | B | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 15 | 15 | 15 | 14 | 0 | 49 | 36 | 31 | 13 |
| N.S. | 1 | 1.00 | 1.00 | 0.93 | 0.00 | 3.27 | 2.40 | 2.07 | 0.87 |
| time (sec) | N/A | 0.011 | 0.152 | 0.179 | 0.000 | 0.348 | 2.393 | 3.773 | 3.763 |

| Problem 32 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 44 | 44 | 55 | 40 | 0 | 106 | 0 | 108 | -1 |
| N.S. | 1 | 1.00 | 1.25 | 0.91 | 0.00 | 2.41 | 0.00 | 2.45 | -0.02 |
| time (sec) | N/A | 0.033 | 0.200 | 0.299 | 0.000 | 0.363 | 0.000 | 3.791 | 0.000 |

| Problem 33 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | A | B | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 24 | 24 | 24 | 20 | 0 | 34 | 68 | 39 | 19 |
| N.S. | 1 | 1.00 | 1.00 | 0.83 | 0.00 | 1.42 | 2.83 | 1.62 | 0.79 |
| time (sec) | N/A | 0.011 | 0.201 | 0.320 | 0.000 | 0.342 | 2.128 | 4.463 | 3.781 |

| Problem 34 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 56 | 56 | 92 | 45 | 0 | 307 | 0 | 133 | -1 |
| N.S. | 1 | 1.00 | 1.64 | 0.80 | 0.00 | 5.48 | 0.00 | 2.38 | -0.02 |
| time (sec) | N/A | 0.030 | 0.128 | 0.367 | 0.000 | 0.348 | 0.000 | 4.051 | 0.000 |

| Problem 35 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F(-1) | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 249 | 249 | 1014 | 1430 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 4.07 | 5.74 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.539 | 1.888 | 0.217 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 36 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 48 | 48 | 46 | 95 | 0 | 85 | 0 | 81 | 49 |
| N.S. | 1 | 1.00 | 0.96 | 1.98 | 0.00 | 1.77 | 0.00 | 1.69 | 1.02 |
| time (sec) | N/A | 0.027 | 0.373 | 0.171 | 0.000 | 0.497 | 0.000 | 3.210 | 3.753 |

| Problem 37 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 17 | 17 | 17 | 94 | 0 | 56 | 0 | 98 | -1 |
| N.S. | 1 | 1.00 | 1.00 | 5.53 | 0.00 | 3.29 | 0.00 | 5.76 | -0.06 |
| time (sec) | N/A | 0.013 | 0.156 | 0.135 | 0.000 | 0.334 | 0.000 | 3.145 | 0.000 |

| Problem 38 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 86 | 86 | 51 | 123 | 0 | 132 | 0 | 163 | -1 |
| N.S. | 1 | 1.00 | 0.59 | 1.43 | 0.00 | 1.53 | 0.00 | 1.90 | -0.01 |
| time (sec) | N/A | 0.111 | 0.194 | 0.349 | 0.000 | 0.341 | 0.000 | 2.441 | 0.000 |

| Problem 39 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F(-2) | F | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 136 | 136 | 108 | 121 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.79 | 0.89 | 0.00 | 0.00 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.069 | 0.066 | 0.176 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 40 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 212 | 212 | 107 | 103 | 0 | 175 | 0 | 117 | -1 |
| N.S. | 1 | 1.00 | 0.50 | 0.49 | 0.00 | 0.83 | 0.00 | 0.55 | -0.00 |
| time (sec) | N/A | 0.071 | 0.165 | 0.090 | 0.000 | 0.365 | 0.000 | 2.685 | 0.000 |

| Problem 41 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 161 | 161 | 95 | 83 | 0 | 157 | 0 | 98 | -1 |
| N.S. | 1 | 1.00 | 0.59 | 0.52 | 0.00 | 0.98 | 0.00 | 0.61 | -0.01 |
| time (sec) | N/A | 0.039 | 0.134 | 0.079 | 0.000 | 0.396 | 0.000 | 3.296 | 0.000 |

| Problem 42 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 148 | 148 | 84 | 65 | 0 | 128 | 0 | 79 | -1 |
| N.S. | 1 | 1.00 | 0.57 | 0.44 | 0.00 | 0.86 | 0.00 | 0.53 | -0.01 |
| time (sec) | N/A | 0.031 | 0.153 | 0.078 | 0.000 | 0.510 | 0.000 | 2.933 | 0.000 |

| Problem 43 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 160 | 160 | 118 | 92 | 0 | 341 | 0 | 102 | -1 |
| N.S. | 1 | 1.00 | 0.74 | 0.58 | 0.00 | 2.13 | 0.00 | 0.64 | -0.01 |
| time (sec) | N/A | 0.075 | 0.182 | 0.104 | 0.000 | 0.431 | 0.000 | 3.221 | 0.000 |

| Problem 44 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 156 | 156 | 109 | 120 | 0 | 333 | 0 | 126 | -1 |
| N.S. | 1 | 1.00 | 0.70 | 0.77 | 0.00 | 2.13 | 0.00 | 0.81 | -0.01 |
| time (sec) | N/A | 0.071 | 0.185 | 0.091 | 0.000 | 0.360 | 0.000 | 5.587 | 0.000 |

| Problem 45 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 161 | 161 | 124 | 141 | 0 | 377 | 0 | 199 | -1 |
| N.S. | 1 | 1.00 | 0.77 | 0.88 | 0.00 | 2.34 | 0.00 | 1.24 | -0.01 |
| time (sec) | N/A | 0.072 | 0.252 | 0.110 | 0.000 | 0.414 | 0.000 | 4.628 | 0.000 |

| Problem 46 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 317 | 317 | 236 | 530 | 0 | 516 | 0 | 368 | -1 |
| N.S. | 1 | 1.00 | 0.74 | 1.67 | 0.00 | 1.63 | 0.00 | 1.16 | -0.00 |
| time (sec) | N/A | 0.191 | 0.629 | 0.159 | 0.000 | 0.393 | 0.000 | 4.869 | 0.000 |

| Problem 47 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 227 | 227 | 176 | 381 | 0 | 394 | 0 | 268 | -1 |
| N.S. | 1 | 1.00 | 0.78 | 1.68 | 0.00 | 1.74 | 0.00 | 1.18 | -0.00 |
| time (sec) | N/A | 0.078 | 0.462 | 0.104 | 0.000 | 0.520 | 0.000 | 4.212 | 0.000 |

| Problem 48 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 198 | 198 | 136 | 257 | 0 | 292 | 0 | 185 | -1 |
| N.S. | 1 | 1.00 | 0.69 | 1.30 | 0.00 | 1.47 | 0.00 | 0.93 | -0.01 |
| time (sec) | N/A | 0.064 | 0.404 | 0.115 | 0.000 | 0.412 | 0.000 | 2.065 | 0.000 |

| Problem 49 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 211 | 211 | 150 | 215 | 0 | 683 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.71 | 1.02 | 0.00 | 3.24 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.112 | 0.477 | 0.129 | 0.000 | 0.806 | 0.000 | 0.000 | 0.000 |

| Problem 50 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 202 | 202 | 151 | 249 | 0 | 691 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.75 | 1.23 | 0.00 | 3.42 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.101 | 0.490 | 0.132 | 0.000 | 0.578 | 0.000 | 0.000 | 0.000 |

| Problem 51 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | C | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 215 | 215 | 155 | 359 | 0 | 745 | 0 | 450 | -1 |
| N.S. | 1 | 1.00 | 0.72 | 1.67 | 0.00 | 3.47 | 0.00 | 2.09 | -0.00 |
| time (sec) | N/A | 0.110 | 0.598 | 0.123 | 0.000 | 0.611 | 0.000 | 5.101 | 0.000 |

| Problem 52 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 452 | 452 | 490 | 1302 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.08 | 2.88 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.265 | 0.518 | 0.169 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 53 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 395 | 395 | 379 | 1245 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.96 | 3.15 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.635 | 0.365 | 0.141 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 54 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 298 | 298 | 264 | 1212 | 0 | 2400 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.89 | 4.07 | 0.00 | 8.05 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.246 | 0.320 | 0.133 | 0.000 | 106.710 | 0.000 | 0.000 | 0.000 |

| Problem 55 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 358 | 358 | 299 | 1316 | 0 | 2266 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.84 | 3.68 | 0.00 | 6.33 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.834 | 0.369 | 0.132 | 0.000 | 12.355 | 0.000 | 0.000 | 0.000 |

| Problem 56 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 382 | 382 | 336 | 1415 | 0 | 5102 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.88 | 3.70 | 0.00 | 13.36 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.897 | 0.401 | 0.132 | 0.000 | 80.296 | 0.000 | 0.000 | 0.000 |

| Problem 57 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 507 | 507 | 533 | 1521 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.05 | 3.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.183 | 0.811 | 0.156 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 58 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F(-1) | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 795 | 795 | 942 | 2367 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.18 | 2.98 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.698 | 1.014 | 0.151 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 59 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 553 | 553 | 755 | 2294 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.37 | 4.15 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.536 | 0.724 | 0.138 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 60 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 484 | 482 | 542 | 2261 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.12 | 4.67 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.764 | 0.568 | 0.141 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 61 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 496 | 496 | 552 | 2379 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.11 | 4.80 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.381 | 0.564 | 0.133 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 62 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 604 | 604 | 497 | 2494 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.82 | 4.13 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.715 | 0.638 | 0.146 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 63 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F(-1) | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 668 | 668 | 617 | 2614 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.92 | 3.91 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.086 | 0.790 | 0.174 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 64 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 380 | 380 | 312 | 713 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.82 | 1.88 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.713 | 0.382 | 0.144 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 65 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 344 | 344 | 226 | 663 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.66 | 1.93 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.351 | 0.354 | 0.119 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 66 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 294 | 294 | 156 | 622 | 0 | 4789 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.53 | 2.12 | 0.00 | 16.29 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.166 | 0.310 | 0.123 | 0.000 | 0.962 | 0.000 | 0.000 | 0.000 |

| Problem 67 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 266 | 266 | 131 | 589 | 0 | 4761 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.49 | 2.21 | 0.00 | 17.90 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.099 | 0.307 | 0.125 | 0.000 | 1.014 | 0.000 | 0.000 | 0.000 |

| Problem 68 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 330 | 330 | 236 | 681 | 0 | 13360 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.72 | 2.06 | 0.00 | 40.48 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.539 | 0.339 | 0.127 | 0.000 | 184.039 | 0.000 | 0.000 | 0.000 |

| Problem 69 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 367 | 367 | 298 | 736 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.81 | 2.01 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.777 | 0.414 | 0.128 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 70 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 457 | 457 | 422 | 827 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.92 | 1.81 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.249 | 0.647 | 0.135 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 71 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 499 | 499 | 405 | 1576 | 0 | 26071 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.81 | 3.16 | 0.00 | 52.25 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.342 | 0.610 | 0.114 | 0.000 | 79.470 | 0.000 | 0.000 | 0.000 |

| Problem 72 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 410 | 410 | 303 | 1525 | 0 | 24806 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.74 | 3.72 | 0.00 | 60.50 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.424 | 0.529 | 0.122 | 0.000 | 111.156 | 0.000 | 0.000 | 0.000 |

| Problem 73 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 411 | 411 | 330 | 1490 | 0 | 24872 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.80 | 3.63 | 0.00 | 60.52 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.515 | 0.522 | 0.135 | 0.000 | 115.298 | 0.000 | 0.000 | 0.000 |

| Problem 74 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 416 | 416 | 346 | 1457 | 0 | 26013 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.83 | 3.50 | 0.00 | 62.53 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.384 | 0.528 | 0.110 | 0.000 | 74.222 | 0.000 | 0.000 | 0.000 |

| Problem 75 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 526 | 526 | 542 | 1565 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.03 | 2.98 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.412 | 0.866 | 0.140 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 76 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 618 | 618 | 684 | 1639 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.11 | 2.65 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.430 | 1.218 | 0.175 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 77 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 392 | 392 | 409 | 863 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.04 | 2.20 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.605 | 0.748 | 0.184 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 78 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 316 | 316 | 320 | 849 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.01 | 2.69 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.304 | 0.569 | 0.149 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 79 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 282 | 282 | 349 | 768 | 0 | 1192 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.24 | 2.72 | 0.00 | 4.23 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.193 | 0.411 | 0.144 | 0.000 | 157.331 | 0.000 | 0.000 | 0.000 |

| Problem 80 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 266 | 266 | 268 | 772 | 0 | 1139 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.01 | 2.90 | 0.00 | 4.28 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.143 | 0.261 | 0.126 | 0.000 | 56.214 | 0.000 | 0.000 | 0.000 |

| Problem 81 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 267 | 267 | 334 | 850 | 0 | 1253 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.25 | 3.18 | 0.00 | 4.69 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.503 | 0.252 | 0.143 | 0.000 | 10.124 | 0.000 | 0.000 | 0.000 |

| Problem 82 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 286 | 286 | 288 | 959 | 0 | 1094 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.01 | 3.35 | 0.00 | 3.83 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.460 | 0.411 | 0.146 | 0.000 | 15.013 | 0.000 | 0.000 | 0.000 |

| Problem 83 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 353 | 353 | 381 | 1143 | 0 | 1485 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.08 | 3.24 | 0.00 | 4.21 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.563 | 0.649 | 0.148 | 0.000 | 78.186 | 0.000 | 0.000 | 0.000 |

| Problem 84 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F(-1) | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 501 | 501 | 734 | 1607 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.47 | 3.21 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.910 | 2.271 | 0.173 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 85 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 417 | 417 | 608 | 1593 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.46 | 3.82 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.629 | 1.875 | 0.151 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 86 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 349 | 349 | 619 | 1473 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.77 | 4.22 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.323 | 1.017 | 0.148 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 87 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 315 | 315 | 524 | 1477 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.66 | 4.69 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.322 | 0.754 | 0.152 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 88 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 469 | 469 | 601 | 1639 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.28 | 3.49 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.837 | 0.803 | 0.131 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 89 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 463 | 463 | 541 | 1787 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.17 | 3.86 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.745 | 0.693 | 0.148 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 90 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 614 | 614 | 587 | 2139 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.96 | 3.48 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.914 | 1.049 | 0.145 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 91 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F(-2) | A | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 189 | 189 | 187 | 568 | 0 | 2579 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.99 | 3.01 | 0.00 | 13.65 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.179 | 0.654 | 0.158 | 0.000 | 94.788 | 0.000 | 0.000 | 0.000 |

| Problem 92 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 75 | 75 | 57 | 102 | 83 | 70 | 0 | 73 | -1 |
| N.S. | 1 | 1.00 | 0.76 | 1.36 | 1.11 | 0.93 | 0.00 | 0.97 | -0.01 |
| time (sec) | N/A | 0.034 | 0.093 | 0.182 | 0.486 | 0.345 | 0.000 | 5.636 | 0.000 |

| Problem 93 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 130 | 130 | 117 | 789 | 0 | 777 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.90 | 6.07 | 0.00 | 5.98 | 0.00 | 0.00 | -0.01 |
| time (sec) | N/A | 0.102 | 0.119 | 0.507 | 0.000 | 0.372 | 0.000 | 0.000 | 0.000 |

| Problem 94 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 369 | 369 | 252 | 513 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.68 | 1.39 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.511 | 0.723 | 0.144 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 95 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 287 | 287 | 210 | 409 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.73 | 1.43 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.408 | 0.393 | 0.130 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 96 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 266 | 266 | 194 | 399 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.73 | 1.50 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.126 | 0.289 | 0.132 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 97 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 220 | 220 | 149 | 354 | 0 | 2753 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.68 | 1.61 | 0.00 | 12.51 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.084 | 0.236 | 0.134 | 0.000 | 0.731 | 0.000 | 0.000 | 0.000 |

| Problem 98 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 220 | 220 | 151 | 358 | 0 | 2641 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.69 | 1.63 | 0.00 | 12.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.086 | 0.252 | 0.124 | 0.000 | 0.782 | 0.000 | 0.000 | 0.000 |

| Problem 99 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | B | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 267 | 267 | 193 | 391 | 0 | 5995 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.72 | 1.46 | 0.00 | 22.45 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.420 | 0.269 | 0.130 | 0.000 | 39.306 | 0.000 | 0.000 | 0.000 |

| Problem 100 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 291 | 291 | 216 | 426 | 0 | 6018 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.74 | 1.46 | 0.00 | 20.68 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.424 | 0.474 | 0.144 | 0.000 | 170.980 | 0.000 | 0.000 | 0.000 |

| Problem 101 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 376 | 376 | 241 | 516 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.64 | 1.37 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.494 | 0.754 | 0.140 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 102 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 466 | 466 | 453 | 1064 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.97 | 2.28 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.850 | 1.247 | 0.131 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 103 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 341 | 341 | 410 | 960 | 0 | 17339 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.20 | 2.82 | 0.00 | 50.85 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.668 | 0.767 | 0.131 | 0.000 | 15.122 | 0.000 | 0.000 | 0.000 |

| Problem 104 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 297 | 297 | 371 | 946 | 0 | 17285 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.25 | 3.19 | 0.00 | 58.20 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.284 | 0.721 | 0.143 | 0.000 | 13.210 | 0.000 | 0.000 | 0.000 |

| Problem 105 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 299 | 299 | 407 | 899 | 0 | 17258 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.36 | 3.01 | 0.00 | 57.72 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.239 | 0.736 | 0.120 | 0.000 | 14.928 | 0.000 | 0.000 | 0.000 |

| Problem 106 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 310 | 310 | 379 | 903 | 0 | 17397 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.22 | 2.91 | 0.00 | 56.12 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.249 | 0.716 | 0.129 | 0.000 | 13.361 | 0.000 | 0.000 | 0.000 |

| Problem 107 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 394 | 394 | 486 | 990 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.23 | 2.51 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.724 | 1.337 | 0.118 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 108 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 454 | 454 | 620 | 1065 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.37 | 2.35 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.761 | 2.271 | 0.145 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 109 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 761 | 761 | 863 | 1666 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.13 | 2.19 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.934 | 1.103 | 0.199 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 110 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 549 | 549 | 642 | 1580 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.17 | 2.88 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 4.554 | 0.696 | 0.193 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 111 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 431 | 431 | 396 | 1547 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.92 | 3.59 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.374 | 0.021 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 112 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 523 | 521 | 467 | 1691 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.89 | 3.23 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.139 | 0.536 | 0.177 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 113 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 736 | 736 | 582 | 1906 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.79 | 2.59 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.111 | 0.742 | 0.187 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 114 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 545 | 545 | 423 | 930 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.78 | 1.71 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.302 | 0.664 | 0.187 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 115 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-2) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 463 | 463 | 318 | 844 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.69 | 1.82 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.088 | 0.454 | 0.170 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 116 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 402 | 402 | 204 | 794 | 0 | 11131 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.51 | 1.98 | 0.00 | 27.69 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.614 | 0.381 | 0.164 | 0.000 | 5.121 | 0.000 | 0.000 | 0.000 |

| Problem 117 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | B | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 374 | 374 | 211 | 761 | 0 | 11127 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.56 | 2.03 | 0.00 | 29.75 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.185 | 0.010 | 0.000 | 0.000 | 4.971 | 0.000 | 0.000 | 0.000 |

| Problem 118 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 451 | 451 | 319 | 859 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.71 | 1.90 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 1.620 | 0.464 | 0.171 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 119 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 543 | 543 | 423 | 955 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.78 | 1.76 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 2.981 | 0.725 | 0.191 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 120 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | A | F | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 679 | 679 | 547 | 1117 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.81 | 1.65 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 8.386 | 1.323 | 0.191 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 121 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 779 | 779 | 690 | 2083 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.89 | 2.67 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 9.357 | 1.234 | 0.155 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 122 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 609 | 609 | 534 | 1992 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.88 | 3.27 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 3.768 | 0.925 | 0.176 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 123 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 609 | 609 | 570 | 1939 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 0.94 | 3.18 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 3.557 | 0.928 | 0.165 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 124 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F(-2) | F(-1) | F | F(-1) | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 666 | 666 | 692 | 1906 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.04 | 2.86 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.938 | 0.071 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 125 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | C | B | F | F(-1) | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 816 | 814 | 1025 | 2059 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.26 | 2.52 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 14.806 | 4.652 | 0.170 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 126 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 140 | 140 | 99 | 159 | 0 | 178 | 0 | 188 | -1 |
| N.S. | 1 | 1.00 | 0.71 | 1.14 | 0.00 | 1.27 | 0.00 | 1.34 | -0.01 |
| time (sec) | N/A | 0.311 | 0.237 | 0.449 | 0.000 | 0.363 | 0.000 | 5.901 | 0.000 |

| Problem 127 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 115 | 115 | 94 | 144 | 0 | 175 | 0 | 185 | -1 |
| N.S. | 1 | 1.00 | 0.82 | 1.25 | 0.00 | 1.52 | 0.00 | 1.61 | -0.01 |
| time (sec) | N/A | 0.255 | 0.197 | 0.410 | 0.000 | 0.446 | 0.000 | 3.579 | 0.000 |

| Problem 128 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 98 | 98 | 77 | 130 | 0 | 161 | 0 | 171 | -1 |
| N.S. | 1 | 1.00 | 0.79 | 1.33 | 0.00 | 1.64 | 0.00 | 1.74 | -0.01 |
| time (sec) | N/A | 0.115 | 0.172 | 0.324 | 0.000 | 0.447 | 0.000 | 3.977 | 0.000 |

| Problem 129 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 68 | 69 | 33 | 92 | 0 | 50 | 0 | 68 | -1 |
| N.S. | 1 | 1.01 | 0.49 | 1.35 | 0.00 | 0.74 | 0.00 | 1.00 | -0.01 |
| time (sec) | N/A | 0.037 | 0.124 | 0.137 | 0.000 | 0.468 | 0.000 | 3.623 | 0.000 |

| Problem 130 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 95 | 95 | 54 | 121 | 0 | 132 | 0 | 165 | -1 |
| N.S. | 1 | 1.00 | 0.57 | 1.27 | 0.00 | 1.39 | 0.00 | 1.74 | -0.01 |
| time (sec) | N/A | 0.066 | 0.162 | 0.325 | 0.000 | 0.395 | 0.000 | 3.462 | 0.000 |

| Problem 131 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 130 | 130 | 90 | 152 | 0 | 170 | 0 | 199 | -1 |
| N.S. | 1 | 1.00 | 0.69 | 1.17 | 0.00 | 1.31 | 0.00 | 1.53 | -0.01 |
| time (sec) | N/A | 0.258 | 0.181 | 0.371 | 0.000 | 0.383 | 0.000 | 3.092 | 0.000 |

| Problem 132 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 151 | 151 | 113 | 169 | 0 | 194 | 0 | 269 | -1 |
| N.S. | 1 | 1.00 | 0.75 | 1.12 | 0.00 | 1.28 | 0.00 | 1.78 | -0.01 |
| time (sec) | N/A | 0.282 | 0.226 | 0.394 | 0.000 | 0.395 | 0.000 | 3.933 | 0.000 |

| Problem 133 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 149 | 149 | 89 | 147 | 155 | 88 | 0 | 85 | 187 |
| N.S. | 1 | 1.00 | 0.60 | 0.99 | 1.04 | 0.59 | 0.00 | 0.57 | 1.26 |
| time (sec) | N/A | 0.057 | 0.498 | 0.138 | 0.511 | 0.381 | 0.000 | 3.383 | 5.085 |

| Problem 134 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 103 | 103 | 74 | 96 | 104 | 73 | 0 | 70 | 136 |
| N.S. | 1 | 1.00 | 0.72 | 0.93 | 1.01 | 0.71 | 0.00 | 0.68 | 1.32 |
| time (sec) | N/A | 0.027 | 0.267 | 0.086 | 0.494 | 0.440 | 0.000 | 3.695 | 4.690 |

| Problem 135 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | B | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 28 | 28 | 30 | 163 | 0 | 53 | 0 | 63 | -1 |
| N.S. | 1 | 1.00 | 1.07 | 5.82 | 0.00 | 1.89 | 0.00 | 2.25 | -0.04 |
| time (sec) | N/A | 0.028 | 0.146 | 0.134 | 0.000 | 0.370 | 0.000 | 5.629 | 0.000 |

| Problem 136 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | A | F | B | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 84 | 84 | 80 | 245 | 0 | 126 | 0 | 159 | -1 |
| N.S. | 1 | 1.00 | 0.95 | 2.92 | 0.00 | 1.50 | 0.00 | 1.89 | -0.01 |
| time (sec) | N/A | 0.047 | 0.287 | 0.123 | 0.000 | 0.378 | 0.000 | 4.421 | 0.000 |

| Problem 137 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | B | F | A | F | A | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 139 | 139 | 95 | 306 | 0 | 186 | 0 | 232 | -1 |
| N.S. | 1 | 1.00 | 0.68 | 2.20 | 0.00 | 1.34 | 0.00 | 1.67 | -0.01 |
| time (sec) | N/A | 0.075 | 0.373 | 0.138 | 0.000 | 0.388 | 0.000 | 4.273 | 0.000 |

| Problem 138 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | B | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 15 | 15 | 13 | 12 | 11 | 11 | 31 | 11 | 15 |
| N.S. | 1 | 1.00 | 0.87 | 0.80 | 0.73 | 0.73 | 2.07 | 0.73 | 1.00 |
| time (sec) | N/A | 0.003 | 9.309 | 0.111 | 0.274 | 0.326 | 0.074 | 3.683 | 3.723 |

| Problem 139 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | A | A | A | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 16 | 16 | 13 | 10 | 9 | 11 | 10 | 11 | 13 |
| N.S. | 1 | 1.00 | 0.81 | 0.62 | 0.56 | 0.69 | 0.62 | 0.69 | 0.81 |
| time (sec) | N/A | 0.004 | 0.004 | 0.079 | 0.273 | 0.357 | 3.345 | 2.622 | 3.664 |

| Problem 140 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 15 | 15 | 13 | 16 | 0 | 11 | 0 | 11 | 15 |
| N.S. | 1 | 1.00 | 0.87 | 1.07 | 0.00 | 0.73 | 0.00 | 0.73 | 1.00 |
| time (sec) | N/A | 0.017 | 0.006 | 0.096 | 0.000 | 0.355 | 0.000 | 2.657 | 3.675 |

| Problem 141 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | A | F | A | F | A | B |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 15 | 15 | 13 | 16 | 0 | 11 | 0 | 11 | 13 |
| N.S. | 1 | 1.00 | 0.87 | 1.07 | 0.00 | 0.73 | 0.00 | 0.73 | 0.87 |
| time (sec) | N/A | 0.032 | 0.005 | 0.073 | 0.000 | 0.344 | 0.000 | 3.372 | 3.607 |

| Problem 142 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F(-1) | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 242 | 242 | 381 | 0 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.57 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.056 | 0.717 | 0.029 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| Problem 143 | Optimal | Rubi | MMA | Maple | Maxima | Fricas | Sympy | Giac | Mupad |
|---|---|---|---|---|---|---|---|---|---|
| grade | A | A | A | F | F | F(-1) | F | F | F |
| verified | N/A | Yes | Yes | TBD | TBD | TBD | TBD | TBD | TBD |
| size | 488 | 488 | 593 | 0 | 0 | 0 | 0 | 0 | -1 |
| N.S. | 1 | 1.00 | 1.22 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | -0.00 |
| time (sec) | N/A | 0.243 | 2.298 | 0.172 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

## 2.3 Detailed conclusion table specific for Rubi results

The following table is specific to Rubi. It gives additional statistics for each integral. the column **steps** is the number of steps used by Rubi to obtain the antiderivative. The **rules** column is the number of unique rules used. The **integrand size** column is the leaf size of the integrand. Finally the ratio $\frac{\text{number of rules}}{\text{integrand size}}$ is given. The larger this ratio is, the harder the integral was to solve. In this test, problem number [143] had the largest ratio of [104]

Table 2.1: Rubi specific breakdown of results for each integral

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 1 | A | 5 | 4 | 1.00 | 25 | 0.160 |
| 2 | A | 5 | 4 | 1.00 | 27 | 0.148 |
| 3 | A | 5 | 4 | 1.00 | 27 | 0.148 |
| 4 | A | 8 | 8 | 1.00 | 27 | 0.296 |
| 5 | A | 9 | 9 | 1.00 | 27 | 0.333 |
| 6 | A | 9 | 6 | 1.00 | 30 | 0.200 |
| 7 | A | 5 | 3 | 1.00 | 30 | 0.100 |
| 8 | A | 6 | 4 | 1.00 | 30 | 0.133 |
| 9 | A | 7 | 5 | 1.00 | 30 | 0.167 |
| 10 | A | 5 | 4 | 1.00 | 23 | 0.174 |
| 11 | A | 5 | 4 | 1.00 | 23 | 0.174 |
| 12 | A | 5 | 4 | 1.00 | 30 | 0.133 |
| 13 | A | 6 | 5 | 0.99 | 28 | 0.179 |
| 14 | A | 6 | 5 | 1.00 | 30 | 0.167 |
| 15 | A | 9 | 5 | 0.98 | 30 | 0.167 |
| 16 | A | 10 | 6 | 0.99 | 30 | 0.200 |
| 17 | A | 5 | 5 | 1.00 | 34 | 0.147 |
| 18 | A | 5 | 5 | 1.00 | 34 | 0.147 |
| 19 | A | 9 | 6 | 1.00 | 32 | 0.188 |
| 20 | A | 10 | 7 | 1.00 | 32 | 0.219 |
| 21 | A | 5 | 3 | 1.00 | 32 | 0.094 |
| 22 | A | 5 | 3 | 1.00 | 29 | 0.103 |
| 23 | A | 5 | 3 | 1.00 | 29 | 0.103 |
| 24 | A | 6 | 6 | 1.00 | 26 | 0.231 |
| 25 | A | 5 | 4 | 1.00 | 30 | 0.133 |

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| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 26 | A | 7 | 6 | 1.00 | 30 | 0.200 |
| 27 | A | 7 | 6 | 1.00 | 30 | 0.200 |
| 28 | A | 5 | 3 | 1.00 | 30 | 0.100 |
| 29 | A | 6 | 4 | 1.00 | 30 | 0.133 |
| 30 | A | 7 | 5 | 1.00 | 30 | 0.167 |
| 31 | A | 2 | 2 | 1.00 | 26 | 0.077 |
| 32 | A | 5 | 5 | 1.00 | 26 | 0.192 |
| 33 | A | 2 | 2 | 1.00 | 24 | 0.083 |
| 34 | A | 5 | 5 | 1.00 | 20 | 0.250 |
| 35 | A | 6 | 5 | 1.00 | 36 | 0.139 |
| 36 | A | 2 | 2 | 1.00 | 36 | 0.056 |
| 37 | A | 2 | 2 | 1.00 | 32 | 0.062 |
| 38 | A | 13 | 9 | 1.00 | 32 | 0.281 |
| 39 | A | 5 | 5 | 1.00 | 38 | 0.132 |
| 40 | A | 6 | 6 | 1.00 | 35 | 0.171 |
| 41 | A | 5 | 5 | 1.00 | 33 | 0.152 |
| 42 | A | 5 | 5 | 1.00 | 32 | 0.156 |
| 43 | A | 8 | 8 | 1.00 | 35 | 0.229 |
| 44 | A | 8 | 8 | 1.00 | 35 | 0.229 |
| 45 | A | 8 | 8 | 1.00 | 35 | 0.229 |
| 46 | A | 6 | 6 | 1.00 | 38 | 0.158 |
| 47 | A | 5 | 5 | 1.00 | 36 | 0.139 |
| 48 | A | 5 | 5 | 1.00 | 35 | 0.143 |
| 49 | A | 7 | 6 | 1.00 | 38 | 0.158 |
| 50 | A | 7 | 6 | 1.00 | 38 | 0.158 |
| 51 | A | 7 | 6 | 1.00 | 38 | 0.158 |
| 52 | A | 9 | 6 | 1.00 | 27 | 0.222 |
| 53 | A | 9 | 6 | 1.00 | 25 | 0.240 |
| 54 | A | 8 | 5 | 1.00 | 24 | 0.208 |
| 55 | A | 12 | 9 | 1.00 | 27 | 0.333 |
| 56 | A | 18 | 12 | 1.00 | 27 | 0.444 |
| 57 | A | 22 | 13 | 1.00 | 27 | 0.482 |
| 58 | A | 10 | 7 | 1.00 | 27 | 0.259 |
| 59 | A | 10 | 7 | 1.00 | 25 | 0.280 |
| 60 | A | 9 | 6 | 1.00 | 24 | 0.250 |

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Table 2.1 – continued from previous page

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 61 | A | 17 | 11 | 1.00 | 27 | 0.407 |
| 62 | A | 21 | 14 | 1.00 | 27 | 0.518 |
| 63 | A | 26 | 15 | 1.00 | 27 | 0.556 |
| 64 | A | 10 | 6 | 1.00 | 27 | 0.222 |
| 65 | A | 8 | 5 | 1.00 | 27 | 0.185 |
| 66 | A | 5 | 3 | 1.00 | 25 | 0.120 |
| 67 | A | 5 | 3 | 1.00 | 24 | 0.125 |
| 68 | A | 10 | 7 | 1.00 | 27 | 0.259 |
| 69 | A | 11 | 8 | 1.00 | 27 | 0.296 |
| 70 | A | 15 | 9 | 1.00 | 27 | 0.333 |
| 71 | A | 10 | 7 | 1.00 | 27 | 0.259 |
| 72 | A | 6 | 4 | 1.00 | 27 | 0.148 |
| 73 | A | 6 | 4 | 1.00 | 25 | 0.160 |
| 74 | A | 6 | 4 | 1.00 | 24 | 0.167 |
| 75 | A | 12 | 9 | 1.00 | 27 | 0.333 |
| 76 | A | 14 | 11 | 1.00 | 27 | 0.407 |
| 77 | A | 15 | 9 | 1.00 | 28 | 0.321 |
| 78 | A | 9 | 6 | 1.00 | 28 | 0.214 |
| 79 | A | 9 | 6 | 1.00 | 26 | 0.231 |
| 80 | A | 8 | 5 | 1.00 | 25 | 0.200 |
| 81 | A | 17 | 9 | 1.00 | 28 | 0.321 |
| 82 | A | 16 | 8 | 1.00 | 28 | 0.286 |
| 83 | A | 20 | 10 | 1.00 | 28 | 0.357 |
| 84 | A | 17 | 10 | 1.00 | 28 | 0.357 |
| 85 | A | 10 | 7 | 1.00 | 28 | 0.250 |
| 86 | A | 10 | 7 | 1.00 | 26 | 0.269 |
| 87 | A | 9 | 6 | 1.00 | 25 | 0.240 |
| 88 | A | 19 | 11 | 1.00 | 28 | 0.393 |
| 89 | A | 18 | 10 | 1.00 | 28 | 0.357 |
| 90 | A | 26 | 13 | 1.00 | 28 | 0.464 |
| 91 | A | 9 | 6 | 1.00 | 24 | 0.250 |
| 92 | A | 8 | 6 | 1.00 | 22 | 0.273 |
| 93 | A | 10 | 9 | 1.00 | 17 | 0.529 |
| 94 | A | 13 | 7 | 1.00 | 28 | 0.250 |
| 95 | A | 10 | 6 | 1.00 | 28 | 0.214 |

Continued on next page

Table 2.1 – continued from previous page

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 96 | A | 8 | 5 | 1.00 | 28 | 0.179 |
| 97 | A | 5 | 3 | 1.00 | 26 | 0.115 |
| 98 | A | 5 | 3 | 1.00 | 25 | 0.120 |
| 99 | A | 9 | 4 | 1.00 | 28 | 0.143 |
| 100 | A | 10 | 5 | 1.00 | 28 | 0.179 |
| 101 | A | 13 | 6 | 1.00 | 28 | 0.214 |
| 102 | A | 13 | 9 | 1.00 | 28 | 0.321 |
| 103 | A | 9 | 6 | 1.00 | 28 | 0.214 |
| 104 | A | 6 | 4 | 1.00 | 28 | 0.143 |
| 105 | A | 6 | 4 | 1.00 | 26 | 0.154 |
| 106 | A | 6 | 4 | 1.00 | 25 | 0.160 |
| 107 | A | 12 | 7 | 1.00 | 28 | 0.250 |
| 108 | A | 12 | 7 | 1.00 | 28 | 0.250 |
| 109 | A | 9 | 6 | 1.00 | 30 | 0.200 |
| 110 | A | 9 | 6 | 1.00 | 28 | 0.214 |
| 111 | A | 8 | 5 | 1.00 | 27 | 0.185 |
| 112 | A | 17 | 9 | 1.00 | 30 | 0.300 |
| 113 | A | 23 | 10 | 1.00 | 30 | 0.333 |
| 114 | A | 12 | 6 | 1.00 | 30 | 0.200 |
| 115 | A | 8 | 5 | 1.00 | 30 | 0.167 |
| 116 | A | 5 | 3 | 1.00 | 28 | 0.107 |
| 117 | A | 5 | 3 | 1.00 | 27 | 0.111 |
| 118 | A | 9 | 4 | 1.00 | 30 | 0.133 |
| 119 | A | 12 | 5 | 1.00 | 30 | 0.167 |
| 120 | A | 16 | 7 | 1.00 | 30 | 0.233 |
| 121 | A | 10 | 7 | 1.00 | 30 | 0.233 |
| 122 | A | 6 | 4 | 1.00 | 30 | 0.133 |
| 123 | A | 6 | 4 | 1.00 | 28 | 0.143 |
| 124 | A | 6 | 4 | 1.00 | 27 | 0.148 |
| 125 | A | 12 | 7 | 1.00 | 30 | 0.233 |
| 126 | A | 24 | 14 | 1.00 | 30 | 0.467 |
| 127 | A | 20 | 13 | 1.00 | 30 | 0.433 |
| 128 | A | 16 | 12 | 1.00 | 30 | 0.400 |
| 129 | A | 6 | 4 | 1.01 | 28 | 0.143 |
| 130 | A | 10 | 8 | 1.00 | 27 | 0.296 |

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Table 2.1 – continued from previous page

| # | grade | number of steps used | number of unique rules | normalized antiderivative leaf size | integrand leaf size | $\frac{\text{number of rules}}{\text{integrand leaf size}}$ |
|---|---|---|---|---|---|---|
| 131 | A | 17 | 11 | 1.00 | 30 | 0.367 |
| 132 | A | 20 | 12 | 1.00 | 30 | 0.400 |
| 133 | A | 8 | 6 | 1.00 | 34 | 0.176 |
| 134 | A | 6 | 5 | 1.00 | 30 | 0.167 |
| 135 | A | 3 | 3 | 1.00 | 34 | 0.088 |
| 136 | A | 5 | 5 | 1.00 | 34 | 0.147 |
| 137 | A | 7 | 7 | 1.00 | 34 | 0.206 |
| 138 | A | 1 | 1 | 1.00 | 17 | 0.059 |
| 139 | A | 1 | 1 | 1.00 | 15 | 0.067 |
| 140 | A | 2 | 2 | 1.00 | 23 | 0.087 |
| 141 | A | 3 | 3 | 1.00 | 21 | 0.143 |
| 142 | A | 2 | 2 | 1.00 | 40 | 0.050 |
| 143 | A | 2 | 2 | 1.00 | 104 | 0.019 |

# Chapter 3

# Listing of integrals

## Local contents

# 3.1 $\int \frac{(A+Bx)(a+bx+cx^2)}{d+fx^2}\,dx$

**Optimal**. Leaf size=94

$$\frac{(bB+Ac)x}{f}+\frac{Bcx^2}{2f}-\frac{(bBd+Acd-aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,f^{3/2}}-\frac{(Bcd-Abf-aBf)\log(d+fx^2)}{2f^2}$$

```
[Out] (A*c+B*b)*x/f+1/2*B*c*x^2/f-1/2*(-A*b*f-B*a*f+B*c*d)*ln(f*x^2+d)/f^2-(-A*a*
f+A*c*d+B*b*d)*arctan(x*f^(1/2)/d^(1/2))/f^(3/2)/d^(1/2)
```

**Rubi** [A]
time = 0.07, antiderivative size = 94, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.160$, Rules used = {1643, 649, 211, 266}

$$-\frac{\text{ArcTan}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)(-aAf+Acd+bBd)}{\sqrt{d}\,f^{3/2}}-\frac{\log(d+fx^2)(-aBf-Abf+Bcd)}{2f^2}+\frac{x(Ac+bB)}{f}+\frac{Bcx^2}{2f}$$

Antiderivative was successfully verified.

```
[In] Int[((A + B*x)*(a + b*x + c*x^2))/(d + f*x^2),x]
```

```
[Out] ((b*B + A*c)*x)/f + (B*c*x^2)/(2*f) - ((b*B*d + A*c*d - a*A*f)*ArcTan[(Sqrt
[f]*x)/Sqrt[d]])/(Sqrt[d]*f^(3/2)) - ((B*c*d - A*b*f - a*B*f)*Log[d + f*x^2
])/(2*f^2)
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 266

```
Int[(x_)^(m_.)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> Simp[Log[RemoveConten
t[a + b*x^n, x]]/(b*n), x] /; FreeQ[{a, b, m, n}, x] && EqQ[m, n - 1]
```

Rule 649

```
Int[((d_) + (e_.)*(x_))/((a_) + (c_.)*(x_)^2), x_Symbol] :> Dist[d, Int[1/(
a + c*x^2), x], x] + Dist[e, Int[x/(a + c*x^2), x], x] /; FreeQ[{a, c, d, e
}, x] &&  !NiceSqrtQ[(-a)*c]
```

Rule 1643

```
Int[(Pq_)*((d_) + (e_.)*(x_))^(m_.)*((a_) + (c_.)*(x_)^2)^(p_.), x_Symbol]
:> Int[ExpandIntegrand[(d + e*x)^m*Pq*(a + c*x^2)^p, x], x] /; FreeQ[{a, c,
```

```
d, e, m}, x] && PolyQ[Pq, x] && IGtQ[p, -2]
```

Rubi steps

$$\int \frac{(A+Bx)(a+bx+cx^2)}{d+fx^2}\,dx = \int \left(\frac{bB+Ac}{f}+\frac{Bcx}{f}-\frac{bBd+Acd-aAf+(Bcd-Abf-aBf)x}{f(d+fx^2)}\right)dx$$

$$= \frac{(bB+Ac)x}{f}+\frac{Bcx^2}{2f}-\frac{\int \frac{bBd+Acd-aAf+(Bcd-Abf-aBf)x}{d+fx^2}\,dx}{f}$$

$$= \frac{(bB+Ac)x}{f}+\frac{Bcx^2}{2f}-\frac{(bBd+Acd-aAf)\int \frac{1}{d+fx^2}\,dx}{f}-\frac{(Bcd-Abf-a}{f}$$

$$= \frac{(bB+Ac)x}{f}+\frac{Bcx^2}{2f}-\frac{(bBd+Acd-aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,f^{3/2}}-\frac{(Bcd-Ab}{}$$

**Mathematica** [A]
time = 0.05, size = 86, normalized size = 0.91

$$\frac{fx(2bB+2Ac+Bcx)-\frac{2\sqrt{f}\,(bBd+Acd-aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}}+(-Bcd+Abf+aBf)\log(d+fx^2)}{2f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2))/(d + f*x^2),x]
```

```
[Out] (f*x*(2*b*B + 2*A*c + B*c*x) - (2*Sqrt[f]*(b*B*d + A*c*d - a*A*f)*ArcTan[(S
qrt[f]*x)/Sqrt[d]])/Sqrt[d] + (-(B*c*d) + A*b*f + a*B*f)*Log[d + f*x^2])/(2
*f^2)
```

**Maple** [A]
time = 0.11, size = 84, normalized size = 0.89

| method | result |
|---|---|
| default | $\frac{\frac{1}{2}Bc\,x^2+Acx+bBx}{f}+\frac{\frac{(Abf+Baf-Bcd)\ln\left(f\,x^2+d\right)}{2f}+\frac{(Aaf-Acd-Bbd)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}}}{f}$ |
| risch | $\frac{Bc\,x^2}{2f}+\frac{Acx}{f}+\frac{bBx}{f}+\frac{\ln\left(Aafd-Ac\,d^2-Bb\,d^2-\sqrt{-df\,(Aaf-Acd-Bbd)^2}\,x\right)Ab}{2f}+\frac{\ln\left(Aafd-Ac\,d^2-Bb\,d^2-\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)/(f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(1/2*B*c*x^2+A*c*x+b*B*x)+1/f*(1/2*(A*b*f+B*a*f-B*c*d)/f*ln(f*x^2+d)+(A
*a*f-A*c*d-B*b*d)/(d*f)^(1/2)*arctan(f*x/(d*f)^(1/2)))
```

**Maxima** [A]
time = 0.51, size = 84, normalized size = 0.89

$$\frac{(Aaf-(Bb+Ac)d)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}\,f}+\frac{Bcx^2+2\,(Bb+Ac)x}{2\,f}-\frac{(Bcd-(Ba+Ab)f)\log\left(fx^2+d\right)}{2\,f^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="maxima")
```

```
[Out] (A*a*f - (B*b + A*c)*d)*arctan(f*x/sqrt(d*f))/(sqrt(d*f)*f) + 1/2*(B*c*x^2
+ 2*(B*b + A*c)*x)/f - 1/2*(B*c*d - (B*a + A*b)*f)*log(f*x^2 + d)/f^2
```

**Fricas** [A]
time = 0.78, size = 200, normalized size = 2.13

$$\left[\frac{Bcdfx^2+2\,(Bb+Ac)dfx-(Aaf-(Bb+Ac)d)\sqrt{-df}\,\log\left(\frac{fx^2-2\,\sqrt{-df}\,x-d}{fx^2+d}\right)-(Bcd^2-(Ba+Ab)df)\log\left(fx^2+d\right)}{2\,df^2},\ \frac{Bcdfx^2+2\,(Bb+Ac)dfx+2\,(Aaf-(Bb+Ac)d)\sqrt{df}\,\arctan\left(\frac{\sqrt{df}\,x}{d}\right)-(Bcd^2-(Ba+Ab)df)\log\left(fx^2+d\right)}{2\,df^2}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/2*(B*c*d*f*x^2 + 2*(B*b + A*c)*d*f*x - (A*a*f - (B*b + A*c)*d)*sqrt(-d*f
)*log((f*x^2 - 2*sqrt(-d*f)*x - d)/(f*x^2 + d)) - (B*c*d^2 - (B*a + A*b)*d*
f)*log(f*x^2 + d))/(d*f^2), 1/2*(B*c*d*f*x^2 + 2*(B*b + A*c)*d*f*x + 2*(A*a
*f - (B*b + A*c)*d)*sqrt(d*f)*arctan(sqrt(d*f)*x/d) - (B*c*d^2 - (B*a + A*b
)*d*f)*log(f*x^2 + d))/(d*f^2)]
```

**Sympy** [B] Leaf count of result is larger than twice the leaf count of optimal. 333 vs. $2(90) = 180$.
time = 0.84, size = 333, normalized size = 3.54

$$\frac{Bcx^2}{2f}+x\left(\frac{Ac}{f}+\frac{Bb}{f}\right)+\left(\frac{Abf+Baf-Bcd}{2f^2}-\frac{\sqrt{-df^5}\,(Aaf-Acd-Bbd)}{2df^4}\right)\log\left(x+\frac{-Abdf-Badf+Bcd^2+2df^2\left(\frac{Abf+Baf-Bcd}{2f^2}-\frac{\sqrt{-df^5}\,(Aaf-Acd-Bbd)}{2df^4}\right)}{Aaf^2-Acdf-Bbdf}\right)+\left(\frac{Abf+Baf-Bcd}{2f^2}+\frac{\sqrt{-df^5}\,(Aaf-Acd-Bbd)}{2df^4}\right)\log\left(x+\frac{-Abdf-Badf+Bcd^2+2df^2\left(\frac{Abf+Baf-Bcd}{2f^2}+\frac{\sqrt{-df^5}\,(Aaf-Acd-Bbd)}{2df^4}\right)}{Aaf^2-Acdf-Bbdf}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)/(f*x**2+d),x)
```

```
[Out] B*c*x**2/(2*f) + x*(A*c/f + B*b/f) + ((A*b*f + B*a*f - B*c*d)/(2*f**2) - sq
rt(-d*f**5)*(A*a*f - A*c*d - B*b*d)/(2*d*f**4))*log(x + (-A*b*d*f - B*a*d*f
```

```
 + B*c*d**2 + 2*d*f**2*((A*b*f + B*a*f - B*c*d)/(2*f**2) - sqrt(-d*f**5)*(A
*a*f - A*c*d - B*b*d)/(2*d*f**4)))/(A*a*f**2 - A*c*d*f - B*b*d*f)) + ((A*b*
f + B*a*f - B*c*d)/(2*f**2) + sqrt(-d*f**5)*(A*a*f - A*c*d - B*b*d)/(2*d*f*
*4))*log(x + (-A*b*d*f - B*a*d*f + B*c*d**2 + 2*d*f**2*((A*b*f + B*a*f - B*
c*d)/(2*f**2) + sqrt(-d*f**5)*(A*a*f - A*c*d - B*b*d)/(2*d*f**4)))/(A*a*f**
2 - A*c*d*f - B*b*d*f))
```

**Giac** [A]
time = 4.09, size = 87, normalized size = 0.93

$$-\frac{(Bbd + Acd - Aaf)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}\,f} - \frac{(Bcd - Baf - Abf)\log\left(fx^2 + d\right)}{2\,f^2} + \frac{Bcfx^2 + 2\,Bbfx + 2\,Acfx}{2\,f^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="giac")
```

```
[Out] -(B*b*d + A*c*d - A*a*f)*arctan(f*x/sqrt(d*f))/(sqrt(d*f)*f) - 1/2*(B*c*d -
 B*a*f - A*b*f)*log(f*x^2 + d)/f^2 + 1/2*(B*c*f*x^2 + 2*B*b*f*x + 2*A*c*f*x
)/f^2
```

**Mupad** [B]
time = 3.44, size = 97, normalized size = 1.03

$$\frac{x\,\left(A\,c+B\,b\right)}{f}-\frac{\mathrm{atan}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)\,\left(A\,c\,d-A\,a\,f+B\,b\,d\right)}{\sqrt{d}\,f^{3/2}}+\frac{B\,c\,x^2}{2\,f}+\frac{\ln\left(f\,x^2+d\right)\,\left(4\,A\,b\,d\,f^3+4\,B\,a\,d\,f^3-4\,B\,c\,d^2\,f^2\right)}{8\,d\,f^4}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2))/(d + f*x^2),x)
```

```
[Out] (x*(A*c + B*b))/f - (atan((f^(1/2)*x)/d^(1/2))*(A*c*d - A*a*f + B*b*d))/(d^
(1/2)*f^(3/2)) + (B*c*x^2)/(2*f) + (log(d + f*x^2)*(4*A*A*b*d*f^3 + 4*B*a*d*f
^3 - 4*B*c*d^2*f^2))/(8*d*f^4)
```

## 3.2 $\int \frac{(A+Bx)(a+bx+cx^2)^2}{d+fx^2}\,dx$

**Optimal**. Leaf size=228

$$\frac{(Ab^2f-Ac(cd-2af)-bB(2cd-2af))\,x}{f^2}+\frac{(2Abcf-B(c^2d-b^2f-2acf))\,x^2}{2f^2}+\frac{c(2bB+Ac)x^3}{3f}+\frac{Bc^2x^4}{4f}-$$

```
[Out] (A*b^2*f-A*c*(-2*a*f+c*d)-b*B*(-2*a*f+2*c*d))*x/f^2+1/2*(2*A*b*c*f-B*(-2*a*
c*f-b^2*f+c^2*d))*x^2/f^2+1/3*c*(A*c+2*B*b)*x^3/f+1/4*B*c^2*x^4/f-1/2*(2*A*
b*f*(-a*f+c*d)-B*(c^2*d^2-2*a*c*d*f-f*(-a^2*f+b^2*d)))*ln(f*x^2+d)/f^3-(A*b
^2*d*f-2*b*B*d*(-a*f+c*d)-A*(-a*f+c*d)^2)*arctan(x*f^(1/2)/d^(1/2))/f^(5/2)
/d^(1/2)
```

**Rubi** [A]
time = 0.19, antiderivative size = 228, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.148$, Rules used = {1026, 649, 211, 266}

$$-\frac{\log(d+fx^2)(2Abf(cd-af)-B(-f(b^2d-a^2f)-2acdf+c^2d^2))}{2f^3}-\frac{\text{ArcTan}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)(-A(cd-af)^2-2bBd(cd-af)+Ab^2df)}{\sqrt{d}\,f^{5/2}}+\frac{x^2(2Abcf-B(-2acf+b^2(-f)+c^2d))}{2f^2}+\frac{x(-Ac(cd-2af)-bB(2cd-2af)+Ab^2f)}{f^2}+\frac{cx^3(Ac+2bB)}{3f}+\frac{Bc^2x^4}{4f}$$

Antiderivative was successfully verified.

```
[In] Int[((A + B*x)*(a + b*x + c*x^2)^2)/(d + f*x^2),x]
```

```
[Out] ((A*b^2*f - A*c*(c*d - 2*a*f) - b*B*(2*c*d - 2*a*f))*x)/f^2 + ((2*A*b*c*f -
 B*(c^2*d - b^2*f - 2*a*c*f))*x^2)/(2*f^2) + (c*(2*b*B + A*c)*x^3)/(3*f) +
(B*c^2*x^4)/(4*f) - ((A*b^2*d*f - 2*b*B*d*(c*d - a*f) - A*(c*d - a*f)^2)*Ar
cTan[(Sqrt[f]*x)/Sqrt[d]])/(Sqrt[d]*f^(5/2)) - ((2*A*b*f*(c*d - a*f) - B*(c
^2*d^2 - 2*a*c*d*f - f*(b^2*d - a^2*f)))*Log[d + f*x^2])/(2*f^3)
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 266

```
Int[(x_)^(m_.)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> Simp[Log[RemoveConten
t[a + b*x^n, x]]/(b*n), x] /; FreeQ[{a, b, m, n}, x] && EqQ[m, n - 1]
```

Rule 649

```
Int[((d_) + (e_.)*(x_))/((a_) + (c_.)*(x_)^2), x_Symbol] :> Dist[d, Int[1/(
a + c*x^2), x], x] + Dist[e, Int[x/(a + c*x^2), x], x] /; FreeQ[{a, c, d, e
}, x] &&  !NiceSqrtQ[(-a)*c]
```

Rule 1026

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Int[ExpandIntegrand[(a + c*x^2)^p*(d + e*x +
 f*x^2)^q*(g + h*x), x], x] /; FreeQ[{a, c, d, e, f, g, h}, x] && NeQ[e^2 -
 4*d*f, 0] && IntegersQ[p, q] && (GtQ[p, 0] || GtQ[q, 0])
```

Rubi steps

$$\int \frac{(A+Bx)\left(a+bx+cx^2\right)^2}{d+fx^2}\,dx = \int \left(\frac{Ab^2f-Ac(cd-2af)-bB(2cd-2af)}{f^2}+\frac{(2Abcf-B(c^2d-b^2f-2\ldots}{f^2}\right.$$

$$= \frac{(Ab^2f-Ac(cd-2af)-bB(2cd-2af))\,x}{f^2}+\frac{(2Abcf-B(c^2d-b^2f-2\ldots}{2f^2}$$

$$= \frac{(Ab^2f-Ac(cd-2af)-bB(2cd-2af))\,x}{f^2}+\frac{(2Abcf-B(c^2d-b^2f-2\ldots}{2f^2}$$

$$= \frac{(Ab^2f-Ac(cd-2af)-bB(2cd-2af))\,x}{f^2}+\frac{(2Abcf-B(c^2d-b^2f-2\ldots}{2f^2}$$

**Mathematica** [A]
time = 0.15, size = 204, normalized size = 0.89

$$\frac{(-Ab^2df+2bBd(cd-af)+A(cd-af)^2)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,f^{5/2}}+\frac{fx(12Abcfx+6b^2f(2A+Bx)+3Bcx(-2cd+4af+cfx^2)+4Ac(-3cd+6af+cfx^2)+4bB(-6cd+6af+2cfx^2))+6(2Abf(-cd+af)+B(c^2d^2-b^2df-2acdf+a^2f^2))\log\left(d+fx^2\right)}{12f^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2)^2)/(d + f*x^2),x]
```

```
[Out] ((-(A*b^2*d*f) + 2*b*B*d*(c*d - a*f) + A*(c*d - a*f)^2)*ArcTan[(Sqrt[f]*x)/
Sqrt[d]])/(Sqrt[d]*f^(5/2)) + (f*x*(12*A*b*c*f*x + 6*b^2*f*(2*A + B*x) + 3*
B*c*x*(-2*c*d + 4*a*f + c*f*x^2) + 4*A*c*(-3*c*d + 6*a*f + c*f*x^2) + 4*b*B
*(-6*c*d + 6*a*f + 2*c*f*x^2)) + 6*(2*A*b*f*(-(c*d) + a*f) + B*(c^2*d^2 - b
^2*d*f - 2*a*c*d*f + a^2*f^2))*Log[d + f*x^2])/(12*f^3)
```

**Maple** [A]
time = 0.16, size = 235, normalized size = 1.03

| method | result |
|---|---|
| default | $\frac{\frac{1}{4}B\,c^2x^4f+\frac{1}{3}A\,c^2f\,x^3+\frac{2}{3}Bbcf\,x^3+Abcf\,x^2+Bacf\,x^2+\frac{1}{2}B\,b^2f\,x^2-\frac{1}{2}B\,c^2d\,x^2+2Aacfx+A\,b^2fx-A\,c^2dx+2Babfx-2Bbcdx}{f^2}+\frac{(2Ac\ldots}{\ldots}$ |

| risch | Expression too large to display |
|---|---|

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f^2*(1/4*B*c^2*x^4*f+1/3*A*c^2*f*x^3+2/3*B*b*c*f*x^3+A*b*c*f*x^2+B*a*c*f*
x^2+1/2*B*b^2*f*x^2-1/2*B*c^2*d*x^2+2*A*a*c*f*x+A*b^2*f*x-A*c^2*d*x+2*B*a*b
*f*x-2*B*b*c*d*x)+1/f^2*(1/2*(2*A*a*b*f^2-2*A*b*c*d*f+B*a^2*f^2-2*B*a*c*d*f
-B*b^2*d*f+B*c^2*d^2)/f*ln(f*x^2+d)+(A*a^2*f^2-2*A*a*c*d*f-A*b^2*d*f+A*c^2*
d^2-2*B*a*b*d*f+2*B*b*c*d^2)/(d*f)^(1/2)*arctan(f*x/(d*f)^(1/2)))
```

**Maxima [A]**
time = 0.50, size = 220, normalized size = 0.96

$$\frac{(Aa^2f^2 + (2\,Bbc + Ac^2)d^2 - (2\,Bab + Ab^2 + 2\,Aac)df)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}\,f^2} + \frac{3\,Bc^2fx^4 + 4\,(2\,Bbc + Ac^2)fx^3 - 6\,(Bc^2d - (Bb^2 + 2\,(Ba + Ab)c)f)x^2 - 12\,((2\,Bbc + Ac^2)d - (2\,Bab + Ab^2 + 2\,Aac)f)x}{12\,f^2} + \frac{(Bc^2d^2 - (Bb^2 + 2\,(Ba + Ab)c)df + (Ba^2 + 2\,Aab)f^2)\log(fx^2 + d)}{2\,f^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="maxima")
```

```
[Out] (A*a^2*f^2 + (2*B*b*c + A*c^2)*d^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f)*arcta
n(f*x/sqrt(d*f))/(sqrt(d*f)*f^2) + 1/12*(3*B*c^2*f*x^4 + 4*(2*B*b*c + A*c^2
)*f*x^3 - 6*(B*c^2*d - (B*b^2 + 2*(B*a + A*b)*c)*f)*x^2 - 12*((2*B*b*c + A*
c^2)*d - (2*B*a*b + A*b^2 + 2*A*a*c)*f)*x)/f^2 + 1/2*(B*c^2*d^2 - (B*b^2 +
2*(B*a + A*b)*c)*d*f + (B*a^2 + 2*A*a*b)*f^2)*log(f*x^2 + d)/f^3
```

**Fricas [A]**
time = 0.33, size = 500, normalized size = 2.19

$$\left[\frac{3\,Bc^2df^2x^4 + 4\,(2\,Bbc + Ac^2)df^2x^3 - 6\,(Bc^2d^2f - (Bb^2 + 2\,(Ba + Ab)c)df^2)x^2 - 6\,(Aa^2f^2 + (2\,Bbc + Ac^2)d^2 - (2\,Bab + Ab^2 + 2\,Aac)df)\sqrt{-df}\log\left(\frac{fx^2 - 2\,\sqrt{-df}\,x - d}{fx^2 + d}\right) - 12\,((2\,Bbc + Ac^2)d^2f - (2\,Bab + Ab^2 + 2\,Aac)df^2)x + 6\,(Bc^2d^3 - (Bb^2 + 2\,(Ba + Ab)c)d^2f + (Ba^2 + 2\,Aab)df^2)\log(fx^2 + d)}{12\,df^3}, \frac{3\,Bc^2df^2x^4 + 4\,(2\,Bbc + Ac^2)df^2x^3 - 6\,(Bc^2d^2f - (Bb^2 + 2\,(Ba + Ab)c)df^2)x^2 + 12\,(Aa^2f^2 + (2\,Bbc + Ac^2)d^2 - (2\,Bab + Ab^2 + 2\,Aac)df)\sqrt{df}\arctan\left(\frac{\sqrt{df}\,x}{d}\right) - 12\,((2\,Bbc + Ac^2)d^2f - (2\,Bab + Ab^2 + 2\,Aac)df^2)x + 6\,(Bc^2d^3 - (Bb^2 + 2\,(Ba + Ab)c)d^2f + (Ba^2 + 2\,Aab)df^2)\log(fx^2 + d)}{12\,df^3}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/12*(3*B*c^2*d*f^2*x^4 + 4*(2*B*b*c + A*c^2)*d*f^2*x^3 - 6*(B*c^2*d^2*f -
 (B*b^2 + 2*(B*a + A*b)*c)*d*f^2)*x^2 - 6*(A*a^2*f^2 + (2*B*b*c + A*c^2)*d^
2 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f)*sqrt(-d*f)*log((f*x^2 - 2*sqrt(-d*f)*x
 - d)/(f*x^2 + d)) - 12*((2*B*b*c + A*c^2)*d^2*f - (2*B*a*b + A*b^2 + 2*A*a
*c)*d*f^2)*x + 6*(B*c^2*d^3 - (B*b^2 + 2*(B*a + A*b)*c)*d^2*f + (B*a^2 + 2*
A*a*b)*d*f^2)*log(f*x^2 + d))/(d*f^3), 1/12*(3*B*c^2*d*f^2*x^4 + 4*(2*B*b*c
 + A*c^2)*d*f^2*x^3 - 6*(B*c^2*d^2*f - (B*b^2 + 2*(B*a + A*b)*c)*d*f^2)*x^2
 + 12*(A*a^2*f^2 + (2*B*b*c + A*c^2)*d^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f)
*sqrt(d*f)*arctan(sqrt(d*f)*x/d) - 12*((2*B*b*c + A*c^2)*d^2*f - (2*B*a*b +
 A*b^2 + 2*A*a*c)*d*f^2)*x + 6*(B*c^2*d^3 - (B*b^2 + 2*(B*a + A*b)*c)*d^2*f
 + (B*a^2 + 2*A*a*b)*d*f^2)*log(f*x^2 + d))/(d*f^3)]
```

**Sympy** [B] Leaf count of result is larger than twice the leaf count of optimal. 933 vs. $2(209) = 418$.
time = 15.74, size = 933, normalized size = 4.09

$$\frac{Bc^2x^4}{4f} + x^3\left(\frac{Ac^2}{3f}+\frac{2Bbc}{3f}\right) + x^2\left(\frac{Abc}{f}+\frac{Bac}{f}+\frac{Bb^2}{2f}-\frac{Bc^2d}{2f^2}\right) + x\left(\frac{2Aac}{f}+\frac{Ab^2}{f}-\frac{Ac^2d}{f^2}+\frac{2Bab}{f}-\frac{2Bbcd}{f^2}\right) + \left(\frac{2Aabf^2-2Abcdf+Ba^2f^2-2Bacdf-Bb^2df+Bc^2d^2}{2f^3} - \frac{\sqrt{-df^7}\,(Aa^2f^2-2Aacdf-Ab^2df+Ac^2d^2-2Babdf+2Bbcd^2)}{2df^6}\right)\log\left(x + \frac{-2Aabdf^2+2Abcd^2f-Ba^2df^2+2Bacd^2f+Bb^2d^2f-Bc^2d^3+2df^3\left(\frac{2Aabf^2-2Abcdf+Ba^2f^2-2Bacdf-Bb^2df+Bc^2d^2}{2f^3} - \frac{\sqrt{-df^7}\,(Aa^2f^2-2Aacdf-Ab^2df+Ac^2d^2-2Babdf+2Bbcd^2)}{2df^6}\right)}{Aa^2f^3-2Aacdf^2-Ab^2df^2+Ac^2d^2f-2Babdf^2+2Bbcd^2f}\right) + \left(\frac{2Aabf^2-2Abcdf+Ba^2f^2-2Bacdf-Bb^2df+Bc^2d^2}{2f^3} + \frac{\sqrt{-df^7}\,(Aa^2f^2-2Aacdf-Ab^2df+Ac^2d^2-2Babdf+2Bbcd^2)}{2df^6}\right)\log\left(x + \frac{-2Aabdf^2+2Abcd^2f-Ba^2df^2+2Bacd^2f+Bb^2d^2f-Bc^2d^3+2df^3\left(\frac{2Aabf^2-2Abcdf+Ba^2f^2-2Bacdf-Bb^2df+Bc^2d^2}{2f^3} + \frac{\sqrt{-df^7}\,(Aa^2f^2-2Aacdf-Ab^2df+Ac^2d^2-2Babdf+2Bbcd^2)}{2df^6}\right)}{Aa^2f^3-2Aacdf^2-Ab^2df^2+Ac^2d^2f-2Babdf^2+2Bbcd^2f}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**2/(f*x**2+d),x)
```

```
[Out] B*c**2*x**4/(4*f) + x**3*(A*c**2/(3*f) + 2*B*b*c/(3*f)) + x**2*(A*b*c/f + B
*a*c/f + B*b**2/(2*f) - B*c**2*d/(2*f**2)) + x*(2*A*a*c/f + A*b**2/f - A*c*
*2*d/f**2 + 2*B*a*b/f - 2*B*b*c*d/f**2) + ((2*A*a*b*f**2 - 2*A*b*c*d*f + B*
a**2*f**2 - 2*B*a*c*d*f - B*b**2*d*f + B*c**2*d**2)/(2*f**3) - sqrt(-d*f**7
)*(A*a**2*f**2 - 2*A*a*c*d*f - A*b**2*d*f + A*c**2*d**2 - 2*B*a*b*d*f + 2*B
*b*c*d**2)/(2*d*f**6))*log(x + (-2*A*a*b*d*f**2 + 2*A*b*c*d**2*f - B*a**2*d
*f**2 + 2*B*a*c*d**2*f + B*b**2*d**2*f - B*c**2*d**3 + 2*d*f**3*((2*A*a*b*f
**2 - 2*A*b*c*d*f + B*a**2*f**2 - 2*B*a*c*d*f - B*b**2*d*f + B*c**2*d**2)/(
2*f**3) - sqrt(-d*f**7)*(A*a**2*f**2 - 2*A*a*c*d*f - A*b**2*d*f + A*c**2*d*
*2 - 2*B*a*b*d*f + 2*B*b*c*d**2)/(2*d*f**6)))/(A*a**2*f**3 - 2*A*a*c*d*f**2
 - A*b**2*d*f**2 + A*c**2*d**2*f - 2*B*a*b*d*f**2 + 2*B*b*c*d**2*f)) + ((2*
A*a*b*f**2 - 2*A*b*c*d*f + B*a**2*f**2 - 2*B*a*c*d*f - B*b**2*d*f + B*c**2*
d**2)/(2*f**3) + sqrt(-d*f**7)*(A*a**2*f**2 - 2*A*a*c*d*f - A*b**2*d*f + A*
c**2*d**2 - 2*B*a*b*d*f + 2*B*b*c*d**2)/(2*d*f**6))*log(x + (-2*A*a*b*d*f**
2 + 2*A*b*c*d**2*f - B*a**2*d*f**2 + 2*B*a*c*d**2*f + B*b**2*d**2*f - B*c**
2*d**3 + 2*d*f**3*((2*A*a*b*f**2 - 2*A*b*c*d*f + B*a**2*f**2 - 2*B*a*c*d*f
- B*b**2*d*f + B*c**2*d**2)/(2*f**3) + sqrt(-d*f**7)*(A*a**2*f**2 - 2*A*a*c
*d*f - A*b**2*d*f + A*c**2*d**2 - 2*B*a*b*d*f + 2*B*b*c*d**2)/(2*d*f**6)))/
(A*a**2*f**3 - 2*A*a*c*d*f**2 - A*b**2*d*f**2 + A*c**2*d**2*f - 2*B*a*b*d*f
**2 + 2*B*b*c*d**2*f))
```

**Giac** [A]
time = 3.92, size = 263, normalized size = 1.15

$$\frac{(2\,Bbcd^2 + Ac^2d^2 - 2\,Babdf - Ab^2df - 2\,Aacdf + Aa^2f^2)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}\,f^2} + \frac{(Bc^2d^2 - Bb^2df - 2\,Bacdf - 2\,Abcdf + Ba^2f^2 + 2\,Aabf^2)\log\left(fx^2+d\right)}{2\,f^3} + \frac{3\,Bc^2f^3x^4 + 8\,Bbcf^3x^3 + 4\,Ac^2f^3x^3 - 6\,Bc^2df^2x^2 + 6\,Bb^2f^3x^2 + 12\,Bacf^3x^2 + 12\,Abcf^3x^2 - 24\,Bbcdf^2x - 12\,Ac^2df^2x + 24\,Babf^3x + 12\,Ab^2f^3x + 24\,Aacf^3x}{12\,f^4}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="giac")
```

```
[Out] (2*B*b*c*d^2 + A*c^2*d^2 - 2*B*a*b*d*f - A*b^2*d*f - 2*A*a*c*d*f + A*a^2*f^
2)*arctan(f*x/sqrt(d*f))/(sqrt(d*f)*f^2) + 1/2*(B*c^2*d^2 - B*b^2*d*f - 2*B
*a*c*d*f - 2*A*b*c*d*f + B*a^2*f^2 + 2*A*a*b*f^2)*log(f*x^2 + d)/f^3 + 1/12
*(3*B*c^2*f^3*x^4 + 8*B*b*c*f^3*x^3 + 4*A*c^2*f^3*x^3 - 6*B*c^2*d*f^2*x^2 +
 6*B*b^2*f^3*x^2 + 12*B*a*c*f^3*x^2 + 12*A*b*c*f^3*x^2 - 24*B*b*c*d*f^2*x -
 12*A*c^2*d*f^2*x + 24*B*a*b*f^3*x + 12*A*b^2*f^3*x + 24*A*a*c*f^3*x)/f^4
```

**Mupad** [B]
time = 0.25, size = 253, normalized size = 1.11

$$x\left(\frac{A b^2+2 B a b+2 A a c}{f}-\frac{d\left(A c^2+2 B b c\right)}{f^2}\right)+x^2\left(\frac{B b^2+2 A c b+2 B a c}{2 f}-\frac{B c^2 d}{2 f^2}\right)+\frac{x^3\left(A c^2+2 B b c\right)}{3 f}+\frac{B c^2 x^4}{4 f}+\frac{\operatorname{atan}\left(\frac{\sqrt{f}\, x}{\sqrt{d}}\right)\left(A a^2 f^2-2 B a b d f-2 A a c d f-A b^2 d f+2 B b c d^2+A c^2 d^2\right)}{\sqrt{d}\, f^{5/2}}+\frac{\ln\left(f x^2+d\right)\left(4 B a^2 d f^5+8 A a b d f^5-8 B a c d^2 f^4-4 B b^2 d^2 f^4-8 A b c d^2 f^4+4 B c^2 d^3 f^3\right)}{8 d f^6}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^2)/(d + f*x^2),x)
```

```
[Out] x*((A*b^2 + 2*A*a*c + 2*B*a*b)/f - (d*(A*c^2 + 2*B*b*c))/f^2) + x^2*((B*b^2
 + 2*A*b*c + 2*B*a*c)/(2*f) - (B*c^2*d)/(2*f^2)) + (x^3*(A*c^2 + 2*B*b*c))/
(3*f) + (B*c^2*x^4)/(4*f) + (atan((f^(1/2)*x)/d^(1/2))*(A*a^2*f^2 + A*c^2*d
^2 + 2*B*b*c*d^2 - A*b^2*d*f - 2*A*a*c*d*f - 2*B*a*b*d*f))/(d^(1/2)*f^(5/2)
) + (log(d + f*x^2)*(4*B*a^2*d*f^5 - 4*B*b^2*d^2*f^4 + 4*B*c^2*d^3*f^3 + 8*
A*a*b*d*f^5 - 8*A*b*c*d^2*f^4 - 8*B*a*c*d^2*f^4))/(8*d*f^6)
```

## 3.3 $\int \frac{(A+Bx)(a+bx+cx^2)^3}{d+fx^2}\,dx$

**Optimal**. Leaf size=441

$$-\frac{(b^3Bdf+3Ab^2f(cd-af)-3bB(cd-af)^2-Ac(c^2d^2-3acdf+3a^2f^2))\,x}{f^3}-\frac{(Abf(3c^2d-b^2f-6acf)-\ldots$$

[Out] -(b^3*B*d*f+3*A*b^2*f*(-a*f+c*d)-3*b*B*(-a*f+c*d)^2-A*c*(3*a^2*f^2-3*a*c*d*f+c^2*d^2))*x/f^3-1/2*(A*b*f*(-6*a*c*f-b^2*f+3*c^2*d)-B*(c^3*d^2-3*a*c^2*d*f+3*a*b^2*f^2-3*c*f*(-a^2*f+b^2*d)))*x^2/f^3+1/3*(b^3*B*f+3*A*b^2*c*f-A*c^2*(-3*a*f+c*d)-3*b*B*c*(-2*a*f+c*d))*x^3/f^2+1/4*c*(3*A*b*c*f-B*(-3*a*c*f-3*b^2*f+c^2*d))*x^4/f^2+1/5*c^2*(A*c+3*B*b)*x^5/f+1/6*B*c^3*x^6/f+1/2*(A*b*f*(3*c^2*d^2-6*a*c*d*f-f*(-3*a^2*f+b^2*d))-B*(-a*f+c*d)*(c^2*d^2-2*a*c*d*f-f*(-a^2*f+3*b^2*d)))*ln(f*x^2+d)/f^4+(b^3*B*d^2*f+3*A*b^2*d*f*(-a*f+c*d)-3*b*B*d*(-a*f+c*d)^2-A*(-a*f+c*d)^3)*arctan(x*f^(1/2)/d^(1/2))/f^(7/2)/d^(1/2)

**Rubi** [A]
time = 0.38, antiderivative size = 441, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.148$, Rules used = {1026, 649, 211, 266}

$$\frac{\log(d+fx^2)\,(Abf(-f(b^2d-3a^2f)-6acdf+3c^2d^2)-B(cd-af)(-f(3b^2d-a^2f)-2acdf+c^2d^2))}{2f^4}-\frac{x^2(Abf(-6acf+b^2(-f)+3c^2d)-B(-3cf(b^2d-a^2f)+3ab^2f^2-3ac^2df+c^3d^2))}{2f^3}-\frac{x(-Ac(3a^2f^2-3acdf+c^2d^2)+3Ab^2f(cd-af)-3bB(cd-af)^2+b^3Bdf)}{f^3}+\frac{\text{ArcTan}\left(\frac{\sqrt{f}x}{\sqrt{d}}\right)(3Ab^2df(cd-af)-A(cd-af)^3-3bBd(cd-af)^2+b^3Bd^2f)}{\sqrt{d}\,f^{7/2}}+\frac{cx^4(3Abcf-B(-3acf-3b^2f+c^2d))}{4f^2}+\frac{x^3(-Ac^2(cd-3af)-3bBc(cd-2af)+3Ab^2cf+b^3Bf)}{3f^2}+\frac{c^2x^5(Ac+3bB)}{5f}+\frac{Bc^3x^6}{6f}$$

Antiderivative was successfully verified.

[In] Int[((A + B*x)*(a + b*x + c*x^2)^3)/(d + f*x^2),x]

[Out] -(((b^3*B*d*f + 3*A*b^2*f*(c*d - a*f) - 3*b*B*(c*d - a*f)^2 - A*c*(c^2*d^2 - 3*a*c*d*f + 3*a^2*f^2))*x)/f^3) - ((A*b*f*(3*c^2*d - b^2*f - 6*a*c*f) - B*(c^3*d^2 - 3*a*c^2*d*f + 3*a*b^2*f^2 - 3*c*f*(b^2*d - a^2*f)))*x^2)/(2*f^3) + ((b^3*B*f + 3*A*b^2*c*f - A*c^2*(c*d - 3*a*f) - 3*b*B*c*(c*d - 2*a*f))*x^3)/(3*f^2) + (c*(3*A*b*c*f - B*(c^2*d - 3*b^2*f - 3*a*c*f))*x^4)/(4*f^2) + (c^2*(3*b*B + A*c)*x^5)/(5*f) + (B*c^3*x^6)/(6*f) + ((b^3*B*d^2*f + 3*A*b^2*d*f*(c*d - a*f) - 3*b*B*d*(c*d - a*f)^2 - A*(c*d - a*f)^3)*ArcTan[(Sqrt[f]*x)/Sqrt[d]])/(Sqrt[d]*f^(7/2)) + ((A*b*f*(3*c^2*d^2 - 6*a*c*d*f - f*(b^2*d - 3*a^2*f)) - B*(c*d - a*f)*(c^2*d^2 - 2*a*c*d*f - f*(3*b^2*d - a^2*f)))*Log[d + f*x^2])/(2*f^4)

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/Rt[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 266

```
Int[(x_)^(m_.)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> Simp[Log[RemoveConten
t[a + b*x^n, x]]/(b*n), x] /; FreeQ[{a, b, m, n}, x] && EqQ[m, n - 1]
```

Rule 649

```
Int[((d_) + (e_.)*(x_))/((a_) + (c_.)*(x_)^2), x_Symbol] :> Dist[d, Int[1/(
a + c*x^2), x], x] + Dist[e, Int[x/(a + c*x^2), x], x] /; FreeQ[{a, c, d, e
}, x] &&  !NiceSqrtQ[(-a)*c]
```

Rule 1026

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Int[ExpandIntegrand[(a + c*x^2)^p*(d + e*x +
 f*x^2)^q*(g + h*x), x], x] /; FreeQ[{a, c, d, e, f, g, h}, x] && NeQ[e^2 -
 4*d*f, 0] && IntegersQ[p, q] && (GtQ[p, 0] || GtQ[q, 0])
```

Rubi steps

$$\int \frac{(A+Bx)\left(a+bx+cx^2\right)^3}{d+fx^2}\,dx = \int \left(-\frac{b^3Bdf+3Ab^2f(cd-af)-3bB(cd-af)^2-Ac(c^2d^2-3acdf+3\right.}{f^3}$$

$$= -\frac{(b^3Bdf+3Ab^2f(cd-af)-3bB(cd-af)^2-Ac(c^2d^2-3acdf+3a^2f}{f^3}$$

$$= -\frac{(b^3Bdf+3Ab^2f(cd-af)-3bB(cd-af)^2-Ac(c^2d^2-3acdf+3a^2f}{f^3}$$

$$= -\frac{(b^3Bdf+3Ab^2f(cd-af)-3bB(cd-af)^2-Ac(c^2d^2-3acdf+3a^2f}{f^3}$$

**Mathematica** [A]
time = 0.33, size = 422, normalized size = 0.96

$$\frac{\left(b^3Bd^2f+3Ab^2df(cd-af)-3bBd(cd-af)^2-A(cd-af)^3\right)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,f^{7/2}}+\frac{fx\left(10b^3f(-6Bd+3Afx+2Bfx^2)+15b^2f(3Bx(-2cd+2af+cfx^2)+4A(-3cd+3af+cfx^2))+3b(15Acfx(-2cd+4af+cfx^2)+4B(15a^2f^2+10acf(-3d+fx^2)+c^2(15d^2-5dfx^2+3f^2x^4)))+c(5Bx(18a^2f^2+9acf(-2d+fx^2)+c^2(6d^2-3dfx^2+2f^2x^4))+4A(45a^2f^2+15acf(-3d+fx^2)+c^2(15d^2-5dfx^2+3f^2x^4)))\right)-30\left(Abf(-3c^2d^2+b^2df+6acdf-3a^2f^2)+B(cd-af)(c^2d^2-3b^2df-2acdf+a^2f^2)\right)\log\left(d+fx^2\right)}{60f^4}$$

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2)^3)/(d + f*x^2),x]
```

```
[Out] ((b^3*B*d^2*f + 3*A*b^2*d*f*(c*d - a*f) - 3*b*B*d*(c*d - a*f)^2 - A*(c*d -
a*f)^3)*ArcTan[(Sqrt[f]*x)/Sqrt[d]])/(Sqrt[d]*f^(7/2)) + (f*x*(10*b^3*f*(-6
*B*d + 3*A*f*x + 2*B*f*x^2) + 15*b^2*f*(3*B*x*(-2*c*d + 2*a*f + c*f*x^2) +
4*A*(-3*c*d + 3*a*f + c*f*x^2)) + 3*b*(15*A*c*f*x*(-2*c*d + 4*a*f + c*f*x^2
```

```
) + 4*B*(15*a^2*f^2 + 10*a*c*f*(-3*d + f*x^2) + c^2*(15*d^2 - 5*d*f*x^2 + 3
*f^2*x^4))) + c*(5*B*x*(18*a^2*f^2 + 9*a*c*f*(-2*d + f*x^2) + c^2*(6*d^2 -
3*d*f*x^2 + 2*f^2*x^4)) + 4*A*(45*a^2*f^2 + 15*a*c*f*(-3*d + f*x^2) + c^2*(
15*d^2 - 5*d*f*x^2 + 3*f^2*x^4)))) - 30*(A*b*f*(-3*c^2*d^2 + b^2*d*f + 6*a*
c*d*f - 3*a^2*f^2) + B*(c*d - a*f)*(c^2*d^2 - 3*b^2*d*f - 2*a*c*d*f + a^2*f
^2))*Log[d + f*x^2])/(60*f^4)
```

**Maple [A]**
time = 0.20, size = 592, normalized size = 1.34

| method | result |
|---|---|
| default | $\frac{A\,b^{2} c\,f^{2} x^{3}+A a\,c^{2} f^{2} x^{3}+\frac{3}{4} A b\,c^{2} f^{2} x^{4}+\frac{3}{4} B a\,c^{2} f^{2} x^{4}+\frac{3}{4} B\,b^{2} c\,f^{2} x^{4}-\frac{1}{4} B\,c^{3} d f\,x^{4}-\frac{1}{3} A\,c^{3} d f\,x^{3}+\frac{3}{2} B\,a^{2} c\,f^{2} x^{2}+\frac{3}{2} B a\,b^{2} f^{2} x^{2}+3 A\,a^{2} c\,f^{2} x \cdots}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^3/(f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f^3*(A*b^2*c*f^2*x^3+A*a*c^2*f^2*x^3+3/4*A*b*c^2*f^2*x^4+3/4*B*a*c^2*f^2*
x^4+3/4*B*b^2*c*f^2*x^4-1/4*B*c^3*d*f*x^4-1/3*A*c^3*d*f*x^3+3/2*B*a^2*c*f^2
*x^2+3/2*B*a*b^2*f^2*x^2+3*A*a^2*c*f^2*x+3*A*a*b^2*f^2*x+3*B*a^2*b*f^2*x-b^
3*B*d*f*x+3*B*b*c^2*d^2*x+3/5*B*b*c^2*f^2*x^5+1/2*B*c^3*d^2*x^2+A*c^3*d^2*x
+1/3*B*b^3*f^2*x^3+1/2*A*b^3*f^2*x^2+1/5*A*c^3*f^2*x^5+1/6*B*c^3*x^6*f^2-3*
A*b^2*c*d*f*x-3/2*B*b^2*c*d*f*x^2-3*A*a*c^2*d*f*x+2*B*a*b*c*f^2*x^3-B*b*c^2
*d*f*x^3+3*A*a*b*c*f^2*x^2-3/2*A*b*c^2*d*f*x^2-3/2*B*a*c^2*d*f*x^2-6*B*a*b*
c*d*f*x)+1/f^3*(1/2*(3*A*a^2*b*f^3-6*A*a*b*c*d*f^2-A*b^3*d*f^2+3*A*b*c^2*d^
2*f+B*a^3*f^3-3*B*a^2*c*d*f^2-3*B*a*b^2*d*f^2+3*B*a*c^2*d^2*f+3*B*b^2*c*d^2
*f-B*c^3*d^3)/f*ln(f*x^2+d)+(A*a^3*f^3-3*A*a^2*c*d*f^2-3*A*a*b^2*d*f^2+3*A*
a*c^2*d^2*f+3*A*b^2*c*d^2*f-A*c^3*d^3-3*B*a^2*b*d*f^2+6*B*a*b*c*d^2*f+B*b^3
*d^2*f-3*B*b*c^2*d^3)/(d*f)^(1/2)*arctan(f*x/(d*f)^(1/2)))
```

**Maxima [A]**
time = 0.52, size = 471, normalized size = 1.07

$$\frac{\left(A a^{3} f^{3} - \left(3 \, B b c^{2} + A c^{3}\right) d^{3} + \left(B b^{3} + 3 \, A a c^{2} + 3 \, \left(2 \, B a b + A b^{2}\right) c\right) d^{2} f - 3 \, \left(B a^{2} b + A a b^{2} + A a^{2} c\right) d f^{2}\right) \arctan\left(\frac{f x}{\sqrt{d f}}\right)}{\sqrt{d f} f^{3}} + \frac{10 \, B c^{3} f^{2} x^{6} + 12 \, \left(3 \, B b c^{2} + A c^{3}\right) f^{2} x^{5} - 15 \, \left(B c^{3} d f - 3 \, \left(B b^{2} c + \left(B a + A b\right) c^{2}\right) f^{2}\right) x^{4} - 20 \, \left(\left(3 \, B b c^{2} + A c^{3}\right) d f - \left(B b^{3} + 3 \, A a c^{2} + 3 \, \left(2 \, B a b + A b^{2}\right) c\right) f^{2}\right) x^{3} + 30 \, \left(B c^{3} d^{2} - 3 \, \left(B b^{2} c + \left(B a + A b\right) c^{2}\right) d f + \left(3 \, B a b^{2} + A b^{3} + 3 \, \left(B a^{2} + 2 \, A a b\right) c\right) f^{2}\right) x^{2} + 60 \, \left(\left(3 \, B b c^{2} + A c^{3}\right) d^{2} - \left(B b^{3} + 3 \, A a c^{2} + 3 \, \left(2 \, B a b + A b^{2}\right) c\right) d f + 3 \, \left(B a^{2} b + A a b^{2} + A a^{2} c\right) f^{2}\right) x}{60 \, f^{3}} - \frac{\left(B c^{3} d^{3} - 3 \, \left(B b^{2} c + \left(B a + A b\right) c^{2}\right) d^{2} f + \left(3 \, B a b^{2} + A b^{3} + 3 \, \left(B a^{2} + 2 \, A a b\right) c\right) d f^{2} - \left(B a^{3} + 3 \, A a^{2} b\right) f^{3}\right) \log\left(f x^{2} + d\right)}{2 \, f^{4}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^3/(f*x^2+d),x, algorithm="maxima")
```

```
[Out] (A*a^3*f^3 - (3*B*b*c^2 + A*c^3)*d^3 + (B*b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*
b^2)*c)*d^2*f - 3*(B*a^2*b + A*a*b^2 + A*a^2*c)*d*f^2)*arctan(f*x/sqrt(d*f)
)/(sqrt(d*f)*f^3) + 1/60*(10*B*c^3*f^2*x^6 + 12*(3*B*b*c^2 + A*c^3)*f^2*x^5
 - 15*(B*c^3*d*f - 3*(B*b^2*c + (B*a + A*b)*c^2)*f^2)*x^4 - 20*((3*B*b*c^2
+ A*c^3)*d*f - (B*b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*f^2)*x^3 + 30*(B
```

```
*c^3*d^2 - 3*(B*b^2*c + (B*a + A*b)*c^2)*d*f + (3*B*a*b^2 + A*b^3 + 3*(B*a^
2 + 2*A*a*b)*c)*f^2)*x^2 + 60*((3*B*b*c^2 + A*c^3)*d^2 - (B*b^3 + 3*A*a*c^2
 + 3*(2*B*a*b + A*b^2)*c)*d*f + 3*(B*a^2*b + A*a*b^2 + A*a^2*c)*f^2)*x)/f^3
 - 1/2*(B*c^3*d^3 - 3*(B*b^2*c + (B*a + A*b)*c^2)*d^2*f + (3*B*a*b^2 + A*b^
3 + 3*(B*a^2 + 2*A*a*b)*c)*d*f^2 - (B*a^3 + 3*A*a^2*b)*f^3)*log(f*x^2 + d)/
f^4
```

**Fricas** [**A**]
time = 0.33, size = 1014, normalized size = 2.30

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^3/(f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/60*(10*B*c^3*d*f^3*x^6 + 12*(3*B*b*c^2 + A*c^3)*d*f^3*x^5 - 15*(B*c^3*d^
2*f^2 - 3*(B*b^2*c + (B*a + A*b)*c^2)*d*f^3)*x^4 - 20*((3*B*b*c^2 + A*c^3)*
d^2*f^2 - (B*b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*d*f^3)*x^3 + 30*(B*c^
3*d^3*f - 3*(B*b^2*c + (B*a + A*b)*c^2)*d^2*f^2 + (3*B*a*b^2 + A*b^3 + 3*(B
*a^2 + 2*A*a*b)*c)*d*f^3)*x^2 - 30*(A*a^3*f^3 - (3*B*b*c^2 + A*c^3)*d^3 + (
B*b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*d^2*f - 3*(B*a^2*b + A*a*b^2 + A
*a^2*c)*d*f^2)*sqrt(-d*f)*log((f*x^2 - 2*sqrt(-d*f)*x - d)/(f*x^2 + d)) + 6
0*((3*B*b*c^2 + A*c^3)*d^3*f - (B*b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*
d^2*f^2 + 3*(B*a^2*b + A*a*b^2 + A*a^2*c)*d*f^3)*x - 30*(B*c^3*d^4 - 3*(B*b
^2*c + (B*a + A*b)*c^2)*d^3*f + (3*B*a*b^2 + A*b^3 + 3*(B*a^2 + 2*A*a*b)*c)
*d^2*f^2 - (B*a^3 + 3*A*a^2*b)*d*f^3)*log(f*x^2 + d))/(d*f^4), 1/60*(10*B*c
^3*d*f^3*x^6 + 12*(3*B*b*c^2 + A*c^3)*d*f^3*x^5 - 15*(B*c^3*d^2*f^2 - 3*(B*
b^2*c + (B*a + A*b)*c^2)*d*f^3)*x^4 - 20*((3*B*b*c^2 + A*c^3)*d^2*f^2 - (B*
b^3 + 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*d*f^3)*x^3 + 30*(B*c^3*d^3*f - 3*(
B*b^2*c + (B*a + A*b)*c^2)*d^2*f^2 + (3*B*a*b^2 + A*b^3 + 3*(B*a^2 + 2*A*a*
b)*c)*d*f^3)*x^2 + 60*(A*a^3*f^3 - (3*B*b*c^2 + A*c^3)*d^3 + (B*b^3 + 3*A*a
*c^2 + 3*(2*B*a*b + A*b^2)*c)*d^2*f - 3*(B*a^2*b + A*a*b^2 + A*a^2*c)*d*f^2
)*sqrt(d*f)*arctan(sqrt(d*f)*x/d) + 60*((3*B*b*c^2 + A*c^3)*d^3*f - (B*b^3
+ 3*A*a*c^2 + 3*(2*B*a*b + A*b^2)*c)*d^2*f^2 + 3*(B*a^2*b + A*a*b^2 + A*a^2
*c)*d*f^3)*x - 30*(B*c^3*d^4 - 3*(B*b^2*c + (B*a + A*b)*c^2)*d^3*f + (3*B*a
*b^2 + A*b^3 + 3*(B*a^2 + 2*A*a*b)*c)*d^2*f^2 - (B*a^3 + 3*A*a^2*b)*d*f^3)*
log(f*x^2 + d))/(d*f^4)]
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**3/(f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac** [A]
time = 4.02, size = 623, normalized size = 1.41

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^3/(f*x^2+d),x, algorithm="giac")
```

```
[Out] -(3*B*b*c^2*d^3 + A*c^3*d^3 - B*b^3*d^2*f - 6*B*a*b*c*d^2*f - 3*A*b^2*c*d^2
*f - 3*A*a*c^2*d^2*f + 3*B*a^2*b*d*f^2 + 3*A*a*b^2*d*f^2 + 3*A*a^2*c*d*f^2
- A*a^3*f^3)*arctan(f*x/sqrt(d*f))/(sqrt(d*f)*f^3) - 1/2*(B*c^3*d^3 - 3*B*b
^2*c*d^2*f - 3*B*a*c^2*d^2*f - 3*A*b*c^2*d^2*f + 3*B*a*b^2*d*f^2 + A*b^3*d*
f^2 + 3*B*a^2*c*d*f^2 + 6*A*a*b*c*d*f^2 - B*a^3*f^3 - 3*A*a^2*b*f^3)*log(f*
x^2 + d)/f^4 + 1/60*(10*B*c^3*f^5*x^6 + 36*B*b*c^2*f^5*x^5 + 12*A*c^3*f^5*x
^5 - 15*B*c^3*d*f^4*x^4 + 45*B*b^2*c*f^5*x^4 + 45*B*a*c^2*f^5*x^4 + 45*A*b*
c^2*f^5*x^4 - 60*B*b*c^2*d*f^4*x^3 - 20*A*c^3*d*f^4*x^3 + 20*B*b^3*f^5*x^3
+ 120*B*a*b*c*f^5*x^3 + 60*A*b^2*c*f^5*x^3 + 60*A*a*c^2*f^5*x^3 + 30*B*c^3*
d^2*f^3*x^2 - 90*B*b^2*c*d*f^4*x^2 - 90*B*a*c^2*d*f^4*x^2 - 90*A*b*c^2*d*f^
4*x^2 + 90*B*a*b^2*f^5*x^2 + 30*A*b^3*f^5*x^2 + 90*B*a^2*c*f^5*x^2 + 180*A*
a*b*c*f^5*x^2 + 180*B*b*c^2*d^2*f^3*x + 60*A*c^3*d^2*f^3*x - 60*B*b^3*d*f^4
*x - 360*B*a*b*c*d*f^4*x - 180*A*b^2*c*d*f^4*x - 180*A*a*c^2*d*f^4*x + 180*
B*a^2*b*f^5*x + 180*A*a*b^2*f^5*x + 180*A*a^2*c*f^5*x)/f^6
```

**Mupad** [B]
time = 3.79, size = 552, normalized size = 1.25

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^3)/(d + f*x^2),x)
```

```
[Out] x^2*((A*b^3 + 3*B*a*b^2 + 3*B*a^2*c + 6*A*a*b*c)/(2*f) - (d*((3*A*b*c^2 + 3
*B*a*c^2 + 3*B*b^2*c)/f - (B*c^3*d)/f^2))/(2*f)) + x*((3*A*a*b^2 + 3*A*a^2*
c + 3*B*a^2*b)/f - (d*((B*b^3 + 3*A*a*c^2 + 3*A*b^2*c + 6*B*a*b*c)/f - (d*(
A*c^3 + 3*B*b*c^2))/f^2))/f) + x^3*((B*b^3 + 3*A*a*c^2 + 3*A*b^2*c + 6*B*a*
b*c)/(3*f) - (d*(A*c^3 + 3*B*b*c^2))/(3*f^2)) + x^4*((3*A*b*c^2 + 3*B*a*c^2
 + 3*B*b^2*c)/(4*f) - (B*c^3*d)/(4*f^2)) + (x^5*(A*c^3 + 3*B*b*c^2))/(5*f)
+ (B*c^3*x^6)/(6*f) + (log(d + f*x^2)*(4*B*a^3*d*f^7 - 4*A*b^3*d^2*f^6 - 4*
B*c^3*d^4*f^4 - 12*B*a*b^2*d^2*f^6 + 12*A*b*c^2*d^3*f^5 + 12*B*a*c^2*d^3*f^
5 - 12*B*a^2*c*d^2*f^6 + 12*B*b^2*c*d^3*f^5 + 12*A*a^2*b*d*f^7 - 24*A*a*b*c
*d^2*f^6))/(8*d*f^8) + (atan((f^(1/2)*x)/d^(1/2))*(A*a^3*f^3 - A*c^3*d^3 -
3*B*b*c^2*d^3 + B*b^3*d^2*f - 3*A*a*b^2*d*f^2 + 3*A*a*c^2*d^2*f - 3*A*a^2*c
*d*f^2 - 3*B*a^2*b*d*f^2 + 3*A*b^2*c*d^2*f + 6*B*a*b*c*d^2*f))/(d^(1/2)*f^(
7/2))
```

## 3.4 $\int \frac{A+Bx}{(a+bx+cx^2)(d+fx^2)}\,dx$

**Optimal**. Leaf size=274

$$\frac{\sqrt{f}\,(bBd-Acd+aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,\left(c^2d^2-2acdf+f\left(b^2d+a^2f\right)\right)}-\frac{\left(Ab^2f+2Ac(cd-af)-bB(cd+af)\right)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}\,\left(c^2d^2-2acdf+f\left(b^2d+a^2f\right)\right)}+$$

```
[Out] 1/2*(A*b*f-B*a*f+B*c*d)*ln(c*x^2+b*x+a)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))
-1/2*(A*b*f-B*a*f+B*c*d)*ln(f*x^2+d)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))-(A
*b^2*f+2*A*c*(-a*f+c*d)-b*B*(a*f+c*d))*arctanh((2*c*x+b)/(-4*a*c+b^2)^(1/2)
)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))/(-4*a*c+b^2)^(1/2)+(A*a*f-A*c*d+B*b*d
)*arctan(x*f^(1/2)/d^(1/2))*f^(1/2)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))/d^(
1/2)
```

**Rubi** [A]
time = 0.18, antiderivative size = 274, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 8, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.296$, Rules used = {1037, 648, 632, 212, 642, 649, 211, 266}

$$\frac{\sqrt{f}\,\text{ArcTan}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)(aAf-Acd+bBd)}{\sqrt{d}\,\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)}+\frac{\log\left(a+bx+cx^2\right)(-aBf+Abf+Bcd)}{2\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)}-\frac{\log\left(d+fx^2\right)(-aBf+Abf+Bcd)}{2\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)}-\frac{\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)\left(2Ac(cd-af)-bB(af+cd)+Ab^2f\right)}{\sqrt{b^2-4ac}\,\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)*(d + f*x^2)),x]
```

```
[Out] (Sqrt[f]*(b*B*d - A*c*d + a*A*f)*ArcTan[(Sqrt[f]*x)/Sqrt[d]])/(Sqrt[d]*(c^2
*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))) - ((A*b^2*f + 2*A*c*(c*d - a*f) - b*
B*(c*d + a*f))*ArcTanh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])/(Sqrt[b^2 - 4*a*c]*(
c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))) + ((B*c*d + A*b*f - a*B*f)*Log[a
+ b*x + c*x^2])/(2*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))) - ((B*c*d + A
*b*f - a*B*f)*Log[d + f*x^2])/(2*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f)))
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 266

```
Int[(x_)^(m_.)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> Simp[Log[RemoveConten
t[a + b*x^n, x]]/(b*n), x] /; FreeQ[{a, b, m, n}, x] && EqQ[m, n - 1]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 649

```
Int[((d_) + (e_.)*(x_))/((a_) + (c_.)*(x_)^2), x_Symbol] :> Dist[d, Int[1/(
a + c*x^2), x], x] + Dist[e, Int[x/(a + c*x^2), x], x] /; FreeQ[{a, c, d, e
}, x] &&  !NiceSqrtQ[(-a)*c]
```

Rule 1037

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*((d_) + (f_.)*
(x_)^2)), x_Symbol] :> With[{q = Simplify[c^2*d^2 + b^2*d*f - 2*a*c*d*f + a
^2*f^2]}, Dist[1/q, Int[Simp[g*c^2*d + g*b^2*f - a*b*h*f - a*g*c*f + c*(h*c
*d + g*b*f - a*h*f)*x, x]/(a + b*x + c*x^2), x], x] + Dist[1/q, Int[Simp[b*
h*d*f - g*c*d*f + a*g*f^2 - f*(h*c*d + g*b*f - a*h*f)*x, x]/(d + f*x^2), x]
, x] /; NeQ[q, 0]] /; FreeQ[{a, b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0
]
```

Rubi steps

$$\int \frac{A+Bx}{(a+bx+cx^2)(d+fx^2)}\,dx = \frac{\int \frac{-abBf+A(c^2d+b^2f-acf)+c(Bcd+Abf-aBf)x}{a+bx+cx^2}\,dx}{c^2d^2-2acdf+f(b^2d+a^2f)} + \frac{\int \frac{f(bBd-Acd+aAf)-f(Bcd+Abf-\cdots}{d+fx^2}}{c^2d^2-2acdf+f(b^2d+\cdots}$$

$$= \frac{(f(bBd-Acd+aAf))\int \frac{1}{d+fx^2}\,dx}{c^2d^2-2acdf+f(b^2d+a^2f)} + \frac{(Bcd+Abf-aBf)\int \frac{b+2cx}{a+bx+cx^2}\,dx}{2(c^2d^2-2acdf+f(b^2d+a^2f))}$$

$$= \frac{\sqrt{f}\,(bBd-Acd+aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,(c^2d^2-2acdf+f(b^2d+a^2f))} + \frac{(Bcd+Abf-aBf)\log(a+\cdots}{2(c^2d^2-2acdf+f(b^2d\cdots}$$

$$= \frac{\sqrt{f}\,(bBd-Acd+aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)}{\sqrt{d}\,(c^2d^2-2acdf+f(b^2d+a^2f))} - \frac{(Ab^2f+2Ac(cd-af)-bB\cdots}{\sqrt{b^2-4ac}\,(c^2d^2-\cdots}$$

**Mathematica [A]**
time = 0.26, size = 212, normalized size = 0.77

$$\frac{2\sqrt{-b^2+4ac}\,\sqrt{f}\,(bBd-Acd+aAf)\tan^{-1}\left(\frac{\sqrt{f}\,x}{\sqrt{d}}\right)+\sqrt{d}\left(2(Ab^2f+2Ac(cd-af)-bB(cd+af))\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)+\sqrt{-b^2+4ac}\,(Bcd+Abf-aBf)(-\log(d+fx^2)+\log(a+x(b+cx)))\right)}{2\sqrt{-b^2+4ac}\,\sqrt{d}\,(c^2d^2-2acdf+f(b^2d+a^2f))}$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)*(d + f*x^2)),x]
```

```
[Out] (2*Sqrt[-b^2 + 4*a*c]*Sqrt[f]*(b*B*d - A*c*d + a*A*f)*ArcTan[(Sqrt[f]*x)/Sq
rt[d]] + Sqrt[d]*(2*(A*b^2*f + 2*A*c*(c*d - a*f) - b*B*(c*d + a*f))*ArcTan[
(b + 2*c*x)/Sqrt[-b^2 + 4*a*c]] + Sqrt[-b^2 + 4*a*c]*(B*c*d + A*b*f - a*B*f
)*(-Log[d + f*x^2] + Log[a + x*(b + c*x)])))/(2*Sqrt[-b^2 + 4*a*c]*Sqrt[d]*
(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f)))
```

**Maple [A]**
time = 0.31, size = 239, normalized size = 0.87

| method | result |
|---|---|
| default | $\frac{f\left(\frac{(-Abf+Baf-Bcd)\ln(fx^2+d)}{2f}+\frac{(Aaf-Acd+Bbd)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{\sqrt{df}}\right)}{a^2f^2-2acdf+b^2df+c^2d^2}+\frac{\frac{(Abcf-Bacf+Bc^2d)\ln(cx^2+bx+a)}{2c}+\frac{2\left(-Aacf+Ab^2f+Ac\cdots\right.}{\cdots}}{a^2f^2-2acd\cdots}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] f/(a^2*f^2-2*a*c*d*f+b^2*d*f+c^2*d^2)*(1/2*(-A*b*f+B*a*f-B*c*d)/f*ln(f*x^2+
d)+(A*a*f-A*c*d+B*b*d)/(d*f)^(1/2)*arctan(f*x/(d*f)^(1/2)))+1/(a^2*f^2-2*a*
```

```
c*d*f+b^2*d*f+c^2*d^2)*(1/2*(A*b*c*f-B*a*c*f+B*c^2*d)/c*ln(c*x^2+b*x+a)+2*(
-A*a*c*f+A*b^2*f+A*c^2*d-B*a*b*f-1/2*(A*b*c*f-B*a*c*f+B*c^2*d)*b/c)/(4*a*c-
b^2)^(1/2)*arctan((2*c*x+b)/(4*a*c-b^2)^(1/2)))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)/(f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac** [**A**]
time = 5.05, size = 266, normalized size = 0.97

$$\frac{(Bcd - Baf + Abf)\log\left(cx^2 + bx + a\right)}{2\,(c^2d^2 + b^2df - 2\,acdf + a^2f^2)} - \frac{(Bcd - Baf + Abf)\log\left(fx^2 + d\right)}{2\,(c^2d^2 + b^2df - 2\,acdf + a^2f^2)} + \frac{(Bbdf - Acdf + Aaf^2)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{(c^2d^2 + b^2df - 2\,acdf + a^2f^2)\sqrt{df}} - \frac{(Bbcd - 2\,Ac^2d + Babf - Ab^2f + 2\,Aacf)\arctan\left(\frac{2\,cx+b}{\sqrt{-b^2+4\,ac}}\right)}{(c^2d^2 + b^2df - 2\,acdf + a^2f^2)\sqrt{-b^2+4\,ac}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d),x, algorithm="giac")
```

```
[Out] 1/2*(B*c*d - B*a*f + A*b*f)*log(c*x^2 + b*x + a)/(c^2*d^2 + b^2*d*f - 2*a*c
*d*f + a^2*f^2) - 1/2*(B*c*d - B*a*f + A*b*f)*log(f*x^2 + d)/(c^2*d^2 + b^2
*d*f - 2*a*c*d*f + a^2*f^2) + (B*b*d*f - A*c*d*f + A*a*f^2)*arctan(f*x/sqrt
(d*f))/((c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*f^2)*sqrt(d*f)) - (B*b*c*d - 2
*A*c^2*d + B*a*b*f - A*b^2*f + 2*A*a*c*f)*arctan((2*c*x + b)/sqrt(-b^2 + 4*
a*c))/((c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*f^2)*sqrt(-b^2 + 4*a*c))
```

**Mupad [B]**
time = 38.32, size = 2500, normalized size = 9.12

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d + f*x^2)*(a + b*x + c*x^2)),x)
```

```
[Out] (log(B^3*c^2*f^2*x + (((B*c*d^2)/2 + (A*b*d*f)/2 - (B*a*d*f)/2 - (A*a*f*(-d
*f)^(1/2))/2 + (A*c*d*(-d*f)^(1/2))/2 - (B*b*d*(-d*f)^(1/2))/2)*((((B*c*d^2
)/2 + (A*b*d*f)/2 - (B*a*d*f)/2 - (A*a*f*(-d*f)^(1/2))/2 + (A*c*d*(-d*f)^(1
/2))/2 - (B*b*d*(-d*f)^(1/2))/2)*(4*A*a^2*c^2*f^4 + 4*A*c^4*d^2*f^2 - c*f^2
*x*(3*A*b^3*f^2 + 4*B*c^3*d^2 - B*a*b^2*f^2 + 4*B*a^2*c*f^2 - 12*A*a*b*c*f^
2 + 12*A*b*c^2*d*f - 8*B*a*c^2*d*f - 3*B*b^2*c*d*f) - 3*A*b^2*c^2*d*f^3 - 4
*B*b*c^3*d^2*f^2 - A*a*b^2*c*f^4 - 8*A*a*c^3*d*f^3 + B*b^3*c*d*f^3 + (2*c*f
^2*((B*c*d^2)/2 + (A*b*d*f)/2 - (B*a*d*f)/2 - (A*a*f*(-d*f)^(1/2))/2 + (A*c
*d*(-d*f)^(1/2))/2 - (B*b*d*(-d*f)^(1/2))/2)*(2*b*c^3*d^3 + 4*c^4*d^3*x - a
^2*b^2*f^3*x + 4*a*b^3*d*f^2 - 2*b^3*c*d^2*f + 4*a^3*c*f^3*x + 3*b^4*d*f^2*
x + 12*a*b*c^2*d^2*f - 14*a^2*b*c*d*f^2 - 4*a*c^3*d^2*f*x - 4*a^2*c^2*d*f^2
*x + 3*b^2*c^2*d^2*f*x - 10*a*b^2*c*d*f^2*x))/(d*(a^2*f^2 + c^2*d^2 + b^2*d
*f - 2*a*c*d*f)) + 4*B*a*b*c^2*d*f^3))/(d*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*
a*c*d*f)) - c*f^2*x*(2*B^2*c^2*d - 4*A^2*c^2*f - B^2*b^2*f + 2*B^2*a*c*f +
2*A*B*b*c*f) + A^2*b*c^2*f^3 + B^2*b*c^2*d*f^2 - 4*A*B*a*c^2*f^3 + A*B*b^2*
c*f^3 - 4*A*B*c^3*d*f^2))/(d*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)) + A
*B^2*c^2*f^2)*(f*((B*a*d)/2 - (A*b*d)/2 + (A*a*(-d*f)^(1/2))/2) - (B*c*d^2)
/2 - (A*c*d*(-d*f)^(1/2))/2 + (B*b*d*(-d*f)^(1/2))/2))/(c^2*d^3 + a^2*d*f^2
 + b^2*d^2*f - 2*a*c*d^2*f) - (log(B^3*c^2*f^2*x + (((B*c*d^2)/2 + (A*b*d*f
)/2 - (B*a*d*f)/2 + (A*a*f*(-d*f)^(1/2))/2 - (A*c*d*(-d*f)^(1/2))/2 + (B*b*
d*(-d*f)^(1/2))/2)*((((B*c*d^2)/2 + (A*b*d*f)/2 - (B*a*d*f)/2 + (A*a*f*(-d*
f)^(1/2))/2 - (A*c*d*(-d*f)^(1/2))/2 + (B*b*d*(-d*f)^(1/2))/2)*(4*A*a^2*c^2
*f^4 + 4*A*c^4*d^2*f^2 - c*f^2*x*(3*A*b^3*f^2 + 4*B*c^3*d^2 - B*a*b^2*f^2 +
 4*B*a^2*c*f^2 - 12*A*a*b*c*f^2 + 12*A*b*c^2*d*f - 8*B*a*c^2*d*f - 3*B*b^2*
c*d*f) - 3*A*b^2*c^2*d*f^3 - 4*B*b*c^3*d^2*f^2 - A*a*b^2*c*f^4 - 8*A*a*c^3*
d*f^3 + B*b^3*c*d*f^3 + (2*c*f^2*((B*c*d^2)/2 + (A*b*d*f)/2 - (B*a*d*f)/2 +
 (A*a*f*(-d*f)^(1/2))/2 - (A*c*d*(-d*f)^(1/2))/2 + (B*b*d*(-d*f)^(1/2))/2)*
(2*b*c^3*d^3 + 4*c^4*d^3*x - a^2*b^2*f^3*x + 4*a*b^3*d*f^2 - 2*b^3*c*d^2*f
+ 4*a^3*c*f^3*x + 3*b^4*d*f^2*x + 12*a*b*c^2*d^2*f - 14*a^2*b*c*d*f^2 - 4*a
*c^3*d^2*f*x - 4*a^2*c^2*d*f^2*x + 3*b^2*c^2*d^2*f*x - 10*a*b^2*c*d*f^2*x))
```

```
/(d*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)) + 4*B*a*b*c^2*d*f^3))/(d*(a^
2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)) - c*f^2*x*(2*B^2*c^2*d - 4*A^2*c^2*
f - B^2*b^2*f + 2*B^2*a*c*f + 2*A*B*b*c*f) + A^2*b*c^2*f^3 + B^2*b*c^2*d*f^
2 - 4*A*B*a*c^2*f^3 + A*B*b^2*c*f^3 - 4*A*B*c^3*d*f^2))/(d*(a^2*f^2 + c^2*d
^2 + b^2*d*f - 2*a*c*d*f)) + A*B^2*c^2*f^2)*(f*((A*b*d)/2 - (B*a*d)/2 + (A*
a*(-d*f)^(1/2))/2) + (B*c*d^2)/2 - (A*c*d*(-d*f)^(1/2))/2 + (B*b*d*(-d*f)^(
1/2))/2))/(c^2*d^3 + a^2*d*f^2 + b^2*d^2*f - 2*a*c*d^2*f) - (log(B^3*c^2*f^
2*x + A*B^2*c^2*f^2 - ((((A*f*(b^2 - 4*a*c)^(3/2) + 2*A*b*f*(4*a*c - b^2) -
 2*B*a*f*(4*a*c - b^2) + 2*B*c*d*(4*a*c - b^2) + 4*A*c^2*d*(b^2 - 4*a*c)^(1
/2) + A*b^2*f*(b^2 - 4*a*c)^(1/2) - 2*B*a*b*f*(b^2 - 4*a*c)^(1/2) - 2*B*b*c
*d*(b^2 - 4*a*c)^(1/2))*(c*f^2*x*(3*A*b^3*f^2 + 4*B*c^3*d^2 - B*a*b^2*f^2 +
 4*B*a^2*c*f^2 - 12*A*a*b*c*f^2 + 12*A*b*c^2*d*f - 8*B*a*c^2*d*f - 3*B*b^2*
c*d*f) - 4*A*c^4*d^2*f^2 - 4*A*a^2*c^2*f^4 + 3*A*b^2*c^2*d*f^3 + 4*B*b*c^3*
d^2*f^2 + A*a*b^2*c*f^4 + 8*A*a*c^3*d*f^3 - B*b^3*c*d*f^3 - 4*B*a*b*c^2*d*f
^3 + (c*f^2*(A*f*(b^2 - 4*a*c)^(3/2) + 2*A*b*f*(4*a*c - b^2) - 2*B*a*f*(4*a
*c - b^2) + 2*B*c*d*(4*a*c - b^2) + 4*A*c^2*d*(b^2 - 4*a*c)^(1/2) + A*b^2*f
*(b^2 - 4*a*c)^(1/2) - 2*B*a*b*f*(b^2 - 4*a*c)^(1/2) - 2*B*b*c*d*(b^2 - 4*a
*c)^(1/2))*(2*b*c^3*d^3 + 4*c^4*d^3*x - a^2*b^2*f^3*x + 4*a*b^3*d*f^2 - 2*b
^3*c*d^2*f + 4*a^3*c*f^3*x + 3*b^4*d*f^2*x + 12*a*b*c^2*d^2*f - 14*a^2*b*c*
d*f^2 - 4*a*c^3*d^2*f*x - 4*a^2*c^2*d*f^2*x + 3*b^2*c^2*d^2*f*x - 10*a*b^2*
c*d*f^2*x))/(2*(4*a*c - b^2)*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f))))/(
4*(4*a*c - b^2)*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)) - c*f^2*x*(2*B^2
*c^2*d - 4*A^2*c^2*f - B^2*b^2*f + 2*B^2*a*c*f + 2*A*B*b*c*f) + A^2*b*c^2*f
^3 + B^2*b*c^2*d*f^2 - 4*A*B*a*c^2*f^3 + A*B*b^2*c*f^3 - 4*A*B*c^3*d*f^2)*(
A*f*(b^2 - 4*a*c)^(3/2) + 2*A*b*f*(4*a*c - b^2) - 2*B*a*f*(4*a*c - b^2) + 2
*B*c*d*(4*a*c - b^2) + 4*A*c^2*d*(b^2 - 4*a*c)^(1/2) + A*b^2*f*(b^2 - 4*a*c
)^(1/2) - 2*B*a*b*f*(b^2 - 4*a*c)^(1/2) - 2*B*b*c*d*(b^2 - 4*a*c)^(1/2)))/(
4*(4*a*c - b^2)*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)))*(A*f*(b^2 - 4*a
*c)^(3/2) + 2*A*b*f*(4*a*c - b^2) - 2*B*a*f*(4*a*c - b^2) + 2*B*c*d*(4*a*c
- b^2) + 4*A*c^2*d*(b^2 - 4*a*c)^(1/2) + A*b^2*f*(b^2 - 4*a*c)^(1/2) - 2*B*
a*b*f*(b^2 - 4*a*c)^(1/2) - 2*B*b*c*d*(b^2 - 4*a*c)^(1/2)))/(b^2*(4*a^2*f^2
 + 4*c^2*d^2 + 4*b^2*d*f - 24*a*c*d*f) - 4*a*c*(4*a^2*f^2 + 4*c^2*d^2 - 8*a
*c*d*f)) + (log(B^3*c^2*f^2*x + A*B^2*c^2*f^2 + ((((A*f*(b^2 - 4*a*c)^(3/2)
 - 2*A*b*f*(4*a*c - b^2) + 2*B*a*f*(4*a*c - b^2) - 2*B*c*d*(4*a*c - b^2) +
4*A*c^2*d*(b^2 - 4*a*c)^(1/2) + A*b^2*f*(b^2 - 4*a*c)^(1/2) - 2*B*a*b*f*(b^
2 - 4*a*c)^(1/2) - 2*B*b*c*d*(b^2 - 4*a*c)^(1/2))*(4*A*a^2*c^2*f^4 + 4*A*c^
4*d^2*f^2 - c*f^2*x*(3*A*b^3*f^2 + 4*B*c^3*d^2 ...
```

## 3.5 $\int \frac{A+Bx}{(a+bx+cx^2)^2(d+fx^2)}\,dx$

**Optimal**. Leaf size=596

$$\frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c\left(Ab^2f+2Ac(cd-af)-bB(cd+af)\right)x}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}-\frac{f^{3/2}(Ab^2df -}{}$$

```
[Out] (A*b*c*(a*f+c*d)-(A*b-B*a)*(-2*a*c*f+b^2*f+2*c^2*d)-c*(A*b^2*f+2*A*c*(-a*f+
c*d)-b*B*(a*f+c*d))*x)/(-4*a*c+b^2)/(b^2*d*f+(-a*f+c*d)^2)/(c*x^2+b*x+a)-(b
^5*B*d*f^2-2*A*b^4*f^2*(-a*f+c*d)-4*A*c^2*(-3*a*f+c*d)*(-a*f+c*d)^2+b^3*B*f
*(-a^2*f^2-4*a*c*d*f+5*c^2*d^2)-4*A*b^2*c*f*(3*a^2*f^2-3*a*c*d*f+2*c^2*d^2)
+2*b*B*c*(3*a^3*f^3+3*a^2*c*d*f^2-7*a*c^2*d^2*f+c^3*d^3))*arctanh((2*c*x+b)
/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(3/2)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))
^2-1/2*f*(2*A*b*f*(-a*f+c*d)+B*(c^2*d^2-2*a*c*d*f-f*(-a^2*f+b^2*d)))*ln(c*x
^2+b*x+a)/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))^2+1/2*f*(2*A*b*f*(-a*f+c*d)+B
*(c^2*d^2-2*a*c*d*f-f*(-a^2*f+b^2*d)))*ln(f*x^2+d)/(c^2*d^2-2*a*c*d*f+f*(a^
2*f+b^2*d))^2-f^(3/2)*(A*b^2*d*f+2*b*B*d*(-a*f+c*d)-A*(-a*f+c*d)^2)*arctan(
x*f^(1/2)/d^(1/2))/(c^2*d^2-2*a*c*d*f+f*(a^2*f+b^2*d))^2/d^(1/2)
```

**Rubi** [A]
time = 1.03, antiderivative size = 596, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 9, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {1032, 1088, 648, 632, 212, 642, 649, 211, 266}

$$-\frac{f^{3/2}\text{ArcTan}\left(\frac{\sqrt{f}x}{\sqrt{d}}\right)\left(-A(cd-af)^2+2bBd(cd-af)+Ab^2df\right)}{\sqrt{d}\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)^2}-\frac{f\log\left(a+bx+cx^2\right)\left(B\left(-f\left(b^2d-a^2f\right)-2acdf+c^2d^2\right)+2Abf(cd-af)\right)}{2\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)^2}+\frac{f\log\left(d+fx^2\right)\left(B\left(-f\left(b^2d-a^2f\right)-2acdf+c^2d^2\right)+2Abf(cd-af)\right)}{2\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)^2}-\frac{\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)\left(-4Ab^2cf\left(3a^2f^2-3acdf+2c^2d^2\right)+b^3Bf\left(-a^2f^2-4acdf+5c^2d^2\right)+2bBc\left(3a^3f^3+3a^2cdf^2-7ac^2d^2f+c^3d^3\right)-2Ab^4f^2(cd-af)-4Ac^2(cd-3af)(cd-af)^2+b^5Bdf^2\right)}{\left(b^2-4ac\right)^{3/2}\left(f\left(a^2f+b^2d\right)-2acdf+c^2d^2\right)^2}+\frac{-(Ab-aB)\left(-2acf+b^2f+2c^2d\right)-cx\left(2Ac(cd-af)-bB(af+cd)+Ab^2f\right)+Abc(af+cd)}{\left(b^2-4ac\right)\left(a+bx+cx^2\right)\left((cd-af)^2+b^2df\right)}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)^2*(d + f*x^2)),x]
```

```
[Out] (A*b*c*(c*d + a*f) - (A*b - a*B)*(2*c^2*d + b^2*f - 2*a*c*f) - c*(A*b^2*f +
 2*A*c*(c*d - a*f) - b*B*(c*d + a*f))*x)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a
*f)^2)*(a + b*x + c*x^2)) - (f^(3/2)*(A*b^2*d*f + 2*b*B*d*(c*d - a*f) - A*(
c*d - a*f)^2)*ArcTan[(Sqrt[f]*x)/Sqrt[d]])/(Sqrt[d]*(c^2*d^2 - 2*a*c*d*f +
f*(b^2*d + a^2*f))^2) - ((b^5*B*d*f^2 - 2*A*b^4*f^2*(c*d - a*f) - 4*A*c^2*(
c*d - 3*a*f)*(c*d - a*f)^2 + b^3*B*f*(5*c^2*d^2 - 4*a*c*d*f - a^2*f^2) - 4*
A*b^2*c*f*(2*c^2*d^2 - 3*a*c*d*f + 3*a^2*f^2) + 2*b*B*c*(c^3*d^3 - 7*a*c^2*
d^2*f + 3*a^2*c*d*f^2 + 3*a^3*f^3))*ArcTanh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])
/((b^2 - 4*a*c)^(3/2)*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))^2) - (f*(2*
A*b*f*(c*d - a*f) + B*(c^2*d^2 - 2*a*c*d*f - f*(b^2*d - a^2*f)))*Log[a + b*
x + c*x^2])/(2*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))^2) + (f*(2*A*b*f*(
c*d - a*f) + B*(c^2*d^2 - 2*a*c*d*f - f*(b^2*d - a^2*f)))*Log[d + f*x^2])/(
2*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))^2)
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 266

```
Int[(x_)^(m_.)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> Simp[Log[RemoveConten
t[a + b*x^n, x]]/(b*n), x] /; FreeQ[{a, b, m, n}, x] && EqQ[m, n - 1]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 649

```
Int[((d_) + (e_.)*(x_))/((a_) + (c_.)*(x_)^2), x_Symbol] :> Dist[d, Int[1/(
a + c*x^2), x], x] + Dist[e, Int[x/(a + c*x^2), x], x] /; FreeQ[{a, c, d, e
}, x] &&  !NiceSqrtQ[(-a)*c]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
```

```
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1088

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*((d_) + (f_.)*(x_)^2)), x_Symbol] :> With[{q = c^2*d^2 + b^2*d*f - 2*a*c*d
*f + a^2*f^2}, Dist[1/q, Int[(A*c^2*d - a*c*C*d + A*b^2*f - a*b*B*f - a*A*c
*f + a^2*C*f + c*(B*c*d - b*C*d + A*b*f - a*B*f)*x)/(a + b*x + c*x^2), x],
x] + Dist[1/q, Int[(c*C*d^2 + b*B*d*f - A*c*d*f - a*C*d*f + a*A*f^2 - f*(B*
c*d - b*C*d + A*b*f - a*B*f)*x)/(d + f*x^2), x], x] /; NeQ[q, 0]] /; FreeQ[
{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{A+Bx}{\left(a+bx+cx^2\right)^2\left(d+fx^2\right)}\,dx = \frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c(Ab^2f+2Ac(cd-a}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}$$

$$= \frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c(Ab^2f+2Ac(cd-a}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}$$

$$= \frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c(Ab^2f+2Ac(cd-a}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}$$

$$= \frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c(Ab^2f+2Ac(cd-a}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}$$

$$= \frac{Abc(cd+af)-(Ab-aB)\left(2c^2d+b^2f-2acf\right)-c(Ab^2f+2Ac(cd-a}{\left(b^2-4ac\right)\left(b^2df+(cd-af)^2\right)\left(a+bx+cx^2\right)}$$

**Mathematica [A]**
time = 1.21, size = 523, normalized size = 0.88

$$\frac{-\frac{2\left(c^2d^2-2acdf+f\left(b^2d+a^2f\right)\right)\left(A\left(b^3f+bc(cd-3af)+b^2cfx+2c^2(cd-af)x\right)+B\left(2a^2cf-bc^2dx-a\left(2c^2d+b^2f+bcfx\right)\right)\right)}{\left(b^2-4ac\right)(a+x(b+cx))}+\frac{2f^{3/2}\left(-Ab^2df+A(cd-af)^2+2bBd(-cd+af)\right)\tan^{-1}\left(\frac{\sqrt{f}x}{\sqrt{d}}\right)}{\sqrt{d}}-\frac{2\left(b^5Bdf^2-4Ac^2(cd-3af)(cd-af)^2+2Ab^4f^2(-cd+af)-b^3Bf\left(-5c^2d^2+4acdf+a^2f^2\right)-4Ab^2cf\left(2c^2d^2-3acdf+3a^2f^2\right)+2bBc\left(c^3d^3-7ac^2d^2f+3a^2cdf^2+3a^3f^3\right)\right)\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)}{\left(-b^2+4ac\right)^{3/2}}+f(2Abf(cd-af)+B(c^2d^2-2acdf+f(-b^2d+a^2f)))\log\left(d+fx^2\right)+f(2Abf(-cd+af)+B(-c^2d^2+2acdf+f(b^2d-a^2f)))\log(a+x(b+cx))}{2\left(c^2d^2-2acdf+f\left(b^2d+a^2f\right)\right)^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)^2*(d + f*x^2)),x]
```

```
[Out] ((-2*(c^2*d^2 - 2*a*c*d*f + f*(b^2*d + a^2*f))*(A*(b^3*f + b*c*(c*d - 3*a*f
) + b^2*c*f*x + 2*c^2*(c*d - a*f)*x) + B*(2*a^2*c*f - b*c^2*d*x - a*(2*c^2*
d + b^2*f + b*c*f*x))))/((b^2 - 4*a*c)*(a + x*(b + c*x))) + (2*f^(3/2)*(-(A
*b^2*d*f) + A*(c*d - a*f)^2 + 2*b*B*d*(-(c*d) + a*f))*ArcTan[(Sqrt[f]*x)/Sq
rt[d]])/Sqrt[d] - (2*(b^5*B*d*f^2 - 4*A*c^2*(c*d - 3*a*f)*(c*d - a*f)^2 + 2
*A*b^4*f^2*(-(c*d) + a*f) - b^3*B*f*(-5*c^2*d^2 + 4*a*c*d*f + a^2*f^2) - 4*
A*b^2*c*f*(2*c^2*d^2 - 3*a*c*d*f + 3*a^2*f^2) + 2*b*B*c*(c^3*d^3 - 7*a*c^2*
d^2*f + 3*a^2*c*d*f^2 + 3*a^3*f^3))*ArcTan[(b + 2*c*x)/Sqrt[-b^2 + 4*a*c]])
/(-b^2 + 4*a*c)^(3/2) + f*(2*A*b*f*(c*d - a*f) + B*(c^2*d^2 - 2*a*c*d*f + f
*(-(b^2*d) + a^2*f)))*Log[d + f*x^2] + f*(2*A*b*f*(-(c*d) + a*f) + B*(-(c^2
*d^2) + 2*a*c*d*f + f*(b^2*d - a^2*f)))*Log[a + x*(b + c*x)])/(2*(c^2*d^2 -
 2*a*c*d*f + f*(b^2*d + a^2*f))^2)
```

**Maple [B]** Leaf count of result is larger than twice the leaf count of optimal. 1253 vs. $2(580) = 1160$.
time = 0.95, size = 1254, normalized size = 2.10

| method | result |
|---|---|
| default | $\frac{f^{2}\left(\frac{\left(-2Aab\,f^{2}+2Abcdf+B\,a^{2}f^{2}-2Bacdf-B\,b^{2}df+B\,c^{2}d^{2}\right)\ln\left(f\,x^{2}+d\right)}{2f}+\frac{\left(A\,a^{2}f^{2}-2Aacdf-A\,b^{2}df+A\,c^{2}d^{2}+2Babdf-2Bbc\,d^{2}\right)\arctan\left(\frac{fx}{\sqrt{df}}\right.}{\sqrt{df}}\right.}{a^{4}f^{4}-4a^{3}cd\,f^{3}+2a^{2}b^{2}d\,f^{3}+6a^{2}c^{2}d^{2}f^{2}-4a\,b^{2}c\,d^{2}f^{2}-4a\,c^{3}d^{3}f+b^{4}d^{2}f^{2}+2b^{2}c^{2}d^{3}f+c^{4}d^{4}}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] f^2/(a^4*f^4-4*a^3*c*d*f^3+2*a^2*b^2*d*f^3+6*a^2*c^2*d^2*f^2-4*a*b^2*c*d^2*
f^2-4*a*c^3*d^3*f+b^4*d^2*f^2+2*b^2*c^2*d^3*f+c^4*d^4)*(1/2*(-2*A*a*b*f^2+2
*A*b*c*d*f+B*a^2*f^2-2*B*a*c*d*f-B*b^2*d*f+B*c^2*d^2)/f*ln(f*x^2+d)+(A*a^2*
f^2-2*A*a*c*d*f-A*b^2*d*f+A*c^2*d^2+2*B*a*b*d*f-2*B*b*c*d^2)/(d*f)^(1/2)*ar
ctan(f*x/(d*f)^(1/2)))-1/(a^4*f^4-4*a^3*c*d*f^3+2*a^2*b^2*d*f^3+6*a^2*c^2*d
^2*f^2-4*a*b^2*c*d^2*f^2-4*a*c^3*d^3*f+b^4*d^2*f^2+2*b^2*c^2*d^3*f+c^4*d^4)
*((c*(2*A*a^3*c*f^3-A*a^2*b^2*f^3-6*A*a^2*c^2*d*f^2+4*A*a*b^2*c*d*f^2+6*A*a
*c^3*d^2*f-A*b^4*d*f^2-3*A*b^2*c^2*d^2*f-2*A*c^4*d^3+B*a^3*b*f^3-B*a^2*b*c*
d*f^2+B*a*b^3*d*f^2-B*a*b*c^2*d^2*f+B*b^3*c*d^2*f+B*b*c^3*d^3)/(4*a*c-b^2)*
x+(3*A*a^3*b*c*f^3-A*a^2*b^3*f^3-7*A*a^2*b*c^2*d*f^2+5*A*a*b^3*c*d*f^2+5*A*
a*b*c^3*d^2*f-A*b^5*d*f^2-2*A*b^3*c^2*d^2*f-A*b*c^4*d^3-2*B*a^4*c*f^3+B*a^3
*b^2*f^3+6*B*a^3*c^2*d*f^2-4*B*a^2*b^2*c*d*f^2-6*B*a^2*c^3*d^2*f+B*a*b^4*d*
f^2+3*B*a*b^2*c^2*d^2*f+2*B*a*c^4*d^3)/(4*a*c-b^2))/(c*x^2+b*x+a)+1/(4*a*c-
b^2)*(1/2*(-8*A*a^2*b*c^2*f^3+2*A*a*b^3*c*f^3+8*A*a*b*c^3*d*f^2-2*A*b^3*c^2
```

```
*d*f^2+4*B*a^3*c^2*f^3-B*a^2*b^2*c*f^3-8*B*a^2*c^3*d*f^2-2*B*a*b^2*c^2*d*f^
2+4*B*a*c^4*d^2*f+B*b^4*c*d*f^2-B*b^2*c^3*d^2*f)/c*ln(c*x^2+b*x+a)+2*(6*A*a
^3*c^2*f^3-10*A*a^2*b^2*c*f^3-14*A*a^2*c^3*d*f^2+2*A*a*b^4*f^3+10*A*a*b^2*c
^2*d*f^2+10*A*a*c^4*d^2*f-2*A*b^4*c*d*f^2-4*A*b^2*c^3*d^2*f-2*A*c^5*d^3+5*B
*a^3*b*c*f^3-B*a^2*b^3*f^3-B*a^2*b*c^2*d*f^2-3*B*a*b^3*c*d*f^2-5*B*a*b*c^3*
d^2*f+b^5*B*d*f^2+2*B*b^3*c^2*d^2*f+B*b*c^4*d^3-1/2*(-8*A*a^2*b*c^2*f^3+2*A
*a*b^3*c*f^3+8*A*a*b*c^3*d*f^2-2*A*b^3*c^2*d*f^2+4*B*a^3*c^2*f^3-B*a^2*b^2*
c*f^3-8*B*a^2*c^3*d*f^2-2*B*a*b^2*c^2*d*f^2+4*B*a*c^4*d^2*f+B*b^4*c*d*f^2-B
*b^2*c^3*d^2*f)*b/c)/(4*a*c-b^2)^(1/2)*arctan((2*c*x+b)/(4*a*c-b^2)^(1/2)))
)
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)**2/(f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 1313 vs. 2(579) = 1158.
time = 4.24, size = 1313, normalized size = 2.20

$$-\frac{(Bc^2d^2f - Bb^2df^2 - 2Bacdf^2 + 2Abcdf^2 + Ba^2f^3 - 2Aabf^3)\log(cx^2+bx+a)}{2(c^4d^4 + 2b^2c^2d^3f - 4ac^3d^3f + b^4d^2f^2 - 4ab^2cd^2f^2 + 6a^2c^2d^2f^2 + 2a^2b^2df^3 - 4a^3cdf^3 + a^4f^4)} + \frac{(Bc^2d^2f - Bb^2df^2 - 2Bacdf^2 + 2Abcdf^2 + Ba^2f^3 - 2Aabf^3)\log(fx^2+d)}{2(c^4d^4 + 2b^2c^2d^3f - 4ac^3d^3f + b^4d^2f^2 - 4ab^2cd^2f^2 + 6a^2c^2d^2f^2 + 2a^2b^2df^3 - 4a^3cdf^3 + a^4f^4)} - \frac{(2Bbcd^2f^2 - Ac^2d^2f^2 - 2Babdf^3 + Ab^2df^3 + 2Aacdf^3 - Aa^2f^4)\arctan\left(\frac{fx}{\sqrt{df}}\right)}{(c^4d^4 + 2b^2c^2d^3f - 4ac^3d^3f + b^4d^2f^2 - 4ab^2cd^2f^2 + 6a^2c^2d^2f^2 + 2a^2b^2df^3 - 4a^3cdf^3 + a^4f^4)\sqrt{df}} + \text{[illegible]}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+d),x, algorithm="giac")
```

```
[Out] -1/2*(B*c^2*d^2*f - B*b^2*d*f^2 - 2*B*a*c*d*f^2 + 2*A*b*c*d*f^2 + B*a^2*f^3
 - 2*A*a*b*f^3)*log(c*x^2 + b*x + a)/(c^4*d^4 + 2*b^2*c^2*d^3*f - 4*a*c^3*d
^3*f + b^4*d^2*f^2 - 4*a*b^2*c*d^2*f^2 + 6*a^2*c^2*d^2*f^2 + 2*a^2*b^2*d*f^
3 - 4*a^3*c*d*f^3 + a^4*f^4) + 1/2*(B*c^2*d^2*f - B*b^2*d*f^2 - 2*B*a*c*d*f
^2 + 2*A*b*c*d*f^2 + B*a^2*f^3 - 2*A*a*b*f^3)*log(f*x^2 + d)/(c^4*d^4 + 2*b
^2*c^2*d^3*f - 4*a*c^3*d^3*f + b^4*d^2*f^2 - 4*a*b^2*c*d^2*f^2 + 6*a^2*c^2*
d^2*f^2 + 2*a^2*b^2*d*f^3 - 4*a^3*c*d*f^3 + a^4*f^4) - (2*B*b*c*d^2*f^2 - A
*c^2*d^2*f^2 - 2*B*a*b*d*f^3 + A*b^2*d*f^3 + 2*A*a*c*d*f^3 - A*a^2*f^4)*arc
tan(f*x/sqrt(d*f))/((c^4*d^4 + 2*b^2*c^2*d^3*f - 4*a*c^3*d^3*f + b^4*d^2*f^
2 - 4*a*b^2*c*d^2*f^2 + 6*a^2*c^2*d^2*f^2 + 2*a^2*b^2*d*f^3 - 4*a^3*c*d*f^3
 + a^4*f^4)*sqrt(d*f)) + (2*B*b*c^4*d^3 - 4*A*c^5*d^3 + 5*B*b^3*c^2*d^2*f -
 14*B*a*b*c^3*d^2*f - 8*A*b^2*c^3*d^2*f + 20*A*a*c^4*d^2*f + B*b^5*d*f^2 -
4*B*a*b^3*c*d*f^2 - 2*A*b^4*c*d*f^2 + 6*B*a^2*b*c^2*d*f^2 + 12*A*a*b^2*c^2*
d*f^2 - 28*A*a^2*c^3*d*f^2 - B*a^2*b^3*f^3 + 2*A*a*b^4*f^3 + 6*B*a^3*b*c*f^
3 - 12*A*a^2*b^2*c*f^3 + 12*A*a^3*c^2*f^3)*arctan((2*c*x + b)/sqrt(-b^2 + 4
*a*c))/((b^2*c^4*d^4 - 4*a*c^5*d^4 + 2*b^4*c^2*d^3*f - 12*a*b^2*c^3*d^3*f +
 16*a^2*c^4*d^3*f + b^6*d^2*f^2 - 8*a*b^4*c*d^2*f^2 + 22*a^2*b^2*c^2*d^2*f^
2 - 24*a^3*c^3*d^2*f^2 + 2*a^2*b^4*d*f^3 - 12*a^3*b^2*c*d*f^3 + 16*a^4*c^2*
d*f^3 + a^4*b^2*f^4 - 4*a^5*c*f^4)*sqrt(-b^2 + 4*a*c)) + (2*B*a*c^4*d^3 - A
*b*c^4*d^3 + 3*B*a*b^2*c^2*d^2*f - 2*A*b^3*c^2*d^2*f - 6*B*a^2*c^3*d^2*f +
5*A*a*b*c^3*d^2*f + B*a*b^4*d*f^2 - A*b^5*d*f^2 - 4*B*a^2*b^2*c*d*f^2 + 5*A
*a*b^3*c*d*f^2 + 6*B*a^3*c^2*d*f^2 - 7*A*a^2*b*c^2*d*f^2 + B*a^3*b^2*f^3 -
A*a^2*b^3*f^3 - 2*B*a^4*c*f^3 + 3*A*a^3*b*c*f^3 + (B*b*c^4*d^3 - 2*A*c^5*d^
3 + B*b^3*c^2*d^2*f - B*a*b*c^3*d^2*f - 3*A*b^2*c^3*d^2*f + 6*A*a*c^4*d^2*f
 + B*a*b^3*c*d*f^2 - A*b^4*c*d*f^2 - B*a^2*b*c^2*d*f^2 + 4*A*a*b^2*c^2*d*f^
2 - 6*A*a^2*c^3*d*f^2 + B*a^3*b*c*f^3 - A*a^2*b^2*c*f^3 + 2*A*a^3*c^2*f^3)*
x)/((c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*f^2)^2*(c*x^2 + b*x + a)*(b^2 - 4*
a*c))
```

**Mupad [B]**
time = 7.53, size = 2500, normalized size = 4.19

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d + f*x^2)*(a + b*x + c*x^2)^2),x)
```

```
[Out] ((A*b^3*f + A*b*c^2*d - 2*B*a*c^2*d - B*a*b^2*f + 2*B*a^2*c*f - 3*A*a*b*c*f
)/((4*a*c - b^2)*(a^2*f^2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)) - (x*(2*A*a*c^2
*f - 2*A*c^3*d + B*b*c^2*d - A*b^2*c*f + B*a*b*c*f))/((4*a*c - b^2)*(a^2*f^
2 + c^2*d^2 + b^2*d*f - 2*a*c*d*f)))/(a + b*x + c*x^2) + symsum(log((x*(4*A
^3*b^3*c^4*f^6 + 16*B^3*a^3*c^4*f^6 - 3*B^3*a^2*b^2*c^3*f^6 + B^3*b^2*c^5*d
```

^2*f^4 - 16*A^3*a*b*c^5*f^6 + 20*A^2*B*a^2*c^5*f^6 - 3*A^2*B*b^4*c^3*f^6 + 4*A^2*B*c^7*d^2*f^4 - 16*B^3*a^2*c^5*d*f^5 + 6*B^3*a*b^2*c^4*d*f^5 - 24*A^2 *B*a*c^6*d*f^5 + 6*A*B^2*a*b^3*c^3*f^6 - 28*A*B^2*a^2*b*c^4*f^6 + 8*A^2*B*a *b^2*c^4*f^6 - 4*A*B^2*b*c^6*d^2*f^4 - 6*A*B^2*b^3*c^4*d*f^5 + 8*A^2*B*b^2* c^5*d*f^5 + 16*A*B^2*a*b*c^5*d*f^5))/(16*a^2*c^6*d^4 + a^4*b^4*f^4 + b^4*c^ 4*d^4 + 16*a^6*c^2*f^4 + b^8*d^2*f^2 - 8*a*b^2*c^5*d^4 - 8*a^5*b^2*c*f^4 + 2*a^2*b^6*d*f^3 - 64*a^3*c^5*d^3*f - 64*a^5*c^3*d*f^3 + 2*b^6*c^2*d^3*f + 9 6*a^4*c^4*d^2*f^2 + 54*a^2*b^4*c^2*d^2*f^2 - 112*a^3*b^2*c^3*d^2*f^2 - 20*a *b^4*c^3*d^3*f - 12*a*b^6*c*d^2*f^2 - 20*a^3*b^4*c*d*f^3 + 64*a^2*b^2*c^4*d ^3*f + 64*a^4*b^2*c^2*d*f^3) - root(2560*a^3*b^2*c^9*d^8*f*z^4 - 1152*a^2*b ^4*c^8*d^8*f*z^4 + 384*a^5*b^8*c*d^3*f^6*z^4 + 384*a*b^8*c^5*d^7*f^2*z^4 + 288*a^3*b^10*c*d^4*f^5*z^4 + 288*a*b^10*c^3*d^6*f^3*z^4 + 224*a^7*b^6*c*d^2 *f^7*z^4 - 192*a^10*b^2*c^2*d*f^8*z^4 + 224*a*b^6*c^7*d^8*f*z^4 + 80*a*b^12 *c*d^5*f^4*z^4 + 48*a^9*b^4*c*d*f^8*z^4 - 33920*a^6*b^2*c^6*d^5*f^4*z^4 + 2 7936*a^5*b^4*c^5*d^5*f^4*z^4 + 26112*a^7*b^2*c^5*d^4*f^5*z^4 + 26112*a^5*b^ 2*c^7*d^6*f^3*z^4 - 20352*a^6*b^4*c^4*d^4*f^5*z^4 - 20352*a^4*b^4*c^6*d^6*f ^3*z^4 - 13080*a^4*b^6*c^4*d^5*f^4*z^4 - 11520*a^8*b^2*c^4*d^3*f^6*z^4 - 11 520*a^4*b^2*c^8*d^7*f^2*z^4 + 8736*a^5*b^6*c^3*d^4*f^5*z^4 + 8736*a^3*b^6*c ^5*d^6*f^3*z^4 + 7488*a^7*b^4*c^3*d^3*f^6*z^4 + 7488*a^3*b^4*c^7*d^7*f^2*z^ 4 + 3840*a^3*b^8*c^3*d^5*f^4*z^4 + 2560*a^9*b^2*c^3*d^2*f^7*z^4 - 2416*a^6* b^6*c^2*d^3*f^6*z^4 - 2416*a^2*b^6*c^6*d^7*f^2*z^4 - 2160*a^4*b^8*c^2*d^4*f ^5*z^4 - 2160*a^2*b^8*c^4*d^6*f^3*z^4 - 1152*a^8*b^4*c^2*d^2*f^7*z^4 - 720* a^2*b^10*c^2*d^5*f^4*z^4 - 16*b^8*c^6*d^8*f*z^4 - 2048*a^4*c^10*d^8*f*z^4 + 256*a^11*c^3*d*f^8*z^4 - 4*a^8*b^6*d*f^8*z^4 + 48*a*b^4*c^9*d^9*z^4 - 24*b ^10*c^4*d^7*f^2*z^4 - 16*b^12*c^2*d^6*f^3*z^4 + 17920*a^7*c^7*d^5*f^4*z^4 - 14336*a^8*c^6*d^4*f^5*z^4 - 14336*a^6*c^8*d^6*f^3*z^4 + 7168*a^9*c^5*d^3*f ^6*z^4 + 7168*a^5*c^9*d^7*f^2*z^4 - 2048*a^10*c^4*d^2*f^7*z^4 - 24*a^4*b^10 *d^3*f^6*z^4 - 16*a^6*b^8*d^2*f^7*z^4 - 16*a^2*b^12*d^4*f^5*z^4 - 192*a^2*b ^2*c^10*d^9*z^4 - 4*b^14*d^5*f^4*z^4 - 4*b^6*c^8*d^9*z^4 + 256*a^3*c^11*d^9 *z^4 + 912*A*B*a^6*b*c^3*d*f^6*z^2 + 192*A*B*a^4*b^5*c*d*f^6*z^2 + 920*A*B* a^4*b^3*c^3*d^2*f^5*z^2 - 480*A*B*a^2*b^5*c^3*d^3*f^4*z^2 - 336*A*B*a^2*b^3 *c^5*d^4*f^3*z^2 - 272*A*B*a^3*b^3*c^4*d^3*f^4*z^2 + 240*A*B*a^3*b^5*c^2*d^ 2*f^5*z^2 + 192*A*B*a*b*c^8*d^6*f*z^2 - 2496*A*B*a^5*b*c^4*d^2*f^5*z^2 + 18 72*A*B*a^4*b*c^5*d^3*f^4*z^2 - 744*A*B*a^5*b^3*c^2*d*f^6*z^2 - 720*A*B*a^2* b*c^7*d^5*f^2*z^2 + 504*A*B*a*b^3*c^6*d^5*f^2*z^2 + 256*A*B*a^3*b*c^6*d^4*f ^3*z^2 + 168*A*B*a*b^7*c^2*d^3*f^4*z^2 - 144*A*B*a^2*b^7*c*d^2*f^5*z^2 + 14 4*A*B*a*b^5*c^4*d^4*f^3*z^2 - 56*B^2*a*b^2*c^7*d^6*f*z^2 - 36*B^2*a^5*b^4*c *d*f^6*z^2 - 16*B^2*a*b^8*c*d^3*f^4*z^2 - 164*A^2*a^3*b^6*c*d*f^6*z^2 - 16* A^2*a*b^8*c*d^2*f^5*z^2 - 96*A*B*b^5*c^5*d^5*f^2*z^2 - 24*A*B*b^7*c^3*d^4*f ^3*z^2 - 580*B^2*a^4*b^2*c^4*d^3*f^4*z^2 + 536*B^2*a^3*b^4*c^3*d^3*f^4*z^2 - 348*B^2*a^4*b^4*c^2*d^2*f^5*z^2 + 316*B^2*a^2*b^2*c^6*d^5*f^2*z^2 + 200*B ^2*a^5*b^2*c^3*d^2*f^5*z^2 - 120*B^2*a^2*b^4*c^4*d^4*f^3*z^2 - 66*B^2*a^2*b ^6*c^2*d^3*f^4*z^2 - 16*B^2*a^3*b^2*c^5*d^4*f^3*z^2 + 1952*A^2*a^4*b^2*c^4* d^2*f^5*z^2 - 1792*A^2*a^3*b^2*c^5*d^3*f^4*z^2 - 1272*A^2*a^3*b^4*c^3*d^2*f ^5*z^2 + 976*A^2*a^2*b^2*c^6*d^4*f^3*z^2 + 960*A^2*a^2*b^4*c^4*d^3*f^4*z^2

```
+ 282*A^2*a^2*b^6*c^2*d^2*f^5*z^2 - 72*A*B*b^3*c^7*d^6*f*z^2 - 16*A*B*b^9*c
*d^3*f^4*z^2 - 16*A*B*a^3*b^7*d*f^6*z^2 + 16*A*B*a*b^9*d^2*f^5*z^2 - 180*B^
2*a*b^4*c^5*d^5*f^2*z^2 + 132*B^2*a^6*b^2*c^2*d*f^6*z^2 + 108*B^2*a^3*b^6*c
*d^2*f^5*z^2 + 20*B^2*a*b^6*c^3*d^4*f^3*z^2 - 736*A^2*a^5*b^2*c^3*d*f^6*z^2
 + 624*A^2*a^4*b^4*c^2*d*f^6*z^2 - 416*A^2*a*b^2*c^7*d^5*f^2*z^2 - 276*A^2*
a*b^4*c^5*d^4*f^3*z^2 - 196*A^2*a*b^6*c^3*d^3*f^4*z^2 + 31*B^2*b^6*c^4*d^5*
f^2*z^2 + 2*B^2*b^8*c^2*d^4*f^3*z^2 - 768*B^2*a^5*c^5*d^3*f^4*z^2 + 512*B^2
*a^6*c^4*d^2*f^5*z^2 + 512*B^2*a^4*c^6*d^4*f^3*z^2 - 128*B^2*a^3*c^7*d^5*f^
2*z^2 + 80*A^2*b^4*c^6*d^5*f^2*z^2 + 31*A^2*b^6*c^4*d^4*f^3*z^2 + 14*A^2*b^
8*c^2*d^3*f^4*z^2 - 1152*A^2*a^3*c^7*d^4*f^3*z^2 + 1008*A^2*a^4*c^6*d^3*f^4
*z^2 + 624*A^2*a^2*c^8*d^5*f^2*z^2 - 288*A^2*a^5*c^5*d^2*f^5*z^2 - 10*B^2*a
^2*b^8*d^2*f^5*z^2 - 48*A^2*a^6*b^2*c^2*f^7*z^2 - 16*A*B*b*c^9*d^7*z^2 + 20
*B^2*b^4*c^6*d^6*f*z^2 - 128*B^2*a^7*c^3*d*f^6*z^2 + 64*A^2*b^2*c^8*d^6*f*z
^2 - 112*A^2*a^6*c^4*d*f^6*z^2 + 3*B^2*a^4*b^6*d*f^6*z^2 + 14*A^2*a^2*b^8*d
*f^6*z^2 + 12*A^2*a^5*b^4*c*f^7*z^2 - 160*A^2*a*c^9*d^6*f*z^2 + 3*B^2*b^10*
d^3*f^4*z^2 - A^2*b^10*d^2*f^5*z^2 + 64*A^2*a^7*c^3*f^7*z^2 + 4*B^2*b^2*c^8
*d^7*z^2 - A^2*a^4*b^6*f^7*z^2 + 16*A^2*c^10*d^...
```

## 3.6 $\int \frac{(A+Bx)\sqrt{a+bx+cx^2}}{d-fx^2}\,dx$

**Optimal**. Leaf size=331

$$-\frac{B\sqrt{a+bx+cx^2}}{f}-\frac{(bB+2Ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}-\frac{\left(B\sqrt{d}-A\sqrt{f}\right)\sqrt{cd-b\sqrt{d}\,\sqrt{f}+\cdots}}{\cdots}$$

```
[Out] -1/2*(2*A*c+B*b)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f/c^(1/
2)-B*(c*x^2+b*x+a)^(1/2)/f-1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^
(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(B
*d^(1/2)-A*f^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/f^(3/2)/d^(1/2)+1/2*a
rctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^
(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(B*d^(1/2)+A*f^(1/2))*(c*d+a*f+b*d
^(1/2)*f^(1/2))^(1/2)/f^(3/2)/d^(1/2)
```

**Rubi** [A]
time = 0.38, antiderivative size = 331, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {1035, 1092, 635, 212, 1047, 738}

$$-\frac{\left(B\sqrt{d}-A\sqrt{f}\right)\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}\,f^{3/2}}+\frac{\left(A\sqrt{f}+B\sqrt{d}\right)\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}\right)}{2\sqrt{d}\,f^{3/2}}-\frac{(2Ac+bB)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}-\frac{B\sqrt{a+bx+cx^2}}{f}$$

Antiderivative was successfully verified.

```
[In] Int[((A + B*x)*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -((B*Sqrt[a + b*x + c*x^2])/f) - ((b*B + 2*A*c)*ArcTanh[(b + 2*c*x)/(2*Sqrt
[c]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[c]*f) - ((B*Sqrt[d] - A*Sqrt[f])*Sqrt[
c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt
[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c
*x^2])])/(2*Sqrt[d]*f^(3/2)) + ((B*Sqrt[d] + A*Sqrt[f])*Sqrt[c*d + b*Sqrt[d
]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f
])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sq
rt[d]*f^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{(A+Bx)\sqrt{a+bx+cx^2}}{d-fx^2}\,dx = -\frac{B\sqrt{a+bx+cx^2}}{f} + \frac{\int \frac{\frac{1}{2}(bBd+2aAf)+(Bcd+Abf+aBf)x+\frac{1}{2}(bB+2Ac)fx^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f}$$

$$= -\frac{B\sqrt{a+bx+cx^2}}{f} - \frac{\int \frac{-\frac{1}{2}(bB+2Ac)df-\frac{1}{2}f(bBd+2aAf)-f(Bcd+Abf+aBf)x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f^2} - \cdots$$

$$= -\frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(bB+2Ac)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f} \cdots$$

$$= -\frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(bB+2Ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f} - \cdots$$

$$= -\frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(bB+2Ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f} - \cdots$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.68, size = 511, normalized size = 1.54

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -1/2*(2*B*Sqrt[a + x*(b + c*x)] - ((b*B + 2*A*c)*Log[f*(b + 2*c*x - 2*Sqrt[
c]*Sqrt[a + x*(b + c*x)])])/Sqrt[c] + RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d
*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*B*d*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1] + A*b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1] - a*B*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*B*f*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*B*Sqrt[c]*d*Log[-(Sqrt[c
]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*A*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1]*#1 - 2*a*A*Sqrt[c]*f*Log[-(Sqrt[c]*x) + Sqrt[a +
b*x + c*x^2] - #1]*#1 + B*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1
]*#1^2 + A*b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a*B*f*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#
1 - a*f*#1 + f*#1^3) & ])/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 802 vs. $2(251) = 502$.
time = 0.17, size = 803, normalized size = 2.43

| method | result |
|---|---|
| default | $\left(Af-B\sqrt{df}\right)\left(\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}+\frac{\left(-2c\sqrt{df}+bf\right)\ln\left(\ldots\right.}{\ldots}\right.$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(A*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*
f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(
-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))
/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f
)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)
/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*
(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(
1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/
f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))+1/2*(-A*f-B*(d*f)^(1
/2))/(d*f)^(1/2)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)
^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((
1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)
^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(
1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*l
n((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*
((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b
*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)
))
```

**Maxima** [F(-2)]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{A\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx-\int \frac{Bx\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(A*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(B*x*sqrt(a
+ b*x + c*x**2)/(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{(A+B\,x)\,\sqrt{c\,x^2+b\,x+a}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^(1/2))/(d - f*x^2),x)
[Out] int(((A + B*x)*(a + b*x + c*x^2)^(1/2))/(d - f*x^2), x)
```

# 3.7 $\int \frac{A+Bx}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=249

$$-\frac{\left(B-\frac{A\sqrt{f}}{\sqrt{d}}\right)\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{f}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}}+\frac{\left(B+\frac{A\sqrt{f}}{\sqrt{d}}\right)\tanh^{-1}\left(\frac{b\sqrt{d}\ \cdots}{2\sqrt{cd+b\cdots}}\right)}{2\sqrt{f}\,\sqrt{cd+b\cdots}}$$

```
[Out] -1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b
*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(B-A*f^(1/2)/d^(1/2))/f^(1/2
)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+
x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^
(1/2))*(B+A*f^(1/2)/d^(1/2))/f^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.13, antiderivative size = 249, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {1047, 738, 212}

$$\frac{\left(\frac{A\sqrt{f}}{\sqrt{d}}+B\right)\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}\right)}{2\sqrt{f}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}-\frac{\left(B-\frac{A\sqrt{f}}{\sqrt{d}}\right)\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{f}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*((B - (A*Sqrt[f])/Sqrt[d])*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqr
t[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x +
c*x^2])])/(Sqrt[f]*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + ((B + (A*Sqrt[f])
/Sqrt[d])*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(
2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[f]*S
qrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{A+Bx}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \frac{1}{2}\left(B-\frac{A\sqrt{f}}{\sqrt{d}}\right)\int \frac{1}{\left(-\sqrt{d}\,\sqrt{f}-fx\right)\sqrt{a+bx+cx^2}}\,dx + \frac{1}{2}\left(B+\frac{A\cdots}{\cdots}\right.\cdots$$

$$= \left(-B-\frac{A\sqrt{f}}{\sqrt{d}}\right)\operatorname{Subst}\left(\int \frac{1}{4cdf+4b\sqrt{d}\,f^{3/2}+4af^2-x^2}\,dx, x, \frac{-b\sqrt{d}\cdots}{\cdots}\right.\cdots$$

$$= -\frac{\left(B-\frac{A\sqrt{f}}{\sqrt{d}}\right)\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{f}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}} + \cdots$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.39, size = 219, normalized size = 0.88

$$-\frac{1}{2}\operatorname{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{Ab\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-aB\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2A\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+B\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f
*#1^4 & , (A*b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*B*Log[-(S
qrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*A*Sqrt[c]*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1]*#1 + B*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 388 vs. 2(185) = 370.
time = 0.14, size = 389, normalized size = 1.56

| method | result |
|---|---|
| default | $-\frac{\left(Af-B\sqrt{df}\right)\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2}c}}{x+\frac{\sqrt{df}}{f}}\right)}{2\sqrt{df}\,f\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(A*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)
*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)
/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c
*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x
+(d*f)^(1/2)/f))-1/2*(-A*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f/((b*(d*f)^(1/2)+f*a
+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(
d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2
*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(
x-(d*f)^(1/2)/f))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 6113 vs. 2(185) = 370.
time = 30.08, size = 6113, normalized size = 24.55

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")

[Out] 1/4*sqrt((B^2*c*d^2 + A^2*a*f^2 + (B^2*a - 2*A*B*b + A^2*c)*d*f + (c^2*d^3*
f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2
*B^2*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)
*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5
- 2*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2
*(a^2*b^2 - 2*a^3*c)*d^2*f^4)))/(c^2*d^3*f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*
f^2))*log(-((B^4*b^2 - 2*A*B^3*b*c)*d^2 - 2*(A*B^3*a*b - A^3*B*b*c)*d*f + (
2*A^3*B*a*b - A^4*b^2)*f^2 + 2*((2*A^3*B*a - A^4*b)*c*f^2 + (B^4*b*c - 2*A*
B^3*c^2)*d^2 - 2*(A*B^3*a*c - A^3*B*c^2)*d*f)*x + 2*((B^3*b^2 - 3*A*B^2*b*c
 + 2*A^2*B*c^2)*d^2*f - (3*A*B^2*a*b - A^2*B*b^2 - (4*A^2*B*a - A^3*b)*c)*d
*f^2 + (2*A^2*B*a^2 - A^3*a*b)*f^3 - (B*c^3*d^4*f - (B*b^2*c - (3*B*a - A*b
)*c^2)*d^3*f^2 - (B*a*b^2 - A*b^3 - (3*B*a^2 - 2*A*a*b)*c)*d^2*f^3 + (B*a^3
 - A*a^2*b)*d*f^4)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2
*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^
2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c
^3)*d^4*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)
*d^2*f^4)))*sqrt(c*x^2 + b*x + a)*sqrt((B^2*c*d^2 + A^2*a*f^2 + (B^2*a - 2*
A*B*b + A^2*c)*d*f + (c^2*d^3*f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)*sqrt((
(B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2
- 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2
)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*
b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^4)))/(c^2*d^3*f +
a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)) - (2*B^2*a*c^2*d^3*f - 2*A^2*a^3*f^4 -
2*(B^2*a*b^2 - 2*B^2*a^2*c + A^2*a*c^2)*d^2*f^2 + 2*(B^2*a^3 + A^2*a*b^2 -
2*A^2*a^2*c)*d*f^3 + (B^2*b*c^2*d^3*f - A^2*a^2*b*f^4 - (B^2*b^3 - 2*B^2*a*
b*c + A^2*b*c^2)*d^2*f^2 + (B^2*a^2*b + A^2*b^3 - 2*A^2*a*b*c)*d*f^3)*x)*sq
rt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*
b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4
*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 -
4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^4)))/x) - 1/4*
sqrt((B^2*c*d^2 + A^2*a*f^2 + (B^2*a - 2*A*B*b + A^2*c)*d*f + (c^2*d^3*f +
a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2
*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f
 + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*
(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^
2*b^2 - 2*a^3*c)*d^2*f^4)))/(c^2*d^3*f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)
)*log(-((B^4*b^2 - 2*A*B^3*b*c)*d^2 - 2*(A*B^3*a*b - A^3*B*b*c)*d*f + (2*A^
3*B*a*b - A^4*b^2)*f^2 + 2*((2*A^3*B*a - A^4*b)*c*f^2 + (B^4*b*c - 2*A*B^3*
c^2)*d^2 - 2*(A*B^3*a*c - A^3*B*c^2)*d*f)*x - 2*((B^3*b^2 - 3*A*B^2*b*c + 2
*A^2*B*c^2)*d^2*f - (3*A*B^2*a*b - A^2*B*b^2 - (4*A^2*B*a - A^3*b)*c)*d*f^2
 + (2*A^2*B*a^2 - A^3*a*b)*f^3 - (B*c^3*d^4*f - (B*b^2*c - (3*B*a - A*b)*c^
2)*d^3*f^2 - (B*a*b^2 - A*b^3 - (3*B*a^2 - 2*A*a*b)*c)*d^2*f^3 + (B*a^3 - A
*a^2*b)*d*f^4)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2*A*B
```

```
^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 -
4*A^3*B*a*b + A^4*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c^3)*
d^4*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d^2
*f^4)))*sqrt(c*x^2 + b*x + a)*sqrt((B^2*c*d^2 + A^2*a*f^2 + (B^2*a - 2*A*B*
b + A^2*c)*d*f + (c^2*d^3*f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2)*sqrt(((B^4
*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*
(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^
2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*b^2*
c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^4)))/(c^2*d^3*f + a^2*
d*f^3 - (b^2 - 2*a*c)*d^2*f^2)) - (2*B^2*a*c^2*d^3*f - 2*A^2*a^3*f^4 - 2*(B
^2*a*b^2 - 2*B^2*a^2*c + A^2*a*c^2)*d^2*f^2 + 2*(B^2*a^3 + A^2*a*b^2 - 2*A^
2*a^2*c)*d*f^3 + (B^2*b*c^2*d^3*f - A^2*a^2*b*f^4 - (B^2*b^3 - 2*B^2*a*b*c
+ A^2*b*c^2)*d^2*f^2 + (B^2*a^2*b + A^2*b^3 - 2*A^2*a*b*c)*d*f^3)*x)*sqrt((
(B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2
- 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2
)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*
b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^4)))/x) + 1/4*sqrt
((B^2*c*d^2 + A^2*a*f^2 + (B^2*a - 2*A*B*b + A^2*c)*d*f - (c^2*d^3*f + a^2*
d*f^3 - (b^2 - 2*a*c)*d^2*f^2)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2
)*d^2 - 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (
4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/(c^4*d^5*f + a^4*d*f^5 - 2*(b^2
*c^2 - 2*a*c^3)*d^4*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 - 2*(a^2*b^
2 - 2*a^3*c)*d^2*f^4)))/(c^2*d^3*f + a^2*d*f^3 - (b^2 - 2*a*c)*d^2*f^2))*lo
g(-((B^4*b^2 - 2*A*B^3*b*c)*d^2 - 2*(A*B^3*a*b ...
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{A}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx-\int \frac{Bx}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(A/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)), x)
 - Integral(B*x/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2))
, x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueWarning, integration
of abs
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+Bx}{(d-fx^2)\sqrt{cx^2+bx+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

## 3.8 $\int \frac{A+Bx}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx$

**Optimal**. Leaf size=381

$$-\frac{2\left(aB(2c^2d-b^2f+2acf)+A(b^3f-bc(cd+3af))+c(Ab^2f+bB(cd-af)-2Ac(cd+af))\,x\right)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}}-\frac{\left(B\sqrt{d}\right.}{}$$

[Out] -1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)*(B*d^(1/2)-A*f^(1/2))/d^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)*(B*d^(1/2)+A*f^(1/2))/d^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)-2*(a*B*(2*a*c*f-b^2*f+2*c^2*d)+A*(b^3*f-b*c*(3*a*f+c*d))+c*(A*b^2*f+b*B*(-a*f+c*d)-2*A*c*(a*f+c*d))*x)/(-4*a*c+b^2)/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)

**Rubi** [A]
time = 0.49, antiderivative size = 380, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {1032, 1047, 738, 212}

$$-\frac{2(cx(-2Ac(af+cd)+bB(cd-af)+Ab^2f)-Abc(3af+cd)+aB(2acf+b^2(-f)+2c^2d)+Ab^3f)}{(b^2-4ac)\sqrt{a+bx+cx^2}\,(b^2df-(af+cd)^2)}-\frac{\sqrt{f}\left(B\sqrt{d}-A\sqrt{f}\right)\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{\sqrt{f}\left(A\sqrt{f}+B\sqrt{d}\right)\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{d}\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

[In] Int[(A + B*x)/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]

[Out] (-2*(A*b^3*f - A*b*c*(c*d + 3*a*f) + a*B*(2*c^2*d - b^2*f + 2*a*c*f) + c*(A*b^2*f + b*B*(c*d - a*f) - 2*A*c*(c*d + a*f))*x))/((b^2 - 4*a*c)*(b^2*d*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) - ((B*Sqrt[d] - A*Sqrt[f])*Sqrt[f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)) + ((B*Sqrt[d] + A*Sqrt[f])*Sqrt[f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2))

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{A+Bx}{(a+bx+cx^2)^{3/2}\,(d-fx^2)}\,dx = -\frac{2(Ab^3f - Abc(cd+3af) + aB(2c^2d - b^2f + 2acf) + c(Ab^2f + bB(c}{(b^2-4ac)\,(b^2df - (cd+af)^2)\,\sqrt{a+bx+c}}$$

$$= -\frac{2(Ab^3f - Abc(cd+3af) + aB(2c^2d - b^2f + 2acf) + c(Ab^2f + bB(c}{(b^2-4ac)\,(b^2df - (cd+af)^2)\,\sqrt{a+bx+c}}$$

$$= -\frac{2(Ab^3f - Abc(cd+3af) + aB(2c^2d - b^2f + 2acf) + c(Ab^2f + bB(c}{(b^2-4ac)\,(b^2df - (cd+af)^2)\,\sqrt{a+bx+c}}$$

$$= -\frac{2(Ab^3f - Abc(cd+3af) + aB(2c^2d - b^2f + 2acf) + c(Ab^2f + bB(c}{(b^2-4ac)\,(b^2df - (cd+af)^2)\,\sqrt{a+bx+c}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.15, size = 641, normalized size = 1.68

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (4*A*(-(b^3*f) + b*c*(c*d + 3*a*f) - b^2*c*f*x + 2*c^2*(c*d + a*f)*x) + 4*B
*(-2*a^2*c*f - b*c^2*d*x + a*(-2*c^2*d + b^2*f + b*c*f*x)) - (b^2 - 4*a*c)*
f*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1
^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*B*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1] - A*b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*B*c*
d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a*A*b*f*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*B*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
+ c*x^2] - #1] - 2*b*B*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1]*#1 + 2*A*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 +
 2*a*A*Sqrt[c]*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - B*c*d*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + A*b*f*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*B*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(2*(
b^2 - 4*a*c)*(-(c^2*d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*Sqrt[a + x*(b + c
*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 933 vs. $2(311) = 622$.

time = 0.14, size = 934, normalized size = 2.45

| method | result |
|---|---|
| default | $\left(Af - B\sqrt{df}\right)\left(\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2} c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}} - \frac{}{\left(-b \right.}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(A*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)
^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1
/2)+f*a+c*d))^(1/2)-(-2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x
+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)
-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/
2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(
1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+
f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2
)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f
)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))+1/2*(-A
*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f*(1/(b*(d*f)^(1/2)+f*a+c*d)*f/((x-(d*f)^(1/2
)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/
f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x-(d*f)^(1/2)/
f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)
+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/
f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^(1/2)+f*a+c*d)*f/((b*(d*f)^(
1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f
)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f
)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^
(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [F(-2)]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueEvaluation time: 3.53
Done
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
[Out] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.9 $\int \frac{A+Bx}{(a+bx+cx^2)^{5/2}(d-fx^2)}\,dx$

**Optimal**. Leaf size=797

$$-\frac{2(aB(2c^2d-b^2f+2acf)+A(b^3f-bc(cd+3af))+c(Ab^2f+bB(cd-af)-2Ac(cd+af))\,x)}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)^{3/2}}-\frac{2(3b^6B\cdots}{}$$

```
[Out] -2/3*(a*B*(2*a*c*f-b^2*f+2*c^2*d)+A*(b^3*f-b*c*(3*a*f+c*d))+c*(A*b^2*f+b*B*
(-a*f+c*d)-2*A*c*(a*f+c*d))*x)/(-4*a*c+b^2)/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*
x+a)^(3/2)-1/2*f^(3/2)*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*
f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(B*d^(1/2)
-A*f^(1/2))/d^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(5/2)+1/2*f^(3/2)*arctanh(1
/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c
*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(B*d^(1/2)+A*f^(1/2))/d^(1/2)/(c*d+a*f+b*d
^(1/2)*f^(1/2))^(5/2)-2/3*(3*b^6*B*d*f^2+24*a^2*B*c^2*f*(a*f+c*d)^2-A*b^5*f
^2*(6*a*f+7*c*d)-b^4*B*f*(-3*a^2*f^2+14*a*c*d*f+7*c^2*d^2)+A*b^3*c*f*(43*a^
2*f^2+46*a*c*d*f+15*c^2*d^2)+2*b^2*B*c*(-11*a^3*f^3+4*a^2*c*d*f^2+5*a*c^2*d
^2*f+2*c^3*d^3)-4*A*b*c^2*(17*a^3*f^3+24*a^2*c*d*f^2+9*a*c^2*d^2*f+2*c^3*d^
3)+c*(3*b^5*B*d*f^2-2*A*b^4*f^2*(3*a*f+4*c*d)-8*A*c^2*(a*f+c*d)^2*(5*a*f+2*
c*d)-b^3*B*f*(-3*a^2*f^2+10*a*c*d*f+17*c^2*d^2)+2*A*b^2*c*f*(19*a^2*f^2+22*
a*c*d*f+15*c^2*d^2)+4*b*B*c*(-5*a^3*f^3+4*a^2*c*d*f^2+11*a*c^2*d^2*f+2*c^3*
d^3))*x)/(-4*a*c+b^2)^2/(c^2*d^2+2*a*c*d*f-f*(-a^2*f+b^2*d))^2/(c*x^2+b*x+a
)^(1/2)
```

**Rubi** [A]
time = 1.14, antiderivative size = 796, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {1032, 1078, 1047, 738, 212}

$$-\frac{\left(B\sqrt{d}-A\sqrt{f}\right)\tanh^{-1}\left(\frac{-2\sqrt{f}\,a+\left(2c\sqrt{d}-b\sqrt{f}\right)x+b\sqrt{d}}{2\sqrt{-\sqrt{d}\sqrt{f}\,b+cd+af}\,\sqrt{cx^2+bx+a}}\right)f^{3/2}}{2\sqrt{d}\left(-\sqrt{d}\sqrt{f}\,b+cd+af\right)^{5/2}}+\frac{\left(\sqrt{f}A+B\sqrt{d}\right)\tanh^{-1}\left(\frac{2\sqrt{f}\,a+\left(\sqrt{f}\,b+2c\sqrt{d}\right)x+b\sqrt{d}}{2\sqrt{\sqrt{d}\sqrt{f}\,b+cd+af}\,\sqrt{cx^2+bx+a}}\right)f^{3/2}}{2\sqrt{d}\left(\sqrt{d}\sqrt{f}\,b+cd+af\right)^{5/2}}-\frac{2(3Bdf^2b^6-Af^2(7cd+6af)b^5-Bf(7c^2d^2+14acfd-3a^2f^2)\,b^4+Acf(15c^2d^2+46acfd+43a^2f^2)\,b^3+2Bc(2c^3d^3+5ac^2fd^2+4a^2cf^2d-11a^3f^3)\,b^2-4Ac^2(2c^3d^3+9ac^2fd^2+24a^2cf^2d+17a^3f^3)\,b+24a^2Bc^2f(cd+af)^2+c(3Bdf^2b^5-2Af^2(4cd+3af)b^4-Bf(17c^2d^2+10acfd-3a^2f^2)\,b^3+2Acf(15c^2d^2+22acfd+19a^2f^2)\,b^2+4Bc(2c^3d^3+11ac^2fd^2+4a^2cf^2d-5a^3f^3)\,b-8Ac^2(cd+af)^2(2cd+5af))\,x)}{3\,(b^2-4ac)^2\left(c^2d^2+2acfd-f\left(b^2d-a^2f\right)\right)^2\sqrt{cx^2+bx+a}}-\frac{2(Afb^3-Ac(cd+3af)b+aB(-fb^2+2c^2d+2acf)+c(Afb^2+B(cd-af)b-2Ac(cd+af))\,x)}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(cx^2+bx+a)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)^(5/2)*(d - f*x^2)),x]
```

```
[Out] (-2*(A*b^3*f - A*b*c*(c*d + 3*a*f) + a*B*(2*c^2*d - b^2*f + 2*a*c*f) + c*(A
*b^2*f + b*B*(c*d - a*f) - 2*A*c*(c*d + a*f))*x))/(3*(b^2 - 4*a*c)*(b^2*d*f
 - (c*d + a*f)^2)*(a + b*x + c*x^2)^(3/2)) - (2*(3*b^6*B*d*f^2 + 24*a^2*B*c
^2*f*(c*d + a*f)^2 - A*b^5*f^2*(7*c*d + 6*a*f) - b^4*B*f*(7*c^2*d^2 + 14*a*
c*d*f - 3*a^2*f^2) + A*b^3*c*f*(15*c^2*d^2 + 46*a*c*d*f + 43*a^2*f^2) + 2*b
^2*B*c*(2*c^3*d^3 + 5*a*c^2*d^2*f + 4*a^2*c*d*f^2 - 11*a^3*f^3) - 4*A*b*c^2
*(2*c^3*d^3 + 9*a*c^2*d^2*f + 24*a^2*c*d*f^2 + 17*a^3*f^3) + c*(3*b^5*B*d*f
```

```
^2 - 2*A*b^4*f^2*(4*c*d + 3*a*f) - 8*A*c^2*(c*d + a*f)^2*(2*c*d + 5*a*f) -
b^3*B*f*(17*c^2*d^2 + 10*a*c*d*f - 3*a^2*f^2) + 2*A*b^2*c*f*(15*c^2*d^2 + 2
2*a*c*d*f + 19*a^2*f^2) + 4*b*B*c*(2*c^3*d^3 + 11*a*c^2*d^2*f + 4*a^2*c*d*f
^2 - 5*a^3*f^3))*x))/(3*(b^2 - 4*a*c)^2*(c^2*d^2 + 2*a*c*d*f - f*(b^2*d - a
^2*f))^2*Sqrt[a + b*x + c*x^2]) - ((B*Sqrt[d] - A*Sqrt[f])*f^(3/2)*ArcTanh[
(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqr
t[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*(c*d - b*Sqrt[d]*Sq
rt[f] + a*f)^(5/2)) + ((B*Sqrt[d] + A*Sqrt[f])*f^(3/2)*ArcTanh[(b*Sqrt[d] +
 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f]
 + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)
^(5/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
```

```
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1078

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1
)*((d + f*x^2)^(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*(
(A*c - a*C)*((-b)*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(2*a*f))
+ c*(A*(2*c^2*d + b^2*f - c*(2*a*f)) - B*(b*c*d + a*b*f) + C*(b^2*d - 2*a*(
c*d - a*f)))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p +
1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Simp[(b*B - 2*A*c - 2*a*C)
*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(C*d + A*f) - b*(B*c*d + a
*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(a*f*(p + 1) - c*d*(p + 2)
) - (2*f*((A*c - a*C)*((-b)*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c
*(2*a*f)))*(p + q + 2) - (b^2*(C*d + A*f) - b*(B*c*d + a*B*f) + 2*(A*c*(c*d
 - a*f) - a*(c*C*d - a*C*f)))*(b*f*(p + 1)))*x - c*f*(b^2*(C*d + A*f) - b*(
B*c*d + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(2*p + 2*q + 5)*x
^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 - 4*a*c,
 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] &&
 ILtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rubi steps

$$\int \frac{A+Bx}{(a+bx+cx^2)^{5/2}(d-fx^2)}\,dx = -\frac{2(Ab^3f-Abc(cd+3af)+aB(2c^2d-b^2f+2acf)+c(Ab^2f+bB(cd}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)}$$

$$= -\frac{2(Ab^3f-Abc(cd+3af)+aB(2c^2d-b^2f+2acf)+c(Ab^2f+bB(cd}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)}$$

$$= -\frac{2(Ab^3f-Abc(cd+3af)+aB(2c^2d-b^2f+2acf)+c(Ab^2f+bB(cd}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)}$$

$$= -\frac{2(Ab^3f-Abc(cd+3af)+aB(2c^2d-b^2f+2acf)+c(Ab^2f+bB(cd}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)}$$

$$= -\frac{2(Ab^3f-Abc(cd+3af)+aB(2c^2d-b^2f+2acf)+c(Ab^2f+bB(cd}{3\,(b^2-4ac)\,(b^2df-(cd+af)^2)\,(a+bx+cx^2)}$$

**Mathematica [A]**
time = 12.76, size = 674, normalized size = 0.85

$$\frac{2\left(\frac{4c(-Ab^2f+bB(-cd+af)+2Ac(cd+af))(b+2cx)}{(b^2-4ac)\sqrt{a+x(b+cx)}}-\frac{3f(b^4Bdf+2c(cd+af)^2(-aB+Acx)+b^3f(-A(cd+2af)+Bcdx)+bc(cd+af)(Acd+5aAf-3Bcdx+aBfx)-b^2(B(c^2d^2+2acdf-a^2f^2)+2aAcf^2x))}{(c^2d^2+2acdf+f(-b^2d+a^2f))\sqrt{a+x(b+cx)}}+\frac{A(b^3f-bc(cd+3af)+b^2cfx-2c^2(cd+af)x)+B(2a^2cf+bc^2dx+a(2c^2d-b^2f-bcfx))}{(a+x(b+cx))^{3/2}}+\frac{3(b^2-4ac)f^{3/2}\left(\frac{(-B\sqrt{d}+A\sqrt{f})(cd+b\sqrt{d}\sqrt{f}+af)^2\tanh^{-1}\left(\frac{-2a\sqrt{f}+2c\sqrt{d}x+b(\sqrt{d}-\sqrt{f}x)}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+x(b+cx)}}\right)}{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}-\frac{(B\sqrt{d}+A\sqrt{f})(cd-b\sqrt{d}\sqrt{f}+af)^2\tanh^{-1}\left(\frac{-2(a\sqrt{f}+c\sqrt{d}x)-b(\sqrt{d}+\sqrt{f}x)}{2\sqrt{cd+b\sqrt{d}\sqrt{f}+af}\sqrt{a+x(b+cx)}}\right)}{\sqrt{cd+b\sqrt{d}\sqrt{f}+af}}\right)}{4\sqrt{d}(-b^2df+(cd+af)^2)}\right)}{3(b^2-4ac)(-b^2df+(cd+af)^2)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)^(5/2)*(d - f*x^2)),x]
```

```
[Out] (2*((4*c*(-(A*b^2*f) + b*B*(-(c*d) + a*f) + 2*A*c*(c*d + a*f))*(b + 2*c*x))
/((b^2 - 4*a*c)*Sqrt[a + x*(b + c*x)]) - (3*f*(b^4*B*d*f + 2*c*(c*d + a*f)^
2*(-(a*B) + A*c*x) + b^3*f*(-(A*(c*d + 2*a*f)) + B*c*d*x) + b*c*(c*d + a*f)
*(A*c*d + 5*a*A*f - 3*B*c*d*x + a*B*f*x) - b^2*(B*(c^2*d^2 + 2*a*c*d*f - a^
2*f^2) + 2*a*A*c*f^2*x)))/((c^2*d^2 + 2*a*c*d*f + f*(-(b^2*d) + a^2*f))*Sqr
t[a + x*(b + c*x)]) + (A*(b^3*f - b*c*(c*d + 3*a*f) + b^2*c*f*x - 2*c^2*(c*
d + a*f)*x) + B*(2*a^2*c*f + b*c^2*d*x + a*(2*c^2*d - b^2*f - b*c*f*x)))/(a
 + x*(b + c*x))^(3/2) + (3*(b^2 - 4*a*c)*f^(3/2)*(((-(B*Sqrt[d]) + A*Sqrt[f
])*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)^2*ArcTanh[(-2*a*Sqrt[f] + 2*c*Sqrt[d]*x
+ b*(Sqrt[d] - Sqrt[f]*x))/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a +
```

```
x*(b + c*x)])])/Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f] - ((B*Sqrt[d] + A*Sqrt[
f])*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^2*ArcTanh[(-2*(a*Sqrt[f] + c*Sqrt[d]*x)
 - b*(Sqrt[d] + Sqrt[f]*x))/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a +
 x*(b + c*x)])])/Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]))/(4*Sqrt[d]*(-(b^2*d*
f) + (c*d + a*f)^2))))/(3*(b^2 - 4*a*c)*(-(b^2*d*f) + (c*d + a*f)^2))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1767 vs. $2(721) = 1442$.
time = 0.14, size = 1768, normalized size = 2.22

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1768 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)^(5/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(A*f-B*(d*f)^(1/2))/(d*f)^(1/2)/f*(1/3*f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(
d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f
)^(1/2)+f*a+c*d))^(3/2)-1/2*(-2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)
*(2/3*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(
1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-
2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(3/2)+
16/3*c/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)^2*(2
*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/((x+(d*f)^(1/2)/f)^2*c+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/
2))+f/(-b*(d*f)^(1/2)+f*a+c*d)*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/
f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a
+c*d))^(1/2)-(-2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^
(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*
(-2*c*(d*f)^(1/2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*
a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d
)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c
*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)
/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))+1/2*(-A*f-B*(
d*f)^(1/2))/(d*f)^(1/2)/f*(1/3/(b*(d*f)^(1/2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)
^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(
3/2)-1/2*(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)*(2/3*(2*c*(x-(d*f)^(
1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^
(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(
1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(3/2)+16/3*c/(4*c*(b*(d*f)^(1/2)+f*a+c*d
)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)^2*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+
b*f)/f)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b
*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))+1/(b*(d*f)^(1/2)+f*a+c*d)*f*(1/(b*(d*f)^(1/
2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)
```

```
/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f
*a+c*d)*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)
+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^
(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^
(1/2)+f*a+c*d)*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a
+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d
)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)
+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(5/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(5/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)**(5/2)/(-f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^(5/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueEvaluation time: 2.96
Done
```

**Mupad** [**F**]

time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{5/2}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(5/2)),x)
```

```
[Out] int((A + B*x)/((d - f*x^2)*(a + b*x + c*x^2)^(5/2)), x)
```

## 3.10 $\int \frac{1+2x}{(-1+x^2)\sqrt{-1+x+x^2}}\,dx$

**Optimal**. Leaf size=47

$$-\frac{1}{2}\tan^{-1}\left(\frac{3+x}{2\sqrt{-1+x+x^2}}\right)+\frac{3}{2}\tanh^{-1}\left(\frac{1-3x}{2\sqrt{-1+x+x^2}}\right)$$

```
[Out] -1/2*arctan(1/2*(3+x)/(x^2+x-1)^(1/2))+3/2*arctanh(1/2*(1-3*x)/(x^2+x-1)^(1
/2))
```

**Rubi** [A]
time = 0.02, antiderivative size = 47, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 23, $\frac{\text{number of rules}}{\text{integrand size}} = 0.174$, Rules used = {1047, 738, 212, 210}

$$\frac{3}{2}\tanh^{-1}\left(\frac{1-3x}{2\sqrt{x^2+x-1}}\right)-\frac{1}{2}\text{ArcTan}\left(\frac{x+3}{2\sqrt{x^2+x-1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(1 + 2*x)/((-1 + x^2)*Sqrt[-1 + x + x^2]),x]
```

```
[Out] -1/2*ArcTan[(3 + x)/(2*Sqrt[-1 + x + x^2])] + (3*ArcTanh[(1 - 3*x)/(2*Sqrt[
-1 + x + x^2])])/2
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
```

```
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\begin{aligned}\int \frac{1+2x}{(-1+x^2)\sqrt{-1+x+x^2}}\,dx &= \frac{1}{2}\int \frac{1}{(1+x)\sqrt{-1+x+x^2}}\,dx + \frac{3}{2}\int \frac{1}{(-1+x)\sqrt{-1+x+x^2}}\,dx\\ &= -\left(3\text{Subst}\left(\int \frac{1}{4-x^2}\,dx, x, \frac{-1+3x}{\sqrt{-1+x+x^2}}\right)\right) - \text{Subst}\left(\int \frac{1}{-4-x^2}\,d\right.\\ &= \frac{1}{2}\tan^{-1}\left(\frac{-3-x}{2\sqrt{-1+x+x^2}}\right) + \frac{3}{2}\tanh^{-1}\left(\frac{1-3x}{2\sqrt{-1+x+x^2}}\right)\end{aligned}$$

**Mathematica [A]**
time = 0.11, size = 37, normalized size = 0.79

$$-\tan^{-1}\left(1+x-\sqrt{-1+x+x^2}\right) - 3\tanh^{-1}\left(1-x+\sqrt{-1+x+x^2}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(1 + 2*x)/((-1 + x^2)*Sqrt[-1 + x + x^2]),x]
[Out] -ArcTan[1 + x - Sqrt[-1 + x + x^2]] - 3*ArcTanh[1 - x + Sqrt[-1 + x + x^2]]
```

**Maple [A]**
time = 0.17, size = 46, normalized size = 0.98

| method | result |
|---|---|
| default | $-\frac{3\,\mathrm{arctanh}\left(\frac{-1+3x}{2\sqrt{(-1+x)^2-2+3x}}\right)}{2}+\frac{\arctan\left(\frac{-3-x}{2\sqrt{(1+x)^2-2-x}}\right)}{2}$ |
| trager | $-\frac{3\ln\left(-\frac{2\sqrt{x^2+x-1}-1+3x}{-1+x}\right)}{2}+\frac{\mathrm{RootOf}\left(\_Z^2+1\right)\ln\left(\frac{x\,\mathrm{RootOf}\left(\_Z^2+1\right)+2\sqrt{x^2+x-1}+3\,\mathrm{RootOf}\left(\_Z^2+1\right)}{1+x}\right)}{2}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x+1)/(x^2-1)/(x^2+x-1)^(1/2),x,method=_RETURNVERBOSE)
[Out] -3/2*arctanh(1/2*(-1+3*x)/((-1+x)^2-2+3*x)^(1/2))+1/2*arctan(1/2*(-3-x)/((1
+x)^2-2-x)^(1/2))
```

**Maxima** [A]
time = 0.55, size = 65, normalized size = 1.38

$$-\frac{1}{2}\arcsin\left(\frac{2\sqrt{5}\,x}{5\,|2\,x+2|}+\frac{6\sqrt{5}}{5\,|2\,x+2|}\right)-\frac{3}{2}\log\left(\frac{2\sqrt{x^2+x-1}}{|2\,x-2|}+\frac{2}{|2\,x-2|}+\frac{3}{2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2-1)/(x^2+x-1)^(1/2),x, algorithm="maxima")
```

```
[Out] -1/2*arcsin(2/5*sqrt(5)*x/abs(2*x + 2) + 6/5*sqrt(5)/abs(2*x + 2)) - 3/2*lo
g(2*sqrt(x^2 + x - 1)/abs(2*x - 2) + 2/abs(2*x - 2) + 3/2)
```

**Fricas** [A]
time = 0.36, size = 46, normalized size = 0.98

$$\arctan\left(-x+\sqrt{x^2+x-1}-1\right)-\frac{3}{2}\log\left(-x+\sqrt{x^2+x-1}+2\right)+\frac{3}{2}\log\left(-x+\sqrt{x^2+x-1}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2-1)/(x^2+x-1)^(1/2),x, algorithm="fricas")
```

```
[Out] arctan(-x + sqrt(x^2 + x - 1) - 1) - 3/2*log(-x + sqrt(x^2 + x - 1) + 2) +
3/2*log(-x + sqrt(x^2 + x - 1))
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{2x+1}{(x-1)\,(x+1)\,\sqrt{x^2+x-1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x**2-1)/(x**2+x-1)**(1/2),x)
```

```
[Out] Integral((2*x + 1)/((x - 1)*(x + 1)*sqrt(x**2 + x - 1)), x)
```

**Giac** [A]
time = 4.31, size = 48, normalized size = 1.02

$$\arctan\left(-x+\sqrt{x^2+x-1}-1\right)-\frac{3}{2}\log\left(\left|-x+\sqrt{x^2+x-1}+2\right|\right)+\frac{3}{2}\log\left(\left|-x+\sqrt{x^2+x-1}\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2-1)/(x^2+x-1)^(1/2),x, algorithm="giac")
```

```
[Out] arctan(-x + sqrt(x^2 + x - 1) - 1) - 3/2*log(abs(-x + sqrt(x^2 + x - 1) + 2
)) + 3/2*log(abs(-x + sqrt(x^2 + x - 1)))
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.02

$$\int \frac{2\,x+1}{\left(x^2-1\right)\,\sqrt{x^2+x-1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x + 1)/((x^2 - 1)*(x + x^2 - 1)^(1/2)),x)
[Out] int((2*x + 1)/((x^2 - 1)*(x + x^2 - 1)^(1/2)), x)
```

# 3.11 $\int \frac{1+2x}{(1+x^2)\sqrt{-1+x+x^2}}\,dx$

**Optimal**. Leaf size=117

$$-\sqrt{\frac{1}{2}\left(2+\sqrt{5}\right)}\tan^{-1}\left(\frac{5+2\sqrt{5}-\sqrt{5}\,x}{\sqrt{10\left(2+\sqrt{5}\right)}\sqrt{-1+x+x^2}}\right)+\sqrt{\frac{1}{2}\left(-2+\sqrt{5}\right)}\tanh^{-1}\left(\frac{5-2\sqrt{\phantom{5}}}{\sqrt{10\left(-2+\sqrt{5}\right.}}\right.$$

[Out] 1/2*arctanh((5-2*5^(1/2)+x*5^(1/2))/(x^2+x-1)^(1/2)/(-20+10*5^(1/2))^(1/2))
*(-4+2*5^(1/2))^(1/2)-1/2*arctan((5+2*5^(1/2)-x*5^(1/2))/(x^2+x-1)^(1/2)/(2
0+10*5^(1/2))^(1/2))*(4+2*5^(1/2))^(1/2)

**Rubi** [A]
time = 0.11, antiderivative size = 117, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 23, $\frac{\text{number of rules}}{\text{integrand size}} = 0.174$,
Rules used = {1050, 1044, 213, 209}

$$\sqrt{\frac{1}{2}\left(\sqrt{5}-2\right)}\tanh^{-1}\left(\frac{\sqrt{5}\,x-2\sqrt{5}+5}{\sqrt{10\left(\sqrt{5}-2\right)}\sqrt{x^2+x-1}}\right)-\sqrt{\frac{1}{2}\left(2+\sqrt{5}\right)}\text{ArcTan}\left(\frac{-\sqrt{5}\,x+2\sqrt{5}+5}{\sqrt{10\left(2+\sqrt{5}\right)}\sqrt{x^2+x-1}}\right)$$

Antiderivative was successfully verified.

[In] Int[(1 + 2*x)/((1 + x^2)*Sqrt[-1 + x + x^2]),x]

[Out] -(Sqrt[(2 + Sqrt[5])/2]*ArcTan[(5 + 2*Sqrt[5] - Sqrt[5]*x)/(Sqrt[10*(2 + Sq
rt[5])]*Sqrt[-1 + x + x^2])]) + Sqrt[(-2 + Sqrt[5])/2]*ArcTanh[(5 - 2*Sqrt[
5] + Sqrt[5]*x)/(Sqrt[10*(-2 + Sqrt[5])]*Sqrt[-1 + x + x^2])]

Rule 209

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 213

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

Rule 1044

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f
_.)*(x_)^2]), x_Symbol] :> Dist[-2*a*g*h, Subst[Int[1/Simp[2*a^2*g*h*c + a*
```

```
e*x^2, x], x], x, Simp[a*h - g*c*x, x]/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[
{a, c, d, e, f, g, h}, x] && EqQ[a*h^2*e + 2*g*h*(c*d - a*f) - g^2*c*e, 0]
```

Rule 1050

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 + a*c*e^2, 2]}, Dist
[1/(2*q), Int[Simp[(-a)*h*e - g*(c*d - a*f - q) + (h*(c*d - a*f + q) - g*c*
e)*x, x]/((a + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[Si
mp[(-a)*h*e - g*(c*d - a*f + q) + (h*(c*d - a*f - q) - g*c*e)*x, x]/((a + c
*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f, g, h}, x] &&
 NeQ[e^2 - 4*d*f, 0] && NegQ[(-a)*c]
```

Rubi steps

$$\int \frac{1+2x}{(1+x^2)\sqrt{-1+x+x^2}}\,dx = -\frac{\int \frac{-\sqrt{5}+\left(-5-2\sqrt{5}\right)x}{(1+x^2)\sqrt{-1+x+x^2}}\,dx}{2\sqrt{5}} + \frac{\int \frac{\sqrt{5}+\left(-5+2\sqrt{5}\right)x}{(1+x^2)\sqrt{-1+x+x^2}}\,dx}{2\sqrt{5}}$$

$$= -\left(\left(-5+2\sqrt{5}\right)\text{Subst}\left(\int \frac{1}{10\left(2-\sqrt{5}\right)+x^2}\,dx, x, \frac{-5+2\sqrt{5}-\sqrt{5}}{\sqrt{-1+x+x^2}}\right.\right.$$

$$= -\sqrt{\frac{1}{2}\left(2+\sqrt{5}\right)}\tan^{-1}\left(\frac{5+2\sqrt{5}-\sqrt{5}x}{\sqrt{10\left(2+\sqrt{5}\right)}\sqrt{-1+x+x^2}}\right) + \sqrt{\frac{1}{2}\left(-2\right.}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.12, size = 106, normalized size = 0.91

$$\frac{1}{2}\text{RootSum}\left[2-4\#1+6\#1^2+\#1^4\&, \frac{3\log\left(-x+\sqrt{-1+x+x^2}-\#1\right)-2\log\left(-x+\sqrt{-1+x+x^2}-\#1\right)\#1+2\log\left(-x+\sqrt{-1+x+x^2}-\#1\right)\#1^2}{-1+3\#1+\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(1 + 2*x)/((1 + x^2)*Sqrt[-1 + x + x^2]),x]
```

```
[Out] RootSum[2 - 4*#1 + 6*#1^2 + #1^4 & , (3*Log[-x + Sqrt[-1 + x + x^2] - #1] -
 2*Log[-x + Sqrt[-1 + x + x^2] - #1]*#1 + 2*Log[-x + Sqrt[-1 + x + x^2] - #
1]*#1^2)/(-1 + 3*#1 + #1^3) & ]/2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 636 vs. 2(86) = 172.
time = 0.55, size = 637, normalized size = 5.44

| method | result |
|---|---|
| trager | $-\operatorname{RootOf}\left(\operatorname{RootOf}\left(16\_Z^4+16\_Z^2+5\right)^2+\_Z^2+1\right)\ln\left(\frac{4\operatorname{RootOf}\left(\operatorname{RootOf}\left(16\_Z^4+16\_Z^2+5\right)^2+\_Z^2\right.}{}\right.$ |
| default | $\frac{\sqrt{\frac{10\left(-\sqrt{5}-2+x\right)^2}{\left(-\sqrt{5}+2-x\right)^2}-\frac{5\sqrt{5}\left(-\sqrt{5}-2+x\right)^2}{\left(-\sqrt{5}+2-x\right)^2}+10+5\sqrt{5}}\,\sqrt{5}\left(\arctan\left(\frac{\sqrt{5}\sqrt{\left(-2+\sqrt{5}\right)\left(-\frac{\left(-\sqrt{5}\right.}{\left(-\sqrt{5}\right.}\right.}}{}\right.\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x+1)/(x^2+1)/(x^2+x-1)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] (10*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2-5*5^(1/2)*(-5^(1/2)-2+x)^2/(-5^(1/2)+
2-x)^2+10+5*5^(1/2))^(1/2)*5^(1/2)*(arctan(1/5*5^(1/2)*((-2+5^(1/2))*(-(-5^
(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+4*5^(1/2)+9))^(1/2)*(20+10*5^(1/2))^(1/2)*(5^
(1/2)*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+2*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2
-5^(1/2)+2)*(-5^(1/2)-2+x)/(-5^(1/2)+2-x)*(-2+5^(1/2))/((-5^(1/2)-2+x)^4/(-
5^(1/2)+2-x)^4-18*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+1))*5^(1/2)+arctanh((10
*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2-5*5^(1/2)*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x
)^2+10+5*5^(1/2))^(1/2)/(20+10*5^(1/2))^(1/2))+2*arctan(1/5*5^(1/2)*((-2+5^
(1/2))*(-(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+4*5^(1/2)+9))^(1/2)*(20+10*5^(1/
2))^(1/2)*(5^(1/2)*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+2*(-5^(1/2)-2+x)^2/(-5
^(1/2)+2-x)^2-5^(1/2)+2)*(-5^(1/2)-2+x)/(-5^(1/2)+2-x)*(-2+5^(1/2))/((-5^(1
/2)-2+x)^4/(-5^(1/2)+2-x)^4-18*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2+1)))/(-5*(
5^(1/2)*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)^2-2*(-5^(1/2)-2+x)^2/(-5^(1/2)+2-x)
^2-5^(1/2)-2)/(1+(-5^(1/2)-2+x)/(-5^(1/2)+2-x))^2)^(1/2)/(1+(-5^(1/2)-2+x)/
(-5^(1/2)+2-x))/(20+10*5^(1/2))^(1/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+1)/(x^2+x-1)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((2*x + 1)/(sqrt(x^2 + x - 1)*(x^2 + 1)), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 758 vs. $2(86) = 172$.
time = 0.37, size = 758, normalized size = 6.48

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+1)/(x^2+x-1)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/20*5^(1/4)*sqrt(4*sqrt(5) + 10)*(2*sqrt(5) - 5)*log(2*x^2 - 2*sqrt(x^2 +
x - 1)*x + 1/5*(5^(1/4)*sqrt(x^2 + x - 1)*(2*sqrt(5) - 5) - 5^(1/4)*(sqrt(5
)*(2*x + 1) - 5*x))*sqrt(4*sqrt(5) + 10) + x + sqrt(5)) - 1/20*5^(1/4)*sqrt
(4*sqrt(5) + 10)*(2*sqrt(5) - 5)*log(2*x^2 - 2*sqrt(x^2 + x - 1)*x - 1/5*(5
^(1/4)*sqrt(x^2 + x - 1)*(2*sqrt(5) - 5) - 5^(1/4)*(sqrt(5)*(2*x + 1) - 5*x
))*sqrt(4*sqrt(5) + 10) + x + sqrt(5)) - 1/5*5^(3/4)*sqrt(4*sqrt(5) + 10)*a
rctan(2/55*sqrt(5)*(sqrt(5)*(2*x - 1) + 3*x + 4) + 1/275*sqrt(10*x^2 - 10*s
qrt(x^2 + x - 1)*x + (5^(1/4)*sqrt(x^2 + x - 1)*(2*sqrt(5) - 5) - 5^(1/4)*(
sqrt(5)*(2*x + 1) - 5*x))*sqrt(4*sqrt(5) + 10) + 5*x + 5*sqrt(5))*((5^(3/4)
*(2*sqrt(5) + 3) + 2*5^(1/4)*(4*sqrt(5) - 5))*sqrt(4*sqrt(5) + 10) + 2*sqrt
(5)*(3*sqrt(5) + 10) - 20*sqrt(5) + 80) - 2/55*sqrt(x^2 + x - 1)*(sqrt(5)*(
2*sqrt(5) + 3) + 8*sqrt(5) - 10) + 1/55*sqrt(5)*(16*x + 3) + 1/275*(5^(3/4)
*(sqrt(5)*(3*x + 4) + 10*x - 5) - sqrt(x^2 + x - 1)*(5^(3/4)*(3*sqrt(5) + 1
0) - 10*5^(1/4)*(sqrt(5) - 4)) - 10*5^(1/4)*(sqrt(5)*(x - 6) - 4*x + 13))*s
qrt(4*sqrt(5) + 10) - 4/11*x + 2/11) - 1/5*5^(3/4)*sqrt(4*sqrt(5) + 10)*arc
tan(-2/55*sqrt(5)*(sqrt(5)*(2*x - 1) + 3*x + 4) + 1/275*sqrt(10*x^2 - 10*sq
rt(x^2 + x - 1)*x - (5^(1/4)*sqrt(x^2 + x - 1)*(2*sqrt(5) - 5) - 5^(1/4)*(s
qrt(5)*(2*x + 1) - 5*x))*sqrt(4*sqrt(5) + 10) + 5*x + 5*sqrt(5))*((5^(3/4)*
(2*sqrt(5) + 3) + 2*5^(1/4)*(4*sqrt(5) - 5))*sqrt(4*sqrt(5) + 10) - 2*sqrt(
5)*(3*sqrt(5) + 10) + 20*sqrt(5) - 80) + 2/55*sqrt(x^2 + x - 1)*(sqrt(5)*(2
*sqrt(5) + 3) + 8*sqrt(5) - 10) - 1/55*sqrt(5)*(16*x + 3) + 1/275*(5^(3/4)*
(sqrt(5)*(3*x + 4) + 10*x - 5) - sqrt(x^2 + x - 1)*(5^(3/4)*(3*sqrt(5) + 10
) - 10*5^(1/4)*(sqrt(5) - 4)) - 10*5^(1/4)*(sqrt(5)*(x - 6) - 4*x + 13))*sq
rt(4*sqrt(5) + 10) + 4/11*x - 2/11)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{2x+1}{(x^2+1)\sqrt{x^2+x-1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x**2+1)/(x**2+x-1)**(1/2),x)
```

```
[Out] Integral((2*x + 1)/((x**2 + 1)*sqrt(x**2 + x - 1)), x)
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 457 vs. 2(86) = 172.
time = 5.18, size = 457, normalized size = 3.91

$$\frac{1}{4}\sqrt{2\sqrt{5}-4}\log\left(16\left(15\sqrt{5}\left(x-\sqrt{x^2+x-1}\right)+33x+5\sqrt{5}-33\sqrt{x^2+x-1}+2\sqrt{5\sqrt{5}+11}+11\right)^2+16\left(5\sqrt{5}\left(x-\sqrt{x^2+x-1}\right)+11x-5\sqrt{5}\sqrt{5\sqrt{5}+11}-15\sqrt{5}-11\sqrt{x^2+x-1}-11\sqrt{5\sqrt{5}+11}-33\right)^2\right)-\frac{1}{4}\sqrt{2\sqrt{5}-4}\log\left(16\left(15\sqrt{5}\left(x-\sqrt{x^2+x-1}\right)+33x+5\sqrt{5}-33\sqrt{x^2+x-1}-2\sqrt{5\sqrt{5}+11}+11\right)^2+16\left(5\sqrt{5}\left(x-\sqrt{x^2+x-1}\right)+11x+5\sqrt{5}\sqrt{5\sqrt{5}+11}-15\sqrt{5}-11\sqrt{x^2+x-1}+11\sqrt{5\sqrt{5}+11}-33\right)^2\right)+\frac{\sqrt{2\sqrt{5}-4}\left(\arctan(3)+\arctan\left(\frac{1}{10}\left(x-\sqrt{x^2+x-1}\right)\left(\sqrt{5}\sqrt{5\sqrt{5}+11}+4\sqrt{5}-5\sqrt{5\sqrt{5}+11}\right)-\frac{7}{10}\sqrt{5}\sqrt{5\sqrt{5}+11}+\frac{1}{5}\sqrt{5}+\frac{3}{2}\sqrt{5\sqrt{5}+11}\right)\right)}{2\left(\sqrt{5}-2\right)}-\frac{\sqrt{2\sqrt{5}-4}\left(\arctan(3)+\arctan\left(-\frac{1}{10}\left(x-\sqrt{x^2+x-1}\right)\left(\sqrt{5}\sqrt{5\sqrt{5}+11}-4\sqrt{5}-5\sqrt{5\sqrt{5}+11}\right)+\frac{7}{10}\sqrt{5}\sqrt{5\sqrt{5}+11}+\frac{1}{5}\sqrt{5}-\frac{3}{2}\sqrt{5\sqrt{5}+11}\right)\right)}{2\left(\sqrt{5}-2\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+1)/(x^2+x-1)^(1/2),x, algorithm="giac")
```

```
[Out] 1/4*sqrt(2*sqrt(5) - 4)*log(16*(15*sqrt(5)*(x - sqrt(x^2 + x - 1)) + 33*x +
 5*sqrt(5) - 33*sqrt(x^2 + x - 1) + 2*sqrt(5*sqrt(5) + 11) + 11)^2 + 16*(5*
sqrt(5)*(x - sqrt(x^2 + x - 1)) + 11*x - 5*sqrt(5)*sqrt(5*sqrt(5) + 11) - 1
5*sqrt(5) - 11*sqrt(x^2 + x - 1) - 11*sqrt(5*sqrt(5) + 11) - 33)^2) - 1/4*s
qrt(2*sqrt(5) - 4)*log(16*(15*sqrt(5)*(x - sqrt(x^2 + x - 1)) + 33*x + 5*sq
rt(5) - 33*sqrt(x^2 + x - 1) - 2*sqrt(5*sqrt(5) + 11) + 11)^2 + 16*(5*sqrt(
5)*(x - sqrt(x^2 + x - 1)) + 11*x + 5*sqrt(5)*sqrt(5*sqrt(5) + 11) - 15*sqr
t(5) - 11*sqrt(x^2 + x - 1) + 11*sqrt(5*sqrt(5) + 11) - 33)^2) + 1/2*sqrt(2
*sqrt(5) - 4)*(arctan(3) + arctan(1/10*(x - sqrt(x^2 + x - 1))*(sqrt(5)*sqr
t(5*sqrt(5) + 11) + 4*sqrt(5) - 5*sqrt(5*sqrt(5) + 11)) - 7/10*sqrt(5)*sqrt
(5*sqrt(5) + 11) + 1/5*sqrt(5) + 3/2*sqrt(5*sqrt(5) + 11)))/(sqrt(5) - 2) -
 1/2*sqrt(2*sqrt(5) - 4)*(arctan(3) + arctan(-1/10*(x - sqrt(x^2 + x - 1))*
(sqrt(5)*sqrt(5*sqrt(5) + 11) - 4*sqrt(5) - 5*sqrt(5*sqrt(5) + 11)) + 7/10*
sqrt(5)*sqrt(5*sqrt(5) + 11) + 1/5*sqrt(5) - 3/2*sqrt(5*sqrt(5) + 11)))/(sq
rt(5) - 2)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{2\,x+1}{\left(x^2+1\right)\,\sqrt{x^2+x-1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x + 1)/((x^2 + 1)*(x + x^2 - 1)^(1/2)),x)
```

```
[Out] int((2*x + 1)/((x^2 + 1)*(x + x^2 - 1)^(1/2)), x)
```

## 3.12 $\int \frac{a-c+bx}{(1+x^2)\sqrt{a+bx+cx^2}}\,dx$

Optimal. Leaf size=484

$$-\frac{\sqrt{a^2+b^2+c\left(c-\sqrt{a^2+b^2-2ac+c^2}\right)-a\left(2c-\sqrt{a^2+b^2-2ac+c^2}\right)}\,\tan^{-1}\left(\frac{\quad}{\sqrt{2}\,\sqrt[4]{a^2+b^2-2ac}}\right.}{\sqrt{2}\,\sqrt[4]{a^2+b^2-}}$$

```
[Out] -1/2*arctan(1/2*(b*(a^2-2*a*c+b^2+c^2)^(1/2)-x*(b^2+(a-c)*(a-c+(a^2-2*a*c+b
^2+c^2)^(1/2))))/(a^2-2*a*c+b^2+c^2)^(1/4)*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(a^2
+b^2+c*(c-(a^2-2*a*c+b^2+c^2)^(1/2))-a*(2*c-(a^2-2*a*c+b^2+c^2)^(1/2)))^(1/
2))*(a^2+b^2+c*(c-(a^2-2*a*c+b^2+c^2)^(1/2))-a*(2*c-(a^2-2*a*c+b^2+c^2)^(1/
2)))^(1/2)/(a^2-2*a*c+b^2+c^2)^(1/4)*2^(1/2)-1/2*arctanh(1/2*(x*(b^2+(a-c)*
(a-c-(a^2-2*a*c+b^2+c^2)^(1/2)))+b*(a^2-2*a*c+b^2+c^2)^(1/2))/(a^2-2*a*c+b^
2+c^2)^(1/4)*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(a^2+b^2+c*(c+(a^2-2*a*c+b^2+c^2)^
(1/2))-a*(2*c+(a^2-2*a*c+b^2+c^2)^(1/2)))^(1/2))*(a^2+b^2+c*(c+(a^2-2*a*c+b
^2+c^2)^(1/2))-a*(2*c+(a^2-2*a*c+b^2+c^2)^(1/2)))^(1/2)/(a^2-2*a*c+b^2+c^2)
^(1/4)*2^(1/2)
```

Rubi [A]
time = 22.90, antiderivative size = 484, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {1050, 1044, 214, 211}

$$-\frac{\sqrt{-a\left(2c-\sqrt{a^2-2ac+b^2+c^2}\right)+c\left(c-\sqrt{a^2-2ac+b^2+c^2}\right)+a^2+b^2}\,\text{ArcTan}\left(\frac{b\sqrt{a^2-2ac+b^2+c^2}-x\left((a-c)\left(\sqrt{a^2-2ac+b^2+c^2}+a-c\right)+b^2\right)}{\sqrt{2}\,\sqrt[4]{a^2-2ac+b^2+c^2}\,\sqrt{-a\left(2c-\sqrt{a^2-2ac+b^2+c^2}\right)+c\left(c-\sqrt{a^2-2ac+b^2+c^2}\right)+a^2+b^2}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{2}\,\sqrt[4]{a^2-2ac+b^2+c^2}}-\frac{\sqrt{-a\left(\sqrt{a^2-2ac+b^2+c^2}+2c\right)+c\left(\sqrt{a^2-2ac+b^2+c^2}+c\right)+a^2+b^2}\,\tanh^{-1}\left(\frac{x\left((a-c)\left(-\sqrt{a^2-2ac+b^2+c^2}+a-c\right)+b^2\right)+b\sqrt{a^2-2ac+b^2+c^2}}{\sqrt{2}\,\sqrt[4]{a^2-2ac+b^2+c^2}\,\sqrt{-a\left(\sqrt{a^2-2ac+b^2+c^2}+2c\right)+c\left(\sqrt{a^2-2ac+b^2+c^2}+c\right)+a^2+b^2}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{2}\,\sqrt[4]{a^2-2ac+b^2+c^2}}$$

Antiderivative was successfully verified.

```
[In] Int[(a - c + b*x)/((1 + x^2)*Sqrt[a + b*x + c*x^2]),x]
```

```
[Out] -((Sqrt[a^2 + b^2 + c*(c - Sqrt[a^2 + b^2 - 2*a*c + c^2]) - a*(2*c - Sqrt[a
^2 + b^2 - 2*a*c + c^2])]*ArcTan[(b*Sqrt[a^2 + b^2 - 2*a*c + c^2] - (b^2 +
(a - c)*(a - c + Sqrt[a^2 + b^2 - 2*a*c + c^2]))*x)/(Sqrt[2]*(a^2 + b^2 - 2
*a*c + c^2)^(1/4)*Sqrt[a^2 + b^2 + c*(c - Sqrt[a^2 + b^2 - 2*a*c + c^2]) -
a*(2*c - Sqrt[a^2 + b^2 - 2*a*c + c^2])]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*
(a^2 + b^2 - 2*a*c + c^2)^(1/4))) - (Sqrt[a^2 + b^2 + c*(c + Sqrt[a^2 + b^2
 - 2*a*c + c^2]) - a*(2*c + Sqrt[a^2 + b^2 - 2*a*c + c^2])]*ArcTanh[(b*Sqrt
[a^2 + b^2 - 2*a*c + c^2] + (b^2 + (a - c)*(a - c - Sqrt[a^2 + b^2 - 2*a*c
+ c^2]))*x)/(Sqrt[2]*(a^2 + b^2 - 2*a*c + c^2)^(1/4)*Sqrt[a^2 + b^2 + c*(c
+ Sqrt[a^2 + b^2 - 2*a*c + c^2]) - a*(2*c + Sqrt[a^2 + b^2 - 2*a*c + c^2])]
*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*(a^2 + b^2 - 2*a*c + c^2)^(1/4))
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 1044

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f
_.)*(x_)^2]), x_Symbol] :> Dist[-2*a*g*h, Subst[Int[1/Simp[2*a^2*g*h*c + a*
e*x^2, x], x], x, Simp[a*h - g*c*x, x]/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[
{a, c, d, e, f, g, h}, x] && EqQ[a*h^2*e + 2*g*h*(c*d - a*f) - g^2*c*e, 0]
```

Rule 1050

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 + a*c*e^2, 2]}, Dist
[1/(2*q), Int[Simp[(-a)*h*e - g*(c*d - a*f - q) + (h*(c*d - a*f + q) - g*c*
e)*x, x]/((a + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[Si
mp[(-a)*h*e - g*(c*d - a*f + q) + (h*(c*d - a*f - q) - g*c*e)*x, x]/((a + c
*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f, g, h}, x] &&
 NeQ[e^2 - 4*d*f, 0] && NegQ[(-a)*c]
```

Rubi steps

$$\int \frac{a-c+bx}{(1+x^2)\sqrt{a+bx+cx^2}}\,dx = -\frac{\int \frac{-b^2-(a-c)\left(a-c+\sqrt{a^2+b^2-2ac+c^2}\right)-b\sqrt{a^2+b^2-2ac+c^2}\,x}{(1+x^2)\sqrt{a+bx+cx^2}}\,dx}{2\sqrt{a^2+b^2-2ac+c^2}} + \frac{\int -}{}$$

$$= \left(b\left(b^2+(a-c)\left(a-c-\sqrt{a^2+b^2-2ac+c^2}\right)\right)\right)\text{Subst}\left(\int \frac{}{-2b\sqrt{a^2+b}}\right.$$

$$= -\frac{\sqrt{a^2+b^2+c\left(c-\sqrt{a^2+b^2-2ac+c^2}\right)-a\left(2c-\sqrt{a^2+b^2-2ac+c}\right.}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 0.38, size = 210, normalized size = 0.43

$$\frac{1}{2}\text{RootSum}\left[a^2+b^2-4b\sqrt{c}\,\#1-2a\#1^2+4c\#1^2+\#1^4\&,\frac{bc\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+2a\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2c^{3/2}\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}+a\#1-2c\#1-\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(a - c + b*x)/((1 + x^2)*Sqrt[a + b*x + c*x^2]),x]
```

```
[Out] RootSum[a^2 + b^2 - 4*b*Sqrt[c]*#1 - 2*a*#1^2 + 4*c*#1^2 + #1^4 & , (b*c*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*a*Sqrt[c]*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*c^(3/2)*Log[-(Sqrt[c]*x) + Sqrt[a + b
*x + c*x^2] - #1]*#1 - b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^
2)/(b*Sqrt[c] + a*#1 - 2*c*#1 - #1^3) & ]/2
```

**Maple** [**B**] result has leaf size over 500,000. Avoiding possible recursion issues.
time = 4.34, size = 6871419, normalized size = 14197.15

| method | result | size |
|---|---|---|
| default | Expression too large to display | 6871419 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((b*x+a-c)/(x^2+1)/(c*x^2+b*x+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] result too large to display
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((b*x+a-c)/(x^2+1)/(c*x^2+b*x+a)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((b*x + a - c)/(sqrt(c*x^2 + b*x + a)*(x^2 + 1)), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((b*x+a-c)/(x^2+1)/(c*x^2+b*x+a)^(1/2),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{a + bx - c}{\left(x^{2} + 1\right) \sqrt{a + bx + cx^{2}}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((b*x+a-c)/(x**2+1)/(c*x**2+b*x+a)**(1/2),x)
[Out] Integral((a + b*x - c)/((x**2 + 1)*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F]**
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((b*x+a-c)/(x^2+1)/(c*x^2+b*x+a)^(1/2),x, algorithm="giac")
[Out] integrate((b*x + a - c)/(sqrt(c*x^2 + b*x + a)*(x^2 + 1)), x)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{a-c+b\,x}{\left(x^2+1\right)\,\sqrt{c\,x^2+b\,x+a}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a - c + b*x)/((x^2 + 1)*(a + b*x + c*x^2)^(1/2)),x)
[Out] int((a - c + b*x)/((x^2 + 1)*(a + b*x + c*x^2)^(1/2)), x)
```

## 3.13 $\int \frac{(A+Bx)(a+bx+cx^2)}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=184

$$-\frac{(Bce-bBf-Acf)x}{f^2}+\frac{Bcx^2}{2f}-\frac{(Af(ce^2-2cdf-bef+2af^2)+B(f(be^2-2bdf-aef)-c(e^3-3def)))}{f^3\sqrt{e^2-4df}}$$

```
[Out] -(-A*c*f-B*b*f+B*c*e)*x/f^2+1/2*B*c*x^2/f-1/2*(A*f*(-b*f+c*e)-B*(a*f^2-b*e*
f-c*d*f+c*e^2))*ln(f*x^2+e*x+d)/f^3-(A*f*(2*a*f^2-b*e*f-2*c*d*f+c*e^2)+B*(f
*(-a*e*f-2*b*d*f+b*e^2)-c*(-3*d*e*f+e^3)))*arctanh((2*f*x+e)/(-4*d*f+e^2)^(
1/2))/f^3/(-4*d*f+e^2)^(1/2)
```

**Rubi** [A]
time = 0.22, antiderivative size = 182, normalized size of antiderivative = 0.99, number of steps used = 6, number of rules used = 5, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {1642, 648, 632, 212, 642}

$$-\frac{\log(d+ex+fx^2)(Bf(be-af)+Af(ce-bf)-Bc(e^2-df))}{2f^3}-\frac{\tanh^{-1}\left(\frac{e+2fx}{\sqrt{e^2-4df}}\right)(Af(2af^2-bef-2cdf+ce^2)+Bf(-aef-2bdf+be^2)-Bc(e^3-3def))}{f^3\sqrt{e^2-4df}}-\frac{x(-Acf-bBf+Bce)}{f^2}+\frac{Bcx^2}{2f}$$

Antiderivative was successfully verified.

```
[In] Int[((A + B*x)*(a + b*x + c*x^2))/(d + e*x + f*x^2),x]
```

```
[Out] -(((B*c*e - b*B*f - A*c*f)*x)/f^2) + (B*c*x^2)/(2*f) - ((B*f*(b*e^2 - 2*b*d
*f - a*e*f) - B*c*(e^3 - 3*d*e*f) + A*f*(c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2)
)*ArcTanh[(e + 2*f*x)/Sqrt[e^2 - 4*d*f]])/(f^3*Sqrt[e^2 - 4*d*f]) - ((B*f*(
b*e - a*f) + A*f*(c*e - b*f) - B*c*(e^2 - d*f))*Log[d + e*x + f*x^2])/(2*f^
3)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
```

```
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 1642

```
Int[(Pq_)*((d_.) + (e_.)*(x_))^(m_.)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Int[ExpandIntegrand[(d + e*x)^m*Pq*(a + b*x + c*x^2)^p, x
], x] /; FreeQ[{a, b, c, d, e, m}, x] && PolyQ[Pq, x] && IGtQ[p, -2]
```

Rubi steps

$$\int \frac{(A+Bx)\left(a+bx+cx^2\right)}{d+ex+fx^2}\,dx = \int \left(-\frac{Bce-bBf-Acf}{f^2}+\frac{Bcx}{f}+\frac{-Af(cd-af)+Bd(ce-bf)-(Bf(\cdots}{f^2\left(d+\cdots\right.}\right.$$

$$= -\frac{(Bce-bBf-Acf)x}{f^2}+\frac{Bcx^2}{2f}+\frac{\int \frac{-Af(cd-af)+Bd(ce-bf)-(Bf(be-af)+Af(ce-\cdots}{d+ex+fx^2}}{f^2}$$

$$= -\frac{(Bce-bBf-Acf)x}{f^2}+\frac{Bcx^2}{2f}+\frac{(-Bf(be-af)-Af(ce-bf)+Bc(e\cdots}{2f^3}$$

$$= -\frac{(Bce-bBf-Acf)x}{f^2}+\frac{Bcx^2}{2f}-\frac{(Bf(be-af)+Af(ce-bf)-Bc(e^2\cdots}{2f^3}$$

$$= -\frac{(Bce-bBf-Acf)x}{f^2}+\frac{Bcx^2}{2f}-\frac{(Bf(be^2-2bdf-aef)-Bc(e^3-3de\cdots}{\cdots}$$

**Mathematica [A]**
time = 0.15, size = 175, normalized size = 0.95

$$\frac{2f(-Bce+bBf+Acf)x+Bcf^2x^2-\frac{2\left(Bf\left(-be^2+2bdf+aef\right)+Bc\left(e^3-3def\right)+Af\left(-ce^2+2cdf+bef-2af^2\right)\right)\tan^{-1}\left(\frac{e+2fx}{\sqrt{-e^2+4df}}\right)}{\sqrt{-e^2+4df}}+\left(Bf(-be+af)+Af(-ce+bf)+Bc\left(e^2-df\right)\right)\log(d+x(e+fx))}{2f^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2))/(d + e*x + f*x^2),x]
```

```
[Out] (2*f*(-(B*c*e) + b*B*f + A*c*f)*x + B*c*f^2*x^2 - (2*(B*f*(-(b*e^2) + 2*b*d
*f + a*e*f) + B*c*(e^3 - 3*d*e*f) + A*f*(-(c*e^2) + 2*c*d*f + b*e*f - 2*a*f
```

```
^2))*ArcTan[(e + 2*f*x)/Sqrt[-e^2 + 4*d*f]])/Sqrt[-e^2 + 4*d*f] + (B*f*(-(b
*e) + a*f) + A*f*(-(c*e) + b*f) + B*c*(e^2 - d*f))*Log[d + x*(e + f*x)])/(2
*f^3)
```

**Maple [A]**
time = 0.21, size = 190, normalized size = 1.03

| method | result |
|---|---|
| default | $\frac{\frac{1}{2}Bcx^2f+Acfx+Bbfx-Bcex}{f^2}+\frac{\frac{\left(Abf^2-Acef+Baf^2-Bbef-Bcdf+Bce^2\right)\ln\left(fx^2+ex+d\right)}{2f}+\frac{2\left(Aaf^2-Acdf-Bbdf+Bcde-\frac{\left(Abf^2-Ace\right.}{}\right.}{}}{f^2}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f^2*(1/2*B*c*x^2*f+A*c*f*x+B*b*f*x-B*c*e*x)+1/f^2*(1/2*(A*b*f^2-A*c*e*f+B
*a*f^2-B*b*e*f-B*c*d*f+B*c*e^2)/f*ln(f*x^2+e*x+d)+2*(A*a*f^2-A*c*d*f-B*b*d*
f+B*c*d*e-1/2*(A*b*f^2-A*c*e*f+B*a*f^2-B*b*e*f-B*c*d*f+B*c*e^2)*e/f)/(4*d*f
-e^2)^(1/2)*arctan((2*f*x+e)/(4*d*f-e^2)^(1/2)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas [A]**
time = 0.43, size = 584, normalized size = 3.17

[illegible]

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] [1/2*(4*B*c*d*f^3*x^2 - 8*B*c*d*f^2*x*e + 8*(B*b + A*c)*d*f^3*x + 2*B*c*f*x
*e^3 + (2*A*a*f^3 - 2*(B*b + A*c)*d*f^2 - B*c*e^3 + (B*b + A*c)*f*e^2 + (3*
B*c*d*f - (B*a + A*b)*f^2)*e)*sqrt(-4*d*f + e^2)*log((2*f^2*x^2 + 2*f*x*e -
```

```
 2*d*f + sqrt(-4*d*f + e^2)*(2*f*x + e) + e^2)/(f*x^2 + x*e + d)) - (B*c*f^
2*x^2 + 2*(B*b + A*c)*f^2*x)*e^2 - (4*B*c*d^2*f^2 - 4*(B*a + A*b)*d*f^3 + 4
*(B*b + A*c)*d*f^2*e + B*c*e^4 - (B*b + A*c)*f*e^3 - (5*B*c*d*f - (B*a + A*
b)*f^2)*e^2)*log(f*x^2 + x*e + d))/(4*d*f^4 - f^3*e^2), 1/2*(4*B*c*d*f^3*x^
2 - 8*B*c*d*f^2*x*e + 8*(B*b + A*c)*d*f^3*x + 2*B*c*f*x*e^3 - 2*(2*A*a*f^3
- 2*(B*b + A*c)*d*f^2 - B*c*e^3 + (B*b + A*c)*f*e^2 + (3*B*c*d*f - (B*a + A
*b)*f^2)*e)*sqrt(4*d*f - e^2)*arctan(-(2*f*x + e)/sqrt(4*d*f - e^2)) - (B*c
*f^2*x^2 + 2*(B*b + A*c)*f^2*x)*e^2 - (4*B*c*d^2*f^2 - 4*(B*a + A*b)*d*f^3
+ 4*(B*b + A*c)*d*f^2*e + B*c*e^4 - (B*b + A*c)*f*e^3 - (5*B*c*d*f - (B*a +
 A*b)*f^2)*e^2)*log(f*x^2 + x*e + d))/(4*d*f^4 - f^3*e^2)]
```

**Sympy** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 1260 vs. $2(175) = 350$.
time = 7.62, size = 1260, normalized size = 6.85

$$\frac{Bcx^2}{2f} + x\left(\frac{Ac}{f} + \frac{Bb}{f} - \frac{Bce}{f^2}\right) + \left(-\frac{\sqrt{-4df+e^2}\left(-2Aaf^3 + Abef^2 + 2Acdf^2 - Ace^2f + Baef^2 + 2Bbdf^2 - Bbe^2f - 3Bcdef + Bce^3\right)}{2f^3\cdot(4df-e^2)} + \frac{Abf^2 - Acef + Baf^2 - Bbef - Bcdf + Bce^2}{2f^3}\right)\log\left(x + \frac{-Aaef^2 + 2Abdf^2 - Acdef + 2Badf^2 - Bbdef - 2Bcd^2f + Bcde^2 - 4df^3\left(-\frac{\sqrt{-4df+e^2}\ [illegible]}{2f^3\cdot(4df-e^2)} [illegible]\right)}{-2Aaf^3 + Abef^2 + 2Acdf^2 - Ace^2 [illegible]}\right) [illegible]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)/(f*x**2+e*x+d),x)
```

```
[Out] B*c*x**2/(2*f) + x*(A*c/f + B*b/f - B*c*e/f**2) + (-sqrt(-4*d*f + e**2)*(-2
*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B*b*d*f
**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2)) + (A*b*f
**2 - A*c*e*f + B*a*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3))*log(x +
(-A*a*e*f**2 + 2*A*b*d*f**2 - A*c*d*e*f + 2*B*a*d*f**2 - B*b*d*e*f - 2*B*c*
d**2*f + B*c*d*e**2 - 4*d*f**3*(-sqrt(-4*d*f + e**2)*(-2*A*a*f**3 + A*b*e*f
**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B*b*d*f**2 - B*b*e**2*f -
3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2)) + (A*b*f**2 - A*c*e*f + B*a
*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3)) + e**2*f**2*(-sqrt(-4*d*f +
 e**2)*(-2*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 +
 2*B*b*d*f**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2)
) + (A*b*f**2 - A*c*e*f + B*a*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3)
))/(-2*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B
*b*d*f**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3)) + (sqrt(-4*d*f + e**2)*(-
2*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B*b*d*
f**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2)) + (A*b*
f**2 - A*c*e*f + B*a*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3))*log(x +
 (-A*a*e*f**2 + 2*A*b*d*f**2 - A*c*d*e*f + 2*B*a*d*f**2 - B*b*d*e*f - 2*B*c
*d**2*f + B*c*d*e**2 - 4*d*f**3*(sqrt(-4*d*f + e**2)*(-2*A*a*f**3 + A*b*e*f
**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B*b*d*f**2 - B*b*e**2*f -
3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2)) + (A*b*f**2 - A*c*e*f + B*a
*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3)) + e**2*f**2*(sqrt(-4*d*f +
e**2)*(-2*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 +
2*B*b*d*f**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3)/(2*f**3*(4*d*f - e**2))
 + (A*b*f**2 - A*c*e*f + B*a*f**2 - B*b*e*f - B*c*d*f + B*c*e**2)/(2*f**3))
```

```
)/(-2*A*a*f**3 + A*b*e*f**2 + 2*A*c*d*f**2 - A*c*e**2*f + B*a*e*f**2 + 2*B*
b*d*f**2 - B*b*e**2*f - 3*B*c*d*e*f + B*c*e**3))
```

**Giac [A]**
time = 2.48, size = 191, normalized size = 1.04

$$\frac{Bcfx^2 + 2\,Bbfx + 2\,Acfx - 2\,Bcxe}{2\,f^2} - \frac{(Bcdf - Baf^2 - Abf^2 + Bbfe + Acfe - Bce^2)\log\left(fx^2 + xe + d\right)}{2\,f^3} - \frac{(2\,Bbdf^2 + 2\,Acdf^2 - 2\,Aaf^3 - 3\,Bcdfe + Baf^2e + Abf^2e - Bbfe^2 - Acfe^2 + Bce^3)\arctan\left(\frac{2\,fx+e}{\sqrt{4\,df - e^2}}\right)}{\sqrt{4\,df - e^2}\,f^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] 1/2*(B*c*f*x^2 + 2*B*b*f*x + 2*A*c*f*x - 2*B*c*x*e)/f^2 - 1/2*(B*c*d*f - B*
a*f^2 - A*b*f^2 + B*b*f*e + A*c*f*e - B*c*e^2)*log(f*x^2 + x*e + d)/f^3 - (
2*B*b*d*f^2 + 2*A*c*d*f^2 - 2*A*a*f^3 - 3*B*c*d*f*e + B*a*f^2*e + A*b*f^2*e
 - B*b*f*e^2 - A*c*f*e^2 + B*c*e^3)*arctan((2*f*x + e)/sqrt(4*d*f - e^2))/(
sqrt(4*d*f - e^2)*f^3)
```

**Mupad [B]**
time = 3.85, size = 273, normalized size = 1.48

$$x\left(\frac{A\,c+B\,b}{f}-\frac{B\,c\,e}{f^2}\right)-\frac{\ln\left(f\,x^2+e\,x+d\right)\,\left(B\,c\,e^4-4\,A\,b\,d\,f^3-4\,B\,a\,d\,f^3-A\,c\,e^3\,f-B\,b\,e^3\,f+A\,b\,e^2\,f^2+B\,a\,e^2\,f^2+4\,B\,c\,d^2\,f^2+4\,A\,c\,d\,e\,f^2+4\,B\,b\,d\,e\,f^2-5\,B\,c\,d\,e^2\,f\right)}{2\,\left(4\,d\,f^4-e^2\,f^3\right)}-\frac{\mathrm{atan}\left(\frac{e}{\sqrt{4\,d\,f-e^2}}+\frac{2\,f\,x}{\sqrt{4\,d\,f-e^2}}\right)\,\left(B\,c\,e^3-2\,A\,a\,f^3+A\,b\,e\,f^2+2\,A\,c\,d\,f^2+B\,a\,e\,f^2+2\,B\,b\,d\,f^2-A\,c\,e^2\,f-B\,b\,e^2\,f-3\,B\,c\,d\,e\,f\right)}{f^3\,\sqrt{4\,d\,f-e^2}}+\frac{B\,c\,x^2}{2\,f}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2))/(d + e*x + f*x^2),x)
```

```
[Out] x*((A*c + B*b)/f - (B*c*e)/f^2) - (log(d + e*x + f*x^2)*(B*c*e^4 - 4*A*b*d*
f^3 - 4*B*a*d*f^3 - A*c*e^3*f - B*b*e^3*f + A*b*e^2*f^2 + B*a*e^2*f^2 + 4*B
*c*d^2*f^2 + 4*A*c*d*e*f^2 + 4*B*b*d*e*f^2 - 5*B*c*d*e^2*f))/(2*(4*d*f^4 -
e^2*f^3)) - (atan(e/(4*d*f - e^2)^(1/2) + (2*f*x)/(4*d*f - e^2)^(1/2))*(B*c
*e^3 - 2*A*a*f^3 + A*b*e*f^2 + 2*A*c*d*f^2 + B*a*e*f^2 + 2*B*b*d*f^2 - A*c*
e^2*f - B*b*e^2*f - 3*B*c*d*e*f))/(f^3*(4*d*f - e^2)^(1/2)) + (B*c*x^2)/(2*
f)
```

## 3.14 $\int \frac{(A+Bx)\left(a+bx+cx^2\right)^2}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=542

$$\frac{(B(ce-bf)\,(f(be-2af)-c(e^2-2df))+Af(b^2f^2-2cf(be-af)+c^2(e^2-df)))\,x}{f^4}-\frac{(Acf(ce-2bf)-B}{}$$

```
[Out] (B*(-b*f+c*e)*(f*(-2*a*f+b*e)-c*(-2*d*f+e^2))+A*f*(b^2*f^2-2*c*f*(-a*f+b*e)
+c^2*(-d*f+e^2)))*x/f^4-1/2*(A*c*f*(-2*b*f+c*e)-B*(b^2*f^2-2*c*f*(-a*f+b*e)
+c^2*(-d*f+e^2)))*x^2/f^3-1/3*c*(-A*c*f-2*B*b*f+B*c*e)*x^3/f^2+1/4*B*c^2*x^
4/f+1/2*(A*f*(-b*f+c*e)*(f*(-2*a*f+b*e)-c*(-2*d*f+e^2))+B*(c^2*(d^2*f^2-3*d
*e^2*f+e^4)-f^2*(2*a*b*e*f-a^2*f^2-b^2*(-d*f+e^2))+2*c*f*(a*f*(-d*f+e^2)-b*
(-2*d*e*f+e^3))))*ln(f*x^2+e*x+d)/f^5-(A*f*(c^2*(2*d^2*f^2-4*d*e^2*f+e^4)-f
^2*(2*a*b*e*f-2*a^2*f^2-b^2*(-2*d*f+e^2))+2*c*f*(a*f*(-2*d*f+e^2)-b*(-3*d*e
*f+e^3)))-B*(c^2*(5*d^2*e*f^2-5*d*e^3*f+e^5)+f^2*(a^2*e*f^2-2*a*b*f*(-2*d*f
+e^2)+b^2*(-3*d*e*f+e^3))+2*c*f*(a*e*f*(-3*d*f+e^2)-b*(2*d^2*f^2-4*d*e^2*f+
e^4))))*arctanh((2*f*x+e)/(-4*d*f+e^2)^(1/2))/f^5/(-4*d*f+e^2)^(1/2)
```

**Rubi** [A]
time = 0.68, antiderivative size = 542, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {1025, 648, 632, 212, 642}

$$\frac{\log\left(d+ex+fx^2\right)\left(B(-f^2(-a^2f^2+2abef-(b^2(e^2-df)))+2cf(af(e^2-df)-b(e^3-2def))+c^2(d^2f^2-3de^2f+e^4))+Af(ce-bf)\,(f(be-2af)-c(e^2-2df))\right)}{2f^5}-\frac{\tanh^{-1}\left(\frac{e+2fx}{\sqrt{e^2-4df}}\right)\left(Af(-f^2(-2a^2f^2+2abef-(b^2(e^2-2df)))+2cf(af(e^2-2df)-b(e^3-3def))+c^2(2d^2f^2-4de^2f+e^4))-B(f^2(a^2ef^2-2abf(e^2-2df)+b^2(e^3-3def))+2cf(aef(e^2-3df)-b(2d^2f^2-4de^2f+e^4))+c^2(5d^2ef^2-5de^3f+e^5))\right)}{f^5\sqrt{e^2-4df}}+\frac{x(Af(-2cf(be-af)+b^2f^2+c^2(e^2-df))+B(ce-bf)\,(f(be-2af)-c(e^2-2df)))}{f^4}-\frac{x^2(Acf(ce-2bf)-B(-2cf(be-af)+b^2f^2+c^2(e^2-df)))}{2f^3}-\frac{cx^3(-Acf-2bBf+Bce)}{3f^2}+\frac{Bc^2x^4}{4f}$$

Antiderivative was successfully verified.

```
[In] Int[((A + B*x)*(a + b*x + c*x^2)^2)/(d + e*x + f*x^2),x]
```

```
[Out] ((B*(c*e - b*f)*(f*(b*e - 2*a*f) - c*(e^2 - 2*d*f)) + A*f*(b^2*f^2 - 2*c*f*
(b*e - a*f) + c^2*(e^2 - d*f)))*x)/f^4 - ((A*c*f*(c*e - 2*b*f) - B*(b^2*f^2
 - 2*c*f*(b*e - a*f) + c^2*(e^2 - d*f)))*x^2)/(2*f^3) - (c*(B*c*e - 2*b*B*f
 - A*c*f)*x^3)/(3*f^2) + (B*c^2*x^4)/(4*f) - ((A*f*(c^2*(e^4 - 4*d*e^2*f +
2*d^2*f^2) - f^2*(2*a*b*e*f - 2*a^2*f^2 - b^2*(e^2 - 2*d*f)) + 2*c*f*(a*f*(
e^2 - 2*d*f) - b*(e^3 - 3*d*e*f))) - B*(c^2*(e^5 - 5*d*e^3*f + 5*d^2*e*f^2)
 + f^2*(a^2*e*f^2 - 2*a*b*f*(e^2 - 2*d*f) + b^2*(e^3 - 3*d*e*f)) + 2*c*f*(a
*e*f*(e^2 - 3*d*f) - b*(e^4 - 4*d*e^2*f + 2*d^2*f^2))))*ArcTanh[(e + 2*f*x)
/Sqrt[e^2 - 4*d*f]])/(f^5*Sqrt[e^2 - 4*d*f]) + ((A*f*(c*e - b*f)*(f*(b*e -
2*a*f) - c*(e^2 - 2*d*f)) + B*(c^2*(e^4 - 3*d*e^2*f + d^2*f^2) - f^2*(2*a*b
*e*f - a^2*f^2 - b^2*(e^2 - d*f)) + 2*c*f*(a*f*(e^2 - d*f) - b*(e^3 - 2*d*e
*f))))*Log[d + e*x + f*x^2])/(2*f^5)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 1025

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Int[ExpandIntegrand[(a + b*x +
c*x^2)^p*(d + e*x + f*x^2)^q*(g + h*x), x], x] /; FreeQ[{a, b, c, d, e, f,
g, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && IGtQ[p, 0] && In
tegerQ[q]
```

Rubi steps

$$\int \frac{(A+Bx)\left(a+bx+cx^2\right)^2}{d+ex+fx^2}\,dx = \int \left(\frac{B(ce-bf)\left(f(be-2af)-c(e^2-2df)\right)+Af(b^2f^2-2cf(be-af)}{f^4}\right.$$

$$= \frac{\left(B(ce-bf)\left(f(be-2af)-c(e^2-2df)\right)+Af(b^2f^2-2cf(be-af)+c}{f^4}$$

$$= \frac{\left(B(ce-bf)\left(f(be-2af)-c(e^2-2df)\right)+Af(b^2f^2-2cf(be-af)+c}{f^4}$$

$$= \frac{\left(B(ce-bf)\left(f(be-2af)-c(e^2-2df)\right)+Af(b^2f^2-2cf(be-af)+c}{f^4}$$

$$= \frac{\left(B(ce-bf)\left(f(be-2af)-c(e^2-2df)\right)+Af(b^2f^2-2cf(be-af)+c}{f^4}$$

**Mathematica [A]**
time = 0.41, size = 535, normalized size = 0.99

$$\frac{12f(-B(ce-bf)(f(-be+2af)+c(e^2-2df))+Af(b^2f^2+2cf(-be+af)+c^2(e^2-df)))x+6f^2(Acf(-ce+2bf)+B(b^2f^2+2cf(-be+af)+c^2(e^2-df)))x^2+4cf^3(-Bce+2bBf+Acf)x^3+3Bc^2f^4x^4-\frac{12(-Af(c^2(e^4-4de^2f+2d^2f^2)+f^2(-2abef+2a^2f^2+b^2(e^2-2df))+2cf(af(e^2-2df)-b(e^3-3def)))+B(c^2(e^5-5de^3f+5d^2ef^2)+f^2(a^2ef^2+2abf(-e^2+2df)+b^2(e^3-3def))-2cf(-aef(e^2-3df)+b(e^4-4de^2f+2d^2f^2))))\tan^{-1}\left(\frac{e+2fx}{\sqrt{-e^2+4df}}\right)}{\sqrt{-e^2+4df}}+6(Af(-ce+bf)(f(-be+2af)+c(e^2-2df))+B(c^2(e^4-3de^2f+d^2f^2)+f^2(-2abef+a^2f^2+b^2(e^2-df))-2cf(af(-e^2+df)+b(e^3-2def))))\log(d+x(e+fx))}{12f^5}$$

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2)^2)/(d + e*x + f*x^2),x]
```

```
[Out] (12*f*(-(B*(c*e - b*f)*(f*(-(b*e) + 2*a*f) + c*(e^2 - 2*d*f))) + A*f*(b^2*f
^2 + 2*c*f*(-(b*e) + a*f) + c^2*(e^2 - d*f)))*x + 6*f^2*(A*c*f*(-(c*e) + 2*
b*f) + B*(b^2*f^2 + 2*c*f*(-(b*e) + a*f) + c^2*(e^2 - d*f)))*x^2 + 4*c*f^3*
(-(B*c*e) + 2*b*B*f + A*c*f)*x^3 + 3*B*c^2*f^4*x^4 - (12*(-(A*f*(c^2*(e^4 -
 4*d*e^2*f + 2*d^2*f^2) + f^2*(-2*a*b*e*f + 2*a^2*f^2 + b^2*(e^2 - 2*d*f))
+ 2*c*f*(a*f*(e^2 - 2*d*f) - b*(e^3 - 3*d*e*f)))) + B*(c^2*(e^5 - 5*d*e^3*f
 + 5*d^2*e*f^2) + f^2*(a^2*e*f^2 + 2*a*b*f*(-e^2 + 2*d*f) + b^2*(e^3 - 3*d*
e*f)) - 2*c*f*(-(a*e*f*(e^2 - 3*d*f)) + b*(e^4 - 4*d*e^2*f + 2*d^2*f^2))))*
ArcTan[(e + 2*f*x)/Sqrt[-e^2 + 4*d*f]])/Sqrt[-e^2 + 4*d*f] + 6*(A*f*(-(c*e)
 + b*f)*(f*(-(b*e) + 2*a*f) + c*(e^2 - 2*d*f)) + B*(c^2*(e^4 - 3*d*e^2*f +
d^2*f^2) + f^2*(-2*a*b*e*f + a^2*f^2 + b^2*(e^2 - d*f)) - 2*c*f*(a*f*(-e^2
+ d*f) + b*(e^3 - 2*d*e*f))))*Log[d + x*(e + f*x)])/(12*f^5)
```

**Maple [A]**
time = 0.27, size = 800, normalized size = 1.48

| method | result |
|---|---|
| default | $\frac{\frac{1}{4}Bc^2x^4f^3+\frac{1}{3}Ac^2f^3x^3+\frac{2}{3}Bbcf^3x^3-\frac{1}{3}Bc^2ef^2x^3+Abcf^3x^2-\frac{1}{2}Ac^2ef^2x^2+Bacf^3x^2+\frac{1}{2}Bb^2f^3x^2-Bbcef^2x^2-\frac{1}{2}Bc^2df^2x^2+\frac{1}{2}B}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f^4*(1/4*B*c^2*x^4*f^3+1/3*A*c^2*f^3*x^3+2/3*B*b*c*f^3*x^3-1/3*B*c^2*e*f^
2*x^3+A*b*c*f^3*x^2-1/2*A*c^2*e*f^2*x^2+B*a*c*f^3*x^2+1/2*B*b^2*f^3*x^2-B*b
*c*e*f^2*x^2-1/2*B*c^2*d*f^2*x^2+1/2*B*c^2*e^2*f*x^2+2*A*a*c*f^3*x+A*b^2*f^
3*x-2*A*b*c*e*f^2*x-A*c^2*d*f^2*x+A*c^2*e^2*f*x+2*B*a*b*f^3*x-2*B*a*c*e*f^2
*x-B*b^2*e*f^2*x-2*B*b*c*d*f^2*x+2*B*b*c*e^2*f*x+2*B*c^2*d*e*f*x-B*c^2*e^3*
x)+1/f^4*(1/2*(2*A*a*b*f^4-2*A*a*c*e*f^3-A*b^2*e*f^3-2*A*b*c*d*f^3+2*A*b*c*
e^2*f^2+2*A*c^2*d*e*f^2-A*c^2*e^3*f+B*a^2*f^4-2*B*a*b*e*f^3-2*B*a*c*d*f^3+2
*B*a*c*e^2*f^2-B*b^2*d*f^3+B*b^2*e^2*f^2+4*B*b*c*d*e*f^2-2*B*b*c*e^3*f+B*c^
2*d^2*f^2-3*B*c^2*d*e^2*f+B*c^2*e^4)/f*ln(f*x^2+e*x+d)+2*(A*a^2*f^4-2*A*a*c
*d*f^3-A*b^2*d*f^3+2*A*b*c*d*e*f^2+A*c^2*d^2*f^2-A*c^2*d*e^2*f-2*B*a*b*d*f^
3+2*B*a*c*d*e*f^2+B*b^2*d*e*f^2+2*B*b*c*d^2*f^2-2*B*b*c*d*e^2*f-2*B*c^2*d^2
```

```
*e*f+B*c^2*d*e^3-1/2*(2*A*a*b*f^4-2*A*a*c*e*f^3-A*b^2*e*f^3-2*A*b*c*d*f^3+2
*A*b*c*e^2*f^2+2*A*c^2*d*e*f^2-A*c^2*e^3*f+B*a^2*f^4-2*B*a*b*e*f^3-2*B*a*c*
d*f^3+2*B*a*c*e^2*f^2-B*b^2*d*f^3+B*b^2*e^2*f^2+4*B*b*c*d*e*f^2-2*B*b*c*e^3
*f+B*c^2*d^2*f^2-3*B*c^2*d*e^2*f+B*c^2*e^4)*e/f)/(4*d*f-e^2)^(1/2)*arctan((
2*f*x+e)/(4*d*f-e^2)^(1/2)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas [A]**
time = 0.51, size = 1921, normalized size = 3.54

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] [1/12*(12*B*c^2*d*f^5*x^4 + 16*(2*B*b*c + A*c^2)*d*f^5*x^3 + 12*B*c^2*f*x*e
^5 - 24*(B*c^2*d^2*f^4 - (B*b^2 + 2*(B*a + A*b)*c)*d*f^5)*x^2 + 6*(2*A*a^2*
f^5 + 2*(2*B*b*c + A*c^2)*d^2*f^3 - 2*(2*B*a*b + A*b^2 + 2*A*a*c)*d*f^4 - B
*c^2*e^5 + (2*B*b*c + A*c^2)*f*e^4 + (5*B*c^2*d*f - (B*b^2 + 2*(B*a + A*b)*
c)*f^2)*e^3 - (4*(2*B*b*c + A*c^2)*d*f^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*f^3)
*e^2 - (5*B*c^2*d^2*f^2 - 3*(B*b^2 + 2*(B*a + A*b)*c)*d*f^3 + (B*a^2 + 2*A*
a*b)*f^4)*e)*sqrt(-4*d*f + e^2)*log((2*f^2*x^2 + 2*f*x*e - 2*d*f + sqrt(-4*
d*f + e^2)*(2*f*x + e) + e^2)/(f*x^2 + x*e + d)) - 48*((2*B*b*c + A*c^2)*d^
2*f^4 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f^5)*x - 6*(B*c^2*f^2*x^2 + 2*(2*B*b*
c + A*c^2)*f^2*x)*e^4 + 2*(2*B*c^2*f^3*x^3 + 3*(2*B*b*c + A*c^2)*f^3*x^2 -
6*(6*B*c^2*d*f^2 - (B*b^2 + 2*(B*a + A*b)*c)*f^3)*x)*e^3 - (3*B*c^2*f^4*x^4
 + 4*(2*B*b*c + A*c^2)*f^4*x^3 - 6*(5*B*c^2*d*f^3 - (B*b^2 + 2*(B*a + A*b)*
c)*f^4)*x^2 - 12*(5*(2*B*b*c + A*c^2)*d*f^3 - (2*B*a*b + A*b^2 + 2*A*a*c)*f
^4)*x)*e^2 - 8*(2*B*c^2*d*f^4*x^3 + 3*(2*B*b*c + A*c^2)*d*f^4*x^2 - 6*(2*B*
c^2*d^2*f^3 - (B*b^2 + 2*(B*a + A*b)*c)*d*f^4)*x)*e + 6*(4*B*c^2*d^3*f^3 -
4*(B*b^2 + 2*(B*a + A*b)*c)*d^2*f^4 + 4*(B*a^2 + 2*A*a*b)*d*f^5 - B*c^2*e^6
 + (2*B*b*c + A*c^2)*f*e^5 + (7*B*c^2*d*f - (B*b^2 + 2*(B*a + A*b)*c)*f^2)*
e^4 - (6*(2*B*b*c + A*c^2)*d*f^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*f^3)*e^3 - (
13*B*c^2*d^2*f^2 - 5*(B*b^2 + 2*(B*a + A*b)*c)*d*f^3 + (B*a^2 + 2*A*a*b)*f^
```

```
4)*e^2 + 4*(2*(2*B*b*c + A*c^2)*d^2*f^3 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f^4
)*e)*log(f*x^2 + x*e + d))/(4*d*f^6 - f^5*e^2), 1/12*(12*B*c^2*d*f^5*x^4 +
16*(2*B*b*c + A*c^2)*d*f^5*x^3 + 12*B*c^2*f*x*e^5 - 24*(B*c^2*d^2*f^4 - (B*
b^2 + 2*(B*a + A*b)*c)*d*f^5)*x^2 - 12*(2*A*a^2*f^5 + 2*(2*B*b*c + A*c^2)*d
^2*f^3 - 2*(2*B*a*b + A*b^2 + 2*A*a*c)*d*f^4 - B*c^2*e^5 + (2*B*b*c + A*c^2
)*f*e^4 + (5*B*c^2*d*f - (B*b^2 + 2*(B*a + A*b)*c)*f^2)*e^3 - (4*(2*B*b*c +
 A*c^2)*d*f^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*f^3)*e^2 - (5*B*c^2*d^2*f^2 - 3
*(B*b^2 + 2*(B*a + A*b)*c)*d*f^3 + (B*a^2 + 2*A*a*b)*f^4)*e)*sqrt(4*d*f - e
^2)*arctan(-(2*f*x + e)/sqrt(4*d*f - e^2)) - 48*((2*B*b*c + A*c^2)*d^2*f^4
- (2*B*a*b + A*b^2 + 2*A*a*c)*d*f^5)*x - 6*(B*c^2*f^2*x^2 + 2*(2*B*b*c + A*
c^2)*f^2*x)*e^4 + 2*(2*B*c^2*f^3*x^3 + 3*(2*B*b*c + A*c^2)*f^3*x^2 - 6*(6*B
*c^2*d*f^2 - (B*b^2 + 2*(B*a + A*b)*c)*f^3)*x)*e^3 - (3*B*c^2*f^4*x^4 + 4*(
2*B*b*c + A*c^2)*f^4*x^3 - 6*(5*B*c^2*d*f^3 - (B*b^2 + 2*(B*a + A*b)*c)*f^4
)*x^2 - 12*(5*(2*B*b*c + A*c^2)*d*f^3 - (2*B*a*b + A*b^2 + 2*A*a*c)*f^4)*x)
*e^2 - 8*(2*B*c^2*d*f^4*x^3 + 3*(2*B*b*c + A*c^2)*d*f^4*x^2 - 6*(2*B*c^2*d^
2*f^3 - (B*b^2 + 2*(B*a + A*b)*c)*d*f^4)*x)*e + 6*(4*B*c^2*d^3*f^3 - 4*(B*b
^2 + 2*(B*a + A*b)*c)*d^2*f^4 + 4*(B*a^2 + 2*A*a*b)*d*f^5 - B*c^2*e^6 + (2*
B*b*c + A*c^2)*f*e^5 + (7*B*c^2*d*f - (B*b^2 + 2*(B*a + A*b)*c)*f^2)*e^4 -
(6*(2*B*b*c + A*c^2)*d*f^2 - (2*B*a*b + A*b^2 + 2*A*a*c)*f^3)*e^3 - (13*B*c
^2*d^2*f^2 - 5*(B*b^2 + 2*(B*a + A*b)*c)*d*f^3 + (B*a^2 + 2*A*a*b)*f^4)*e^2
 + 4*(2*(2*B*b*c + A*c^2)*d^2*f^3 - (2*B*a*b + A*b^2 + 2*A*a*c)*d*f^4)*e)*l
og(f*x^2 + x*e + d))/(4*d*f^6 - f^5*e^2)]
```

**Sympy** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 4663 vs. $2(520) = 1040$.
time = 91.26, size = 4663, normalized size = 8.60

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**2/(f*x**2+e*x+d),x)
```

```
[Out] B*c**2*x**4/(4*f) + x**3*(A*c**2/(3*f) + 2*B*b*c/(3*f) - B*c**2*e/(3*f**2))
 + x**2*(A*b*c/f - A*c**2*e/(2*f**2) + B*a*c/f + B*b**2/(2*f) - B*b*c*e/f**
2 - B*c**2*d/(2*f**2) + B*c**2*e**2/(2*f**3)) + x*(2*A*a*c/f + A*b**2/f - 2
*A*b*c*e/f**2 - A*c**2*d/f**2 + A*c**2*e**2/f**3 + 2*B*a*b/f - 2*B*a*c*e/f*
*2 - B*b**2*e/f**2 - 2*B*b*c*d/f**2 + 2*B*b*c*e**2/f**3 + 2*B*c**2*d*e/f**3
 - B*c**2*e**3/f**4) + (-sqrt(-4*d*f + e**2)*(-2*A*a**2*f**5 + 2*A*a*b*e*f*
*4 + 4*A*a*c*d*f**4 - 2*A*a*c*e**2*f**3 + 2*A*b**2*d*f**4 - A*b**2*e**2*f**
3 - 6*A*b*c*d*e*f**3 + 2*A*b*c*e**3*f**2 - 2*A*c**2*d**2*f**3 + 4*A*c**2*d*
e**2*f**2 - A*c**2*e**4*f + B*a**2*e*f**4 + 4*B*a*b*d*f**4 - 2*B*a*b*e**2*f
**3 - 6*B*a*c*d*e*f**3 + 2*B*a*c*e**3*f**2 - 3*B*b**2*d*e*f**3 + B*b**2*e**
3*f**2 - 4*B*b*c*d**2*f**3 + 8*B*b*c*d*e**2*f**2 - 2*B*b*c*e**4*f + 5*B*c**
2*d**2*e*f**2 - 5*B*c**2*d*e**3*f + B*c**2*e**5)/(2*f**5*(4*d*f - e**2)) +
(2*A*a*b*f**4 - 2*A*a*c*e*f**3 - A*b**2*e*f**3 - 2*A*b*c*d*f**3 + 2*A*b*c*e
```

```
**2*f**2 + 2*A*c**2*d*e*f**2 - A*c**2*e**3*f + B*a**2*f**4 - 2*B*a*b*e*f**3
 - 2*B*a*c*d*f**3 + 2*B*a*c*e**2*f**2 - B*b**2*d*f**3 + B*b**2*e**2*f**2 +
4*B*b*c*d*e*f**2 - 2*B*b*c*e**3*f + B*c**2*d**2*f**2 - 3*B*c**2*d*e**2*f +
B*c**2*e**4)/(2*f**5))*log(x + (-A*a**2*e*f**4 + 4*A*a*b*d*f**4 - 2*A*a*c*d
*e*f**3 - A*b**2*d*e*f**3 - 4*A*b*c*d**2*f**3 + 2*A*b*c*d*e**2*f**2 + 3*A*c
**2*d**2*e*f**2 - A*c**2*d*e**3*f + 2*B*a**2*d*f**4 - 2*B*a*b*d*e*f**3 - 4*
B*a*c*d**2*f**3 + 2*B*a*c*d*e**2*f**2 - 2*B*b**2*d**2*f**3 + B*b**2*d*e**2*
f**2 + 6*B*b*c*d**2*e*f**2 - 2*B*b*c*d*e**3*f + 2*B*c**2*d**3*f**2 - 4*B*c*
*2*d**2*e**2*f + B*c**2*d*e**4 - 4*d*f**5*(-sqrt(-4*d*f + e**2)*(-2*A*a**2*
f**5 + 2*A*a*b*e*f**4 + 4*A*a*c*d*f**4 - 2*A*a*c*e**2*f**3 + 2*A*b**2*d*f**
4 - A*b**2*e**2*f**3 - 6*A*b*c*d*e*f**3 + 2*A*b*c*e**3*f**2 - 2*A*c**2*d**2
*f**3 + 4*A*c**2*d*e**2*f**2 - A*c**2*e**4*f + B*a**2*e*f**4 + 4*B*a*b*d*f*
*4 - 2*B*a*b*e**2*f**3 - 6*B*a*c*d*e*f**3 + 2*B*a*c*e**3*f**2 - 3*B*b**2*d*
e*f**3 + B*b**2*e**3*f**2 - 4*B*b*c*d**2*f**3 + 8*B*b*c*d*e**2*f**2 - 2*B*b
*c*e**4*f + 5*B*c**2*d**2*e*f**2 - 5*B*c**2*d*e**3*f + B*c**2*e**5)/(2*f**5
*(4*d*f - e**2)) + (2*A*a*b*f**4 - 2*A*a*c*e*f**3 - A*b**2*e*f**3 - 2*A*b*c
*d*f**3 + 2*A*b*c*e**2*f**2 + 2*A*c**2*d*e*f**2 - A*c**2*e**3*f + B*a**2*f*
*4 - 2*B*a*b*e*f**3 - 2*B*a*c*d*f**3 + 2*B*a*c*e**2*f**2 - B*b**2*d*f**3 +
B*b**2*e**2*f**2 + 4*B*b*c*d*e*f**2 - 2*B*b*c*e**3*f + B*c**2*d**2*f**2 - 3
*B*c**2*d*e**2*f + B*c**2*e**4)/(2*f**5)) + e**2*f**4*(-sqrt(-4*d*f + e**2)
*(-2*A*a**2*f**5 + 2*A*a*b*e*f**4 + 4*A*a*c*d*f**4 - 2*A*a*c*e**2*f**3 + 2*
A*b**2*d*f**4 - A*b**2*e**2*f**3 - 6*A*b*c*d*e*f**3 + 2*A*b*c*e**3*f**2 - 2
*A*c**2*d**2*f**3 + 4*A*c**2*d*e**2*f**2 - A*c**2*e**4*f + B*a**2*e*f**4 +
4*B*a*b*d*f**4 - 2*B*a*b*e**2*f**3 - 6*B*a*c*d*e*f**3 + 2*B*a*c*e**3*f**2 -
 3*B*b**2*d*e*f**3 + B*b**2*e**3*f**2 - 4*B*b*c*d**2*f**3 + 8*B*b*c*d*e**2*
f**2 - 2*B*b*c*e**4*f + 5*B*c**2*d**2*e*f**2 - 5*B*c**2*d*e**3*f + B*c**2*e
**5)/(2*f**5*(4*d*f - e**2)) + (2*A*a*b*f**4 - 2*A*a*c*e*f**3 - A*b**2*e*f*
*3 - 2*A*b*c*d*f**3 + 2*A*b*c*e**2*f**2 + 2*A*c**2*d*e*f**2 - A*c**2*e**3*f
 + B*a**2*f**4 - 2*B*a*b*e*f**3 - 2*B*a*c*d*f**3 + 2*B*a*c*e**2*f**2 - B*b*
*2*d*f**3 + B*b**2*e**2*f**2 + 4*B*b*c*d*e*f**2 - 2*B*b*c*e**3*f + B*c**2*d
**2*f**2 - 3*B*c**2*d*e**2*f + B*c**2*e**4)/(2*f**5)))/(-2*A*a**2*f**5 + 2*
A*a*b*e*f**4 + 4*A*a*c*d*f**4 - 2*A*a*c*e**2*f**3 + 2*A*b**2*d*f**4 - A*b**
2*e**2*f**3 - 6*A*b*c*d*e*f**3 + 2*A*b*c*e**3*f**2 - 2*A*c**2*d**2*f**3 + 4
*A*c**2*d*e**2*f**2 - A*c**2*e**4*f + B*a**2*e*f**4 + 4*B*a*b*d*f**4 - 2*B*
a*b*e**2*f**3 - 6*B*a*c*d*e*f**3 + 2*B*a*c*e**3*f**2 - 3*B*b**2*d*e*f**3 +
B*b**2*e**3*f**2 - 4*B*b*c*d**2*f**3 + 8*B*b*c*d*e**2*f**2 - 2*B*b*c*e**4*f
 + 5*B*c**2*d**2*e*f**2 - 5*B*c**2*d*e**3*f + B*c**2*e**5)) + (sqrt(-4*d*f
+ e**2)*(-2*A*a**2*f**5 + 2*A*a*b*e*f**4 + 4*A*a*c*d*f**4 - 2*A*a*c*e**2*f*
*3 + 2*A*b**2*d*f**4 - A*b**2*e**2*f**3 - 6*A*b*c*d*e*f**3 + 2*A*b*c*e**3*f
**2 - 2*A*c**2*d**2*f**3 + 4*A*c**2*d*e**2*f**2 - A*c**2*e**4*f + B*a**2*e*
f**4 + 4*B*a*b*d*f**4 - 2*B*a*b*e**2*f**3 - 6*B*a*c*d*e*f**3 + 2*B*a*c*e**3
*f**2 - 3*B*b**2*d*e*f**3 + B*b**2*e**3*f**2 - 4*B*b*c*d**2*f**3 + 8*B*b*c*
d*e**2*f**2 - 2*B*b*c*e**4*f + 5*B*c**2*d**2*e*f**2 - 5*B*c**2*d*e**3*f + B
*c**2*e**5)/(2*f**5*(4*d*f - e**2)) + (2*A*a*b*f**4 - 2*A*a*c*e*f**3 - A*b*
*2*e*f**3 - 2*A*b*c*d*f**3 + 2*A*b*c*e**2*f**2 + 2*A*c**2*d*e*f**2 - A*c**2
```

```
*e**3*f + B*a**2*f**4 - 2*B*a*b*e*f**3 - 2*B*a*c*d*f**3 + 2*B*a*c*e**2*f**2
 - B*b**2*d*f**3 + B*b**2*e**2*f**2 + 4*B*b*c*d*e*f**2 - 2*B*b*c*e**3*f + B
*c**2*d**2*f**2 - 3*B*c**2*d*e**2*f + B*c**2*e**4)/(2*f**5))*log(x + (-A*a*
*2*e*f**4 + 4*A*a*b*d*f**4 - 2*A*a*c*d*e*f**3 - A*b**2*d*e*f**3 - 4*A*b*c*d
**2*f**3 + 2*A*b*c*d*e**2*f**2 + 3*A*c**2*d**2*e*f**2 - A*c**2*d*e**3*f + 2
*B*a**2*d*f**4 - 2*B*a*b*d*e*f**3 - 4*B*a*c*d**2*f**3 + 2*B*a*c*d*e**2*f**2
 - 2*B*b**2*d**2*f**3 + B*b**2*d*e**2*f**2 + 6*B*b*c*d**2*e*f**2 - 2*B*b*c*
d*e**3*f + 2*B*c**2*d**3*f**2 - 4*B*c**2*d**2*e**2*f + B*c**2*d*e**4 - 4*d*
f**5*(sqrt(-4*d*f + e**2)*(-2*A*a**2*f**5 + 2*A...
```

**Giac [A]**
time = 3.08, size = 738, normalized size = 1.36

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] 1/12*(3*B*c^2*f^3*x^4 + 8*B*b*c*f^3*x^3 + 4*A*c^2*f^3*x^3 - 4*B*c^2*f^2*x^3
*e - 6*B*c^2*d*f^2*x^2 + 6*B*b^2*f^3*x^2 + 12*B*a*c*f^3*x^2 + 12*A*b*c*f^3*
x^2 - 12*B*b*c*f^2*x^2*e - 6*A*c^2*f^2*x^2*e - 24*B*b*c*d*f^2*x - 12*A*c^2*
d*f^2*x + 24*B*a*b*f^3*x + 12*A*b^2*f^3*x + 24*A*a*c*f^3*x + 6*B*c^2*f*x^2*
e^2 + 24*B*c^2*d*f*x*e - 12*B*b^2*f^2*x*e - 24*B*a*c*f^2*x*e - 24*A*b*c*f^2
*x*e + 24*B*b*c*f*x*e^2 + 12*A*c^2*f*x*e^2 - 12*B*c^2*x*e^3)/f^4 + 1/2*(B*c
^2*d^2*f^2 - B*b^2*d*f^3 - 2*B*a*c*d*f^3 - 2*A*b*c*d*f^3 + B*a^2*f^4 + 2*A*
a*b*f^4 + 4*B*b*c*d*f^2*e + 2*A*c^2*d*f^2*e - 2*B*a*b*f^3*e - A*b^2*f^3*e -
 2*A*a*c*f^3*e - 3*B*c^2*d*f*e^2 + B*b^2*f^2*e^2 + 2*B*a*c*f^2*e^2 + 2*A*b*
c*f^2*e^2 - 2*B*b*c*f*e^3 - A*c^2*f*e^3 + B*c^2*e^4)*log(f*x^2 + x*e + d)/f
^5 + (4*B*b*c*d^2*f^3 + 2*A*c^2*d^2*f^3 - 4*B*a*b*d*f^4 - 2*A*b^2*d*f^4 - 4
*A*a*c*d*f^4 + 2*A*a^2*f^5 - 5*B*c^2*d^2*f^2*e + 3*B*b^2*d*f^3*e + 6*B*a*c*
d*f^3*e + 6*A*b*c*d*f^3*e - B*a^2*f^4*e - 2*A*a*b*f^4*e - 8*B*b*c*d*f^2*e^2
 - 4*A*c^2*d*f^2*e^2 + 2*B*a*b*f^3*e^2 + A*b^2*f^3*e^2 + 2*A*a*c*f^3*e^2 +
5*B*c^2*d*f*e^3 - B*b^2*f^2*e^3 - 2*B*a*c*f^2*e^3 - 2*A*b*c*f^2*e^3 + 2*B*b
*c*f*e^4 + A*c^2*f*e^4 - B*c^2*e^5)*arctan((2*f*x + e)/sqrt(4*d*f - e^2))/(
sqrt(4*d*f - e^2)*f^5)
```

**Mupad [B]**
time = 4.85, size = 893, normalized size = 1.65

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^2)/(d + e*x + f*x^2),x)
```

```
[Out] x^3*((A*c^2 + 2*B*b*c)/(3*f) - (B*c^2*e)/(3*f^2)) + x*((A*b^2 + 2*A*a*c + 2
*B*a*b)/f - (d*((A*c^2 + 2*B*b*c)/f - (B*c^2*e)/f^2))/f + (e*((e*((A*c^2 +
```

```
2*B*b*c)/f - (B*c^2*e)/f^2))/f - (B*b^2 + 2*A*b*c + 2*B*a*c)/f + (B*c^2*d)/
f^2))/f) - x^2*((e*((A*c^2 + 2*B*b*c)/f - (B*c^2*e)/f^2))/(2*f) - (B*b^2 +
2*A*b*c + 2*B*a*c)/(2*f) + (B*c^2*d)/(2*f^2)) - (log(d + e*x + f*x^2)*(B*c^
2*e^6 - 4*B*a^2*d*f^5 - A*c^2*e^5*f - A*b^2*e^3*f^3 + B*a^2*e^2*f^4 + 4*B*b
^2*d^2*f^4 + B*b^2*e^4*f^2 - 4*B*c^2*d^3*f^3 + 6*A*c^2*d*e^3*f^2 - 8*A*c^2*
d^2*e*f^3 - 5*B*b^2*d*e^2*f^3 - 8*A*a*b*d*f^5 - 2*B*b*c*e^5*f + 13*B*c^2*d^
2*e^2*f^2 + 2*A*a*b*e^2*f^4 - 2*A*a*c*e^3*f^3 + 8*A*b*c*d^2*f^4 - 2*B*a*b*e
^3*f^3 + 8*B*a*c*d^2*f^4 + 2*A*b*c*e^4*f^2 + 2*B*a*c*e^4*f^2 + 4*A*b^2*d*e*
f^4 - 7*B*c^2*d*e^4*f - 10*A*b*c*d*e^2*f^3 - 10*B*a*c*d*e^2*f^3 + 12*B*b*c*
d*e^3*f^2 - 16*B*b*c*d^2*e*f^3 + 8*A*a*c*d*e*f^4 + 8*B*a*b*d*e*f^4))/(2*(4*
d*f^6 - e^2*f^5)) + (B*c^2*x^4)/(4*f) + (atan(e/(4*d*f - e^2)^(1/2) + (2*f*
x)/(4*d*f - e^2)^(1/2))*(2*A*a^2*f^5 - B*c^2*e^5 - 2*A*b^2*d*f^4 - B*a^2*e*
f^4 + A*c^2*e^4*f + A*b^2*e^2*f^3 + 2*A*c^2*d^2*f^3 - B*b^2*e^3*f^2 - 4*A*c
^2*d*e^2*f^2 - 5*B*c^2*d^2*e*f^2 - 2*A*a*b*e*f^4 - 4*A*a*c*d*f^4 - 4*B*a*b*
d*f^4 + 2*B*b*c*e^4*f + 2*A*a*c*e^2*f^3 + 2*B*a*b*e^2*f^3 - 2*A*b*c*e^3*f^2
 - 2*B*a*c*e^3*f^2 + 4*B*b*c*d^2*f^3 + 3*B*b^2*d*e*f^3 + 5*B*c^2*d*e^3*f -
8*B*b*c*d*e^2*f^2 + 6*A*b*c*d*e*f^3 + 6*B*a*c*d*e*f^3))/(f^5*(4*d*f - e^2)^
(1/2))
```

# 3.15 $\int \frac{A+Bx}{(a+bx+cx^2)(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=406

$$-\frac{\left(Ab^2f+2c(Acd+aBe-aAf)-b(Bcd+Ace+aBf)\right)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{\sqrt{b^2-4ac}\,\left(c^2d^2+f\left(b^2d-abe+a^2f\right)-c\left(bde-a\left(e^2-2df\right)\right)\right)}+\frac{(B(cde-2bdf+aef)-}{\sqrt{e^2-4df}\,\left(c^2d^2\right.}$$

```
[Out] 1/2*(A*b*f-A*c*e-B*a*f+B*c*d)*ln(c*x^2+b*x+a)/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d
)-c*(b*d*e-a*(-2*d*f+e^2)))-1/2*(A*b*f-A*c*e-B*a*f+B*c*d)*ln(f*x^2+e*x+d)/(
c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))-(A*b^2*f+2*c*(-A*a*
f+A*c*d+B*a*e)-b*(A*c*e+B*a*f+B*c*d))*arctanh((2*c*x+b)/(-4*a*c+b^2)^(1/2))
/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))/(-4*a*c+b^2)^(1/2
)+(B*(a*e*f-2*b*d*f+c*d*e)-A*(2*a*f^2-b*e*f-2*c*d*f+c*e^2))*arctanh((2*f*x+
e)/(-4*d*f+e^2)^(1/2))/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^
2)))/(-4*d*f+e^2)^(1/2)
```

**Rubi** [A]
time = 0.29, antiderivative size = 398, normalized size of antiderivative = 0.98, number of steps used = 9, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {1036, 648, 632, 212, 642}

$$\frac{\tanh^{-1}\left(\frac{e+2fx}{\sqrt{e^2-4df}}\right)(B(aef-2bdf+cde)-A(2af^2-bef-2cdf+ce^2))}{\sqrt{e^2-4df}\,\left(f\left(a^2f-abe+b^2d\right)+ac\left(e^2-2df\right)-bcde+c^2d^2\right)}+\frac{\log\left(a+bx+cx^2\right)(-aBf+Abf-Ace+Bcd)}{2\left(f\left(a^2f-abe+b^2d\right)+ac\left(e^2-2df\right)-bcde+c^2d^2\right)}-\frac{\log\left(d+ex+fx^2\right)(-aBf+Abf-Ace+Bcd)}{2\left(f\left(a^2f-abe+b^2d\right)+ac\left(e^2-2df\right)-bcde+c^2d^2\right)}-\frac{\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)(-b(aBf+Ace+Bcd)+2c(-aAf+aBe+Acd)+Ab^2f)}{\sqrt{b^2-4ac}\,\left(f\left(a^2f-abe+b^2d\right)+ac\left(e^2-2df\right)-bcde+c^2d^2\right)}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)*(d + e*x + f*x^2)),x]
```

```
[Out] -(((A*b^2*f + 2*c*(A*c*d + a*B*e - a*A*f) - b*(B*c*d + A*c*e + a*B*f))*ArcT
anh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])/(Sqrt[b^2 - 4*a*c]*(c^2*d^2 - b*c*d*e +
 f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f)))) + ((B*(c*d*e - 2*b*d*f +
a*e*f) - A*(c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2))*ArcTanh[(e + 2*f*x)/Sqrt[e^
2 - 4*d*f]])/(Sqrt[e^2 - 4*d*f]*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2
*f) + a*c*(e^2 - 2*d*f))) + ((B*c*d - A*c*e + A*b*f - a*B*f)*Log[a + b*x +
c*x^2])/(2*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*
f))) - ((B*c*d - A*c*e + A*b*f - a*B*f)*Log[d + e*x + f*x^2])/(2*(c^2*d^2 -
 b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f)))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 1036

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*((d_) + (e_.)*
(x_) + (f_.)*(x_)^2)), x_Symbol] :> With[{q = Simplify[c^2*d^2 - b*c*d*e +
a*c*e^2 + b^2*d*f - 2*a*c*d*f - a*b*e*f + a^2*f^2]}, Dist[1/q, Int[Simp[g*c
^2*d - g*b*c*e + a*h*c*e + g*b^2*f - a*b*h*f - a*g*c*f + c*(h*c*d - g*c*e +
 g*b*f - a*h*f)*x, x]/(a + b*x + c*x^2), x], x] + Dist[1/q, Int[Simp[(-h)*c
*d*e + g*c*e^2 + b*h*d*f - g*c*d*f - g*b*e*f + a*g*f^2 - f*(h*c*d - g*c*e +
 g*b*f - a*h*f)*x, x]/(d + e*x + f*x^2), x], x] /; NeQ[q, 0]] /; FreeQ[{a,
b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\begin{aligned}
\int \frac{A+Bx}{(a+bx+cx^2)(d+ex+fx^2)}\,dx &= \frac{\int \frac{aB(ce-bf)+A\left(c^2d+b^2f-c(be+af)\right)+c(Bcd-Ace+Abf-aBf)x}{a+bx+cx^2}\,dx}{c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)}+\frac{\int \frac{-Af(be-af)\cdots}{\cdots}}{c^2d^2-bc\cdots}\\
&= \frac{(Bcd-Ace+Abf-aBf)\int \frac{b+2cx}{a+bx+cx^2}\,dx}{2\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)\right)}-\frac{(Bcd\cdots}{2\left(c^2d^2-bc\cdots\right.}\\
&= \frac{(Bcd-Ace+Abf-aBf)\log\left(a+bx+cx^2\right)}{2\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)\right)}-\frac{(Bcd-\cdots}{2\left(c^2d^2-bc\cdots\right.}\\
&= -\frac{\left(Ab^2f+2c(Acd+aBe-aAf)-b(Bcd+Ace+aBf)\right)\tanh^{-1}\left(\cdots\right.}{\sqrt{b^2-4ac}\,\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-\cdots\right.\right.}
\end{aligned}$$

**Mathematica** [A]
time = 0.31, size = 267, normalized size = 0.66

$$\frac{\frac{2(Ab^2f+2c(Acd+aBe-aAf)-b(Bcd+Ace+aBf))\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)}{\sqrt{-b^2+4ac}}-\frac{2(B(cde-2bdf+aef)+A(-ce^2+2cdf+bef-2af^2))\tan^{-1}\left(\frac{e+2fx}{\sqrt{-e^2+4df}}\right)}{\sqrt{-e^2+4df}}+(Bcd-Ace+Abf-aBf)\log(a+x(b+cx))+(-Bcd+Ace-Abf+aBf)\log(d+x(e+fx))}{2\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)*(d + e*x + f*x^2)),x]
```

```
[Out] ((2*(A*b^2*f + 2*c*(A*c*d + a*B*e - a*A*f) - b*(B*c*d + A*c*e + a*B*f))*Arc
Tan[(b + 2*c*x)/Sqrt[-b^2 + 4*a*c]])/Sqrt[-b^2 + 4*a*c] - (2*(B*(c*d*e - 2*
b*d*f + a*e*f) + A*(-(c*e^2) + 2*c*d*f + b*e*f - 2*a*f^2))*ArcTan[(e + 2*f*
x)/Sqrt[-e^2 + 4*d*f]])/Sqrt[-e^2 + 4*d*f] + (B*c*d - A*c*e + A*b*f - a*B*f
)*Log[a + x*(b + c*x)] + (-(B*c*d) + A*c*e - A*b*f + a*B*f)*Log[d + x*(e +
f*x)])/(2*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f
)))
```

**Maple** [A]
time = 0.84, size = 384, normalized size = 0.95

| method | result |
|---|---|
| default | $\frac{\frac{\left(-Ab\,f^2+Acef+Ba\,f^2-Bcdf\right)\ln\left(f\,x^2+ex+d\right)}{2f}+\frac{2\left(Aa\,f^2-Abef-Acdf+Ac\,e^2+Bbdf-Bcde-\frac{\left(-Ab\,f^2+Acef+Ba\,f^2-Bcdf\right)e}{2f}\right)\arctan\left(\frac{2fx}{\sqrt{4df}}\right.}{\sqrt{4df-e^2}}}{a^2f^2-abef-2acdf+ac\,e^2+b^2df-bcde+c^2d^2}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/(a^2*f^2-a*b*e*f-2*a*c*d*f+a*c*e^2+b^2*d*f-b*c*d*e+c^2*d^2)*(1/2*(-A*b*f^
2+A*c*e*f+B*a*f^2-B*c*d*f)/f*ln(f*x^2+e*x+d)+2*(A*a*f^2-A*b*e*f-A*c*d*f+A*c
*e^2+B*b*d*f-B*c*d*e-1/2*(-A*b*f^2+A*c*e*f+B*a*f^2-B*c*d*f)*e/f)/(4*d*f-e^2
)^(1/2)*arctan((2*f*x+e)/(4*d*f-e^2)^(1/2)))+1/(a^2*f^2-a*b*e*f-2*a*c*d*f+a
*c*e^2+b^2*d*f-b*c*d*e+c^2*d^2)*(1/2*(A*b*c*f-A*c^2*e-B*a*c*f+B*c^2*d)/c*ln
(c*x^2+b*x+a)+2*(-A*a*c*f+A*b^2*f-A*b*c*e+A*c^2*d-B*a*b*f+B*a*c*e-1/2*(A*b*
c*f-A*c^2*e-B*a*c*f+B*c^2*d)*b/c)/(4*a*c-b^2)^(1/2)*arctan((2*c*x+b)/(4*a*c
-b^2)^(1/2)))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)/(f*x**2+e*x+d),x)
```

```
[Out] Timed out
```

**Giac** [**A**]
time = 2.06, size = 416, normalized size = 1.02

$$\frac{(Bcd - Baf + Abf - Ace)\log\left(cx^2 + bx + a\right)}{2\left(c^2d^2 + b^2df - 2\,acdf + a^2f^2 - bcde - abfe + ace^2\right)} - \frac{(Bcd - Baf + Abf - Ace)\log\left(fx^2 + xe + d\right)}{2\left(c^2d^2 + b^2df - 2\,acdf + a^2f^2 - bcde - abfe + ace^2\right)} - \frac{\left(Bbcd - 2\,Ac^2d + Babf - Ab^2f + 2\,Aacf - 2\,Bace + Abce\right)\arctan\left(\frac{2\,cx+b}{\sqrt{-b^2+4\,ac}}\right)}{\left(c^2d^2 + b^2df - 2\,acdf + a^2f^2 - bcde - abfe + ace^2\right)\sqrt{-b^2+4\,ac}} + \frac{\left(2\,Bbdf - 2\,Acdf + 2\,Aaf^2 - Bcde - Bafe - Abfe + Ace^2\right)\arctan\left(\frac{2\,fx+e}{\sqrt{4\,df-e^2}}\right)}{\left(c^2d^2 + b^2df - 2\,acdf + a^2f^2 - bcde - abfe + ace^2\right)\sqrt{4\,df-e^2}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] 1/2*(B*c*d - B*a*f + A*b*f - A*c*e)*log(c*x^2 + b*x + a)/(c^2*d^2 + b^2*d*f
 - 2*a*c*d*f + a^2*f^2 - b*c*d*e - a*b*f*e + a*c*e^2) - 1/2*(B*c*d - B*a*f
+ A*b*f - A*c*e)*log(f*x^2 + x*e + d)/(c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*
f^2 - b*c*d*e - a*b*f*e + a*c*e^2) - (B*b*c*d - 2*A*c^2*d + B*a*b*f - A*b^2
*f + 2*A*a*c*f - 2*B*a*c*e + A*b*c*e)*arctan((2*c*x + b)/sqrt(-b^2 + 4*a*c)
)/((c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*f^2 - b*c*d*e - a*b*f*e + a*c*e^2)*
sqrt(-b^2 + 4*a*c)) + (2*B*b*d*f - 2*A*c*d*f + 2*A*a*f^2 - B*c*d*e - B*a*f*
e - A*b*f*e + A*c*e^2)*arctan((2*f*x + e)/sqrt(4*d*f - e^2))/((c^2*d^2 + b^
2*d*f - 2*a*c*d*f + a^2*f^2 - b*c*d*e - a*b*f*e + a*c*e^2)*sqrt(4*d*f - e^2
))
```

**Mupad [F(-1)]**
time = 0.00, size = -1, normalized size = -0.00

Hanged

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a + b*x + c*x^2)*(d + e*x + f*x^2)),x)
[Out] \text{Hanged}
```

# 3.16 $\int \frac{A+Bx}{\left(a+bx+cx^2\right)^2\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=1075

$$-\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+c(Ab^2f+2c(Acd+aBe-aAf)-b(B}{\left(b^2-4ac\right)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\left(a+bx+cx^2\right)}$$

```
[Out] (-A*c*(2*a*c*e-b*(a*f+c*d))-(A*b-B*a)*(2*c^2*d+b^2*f-c*(2*a*f+b*e))-c*(A*b^
2*f+2*c*(-A*a*f+A*c*d+B*a*e)-b*(A*c*e+B*a*f+B*c*d))*x)/(-4*a*c+b^2)/((-a*f+
c*d)^2-(-a*e+b*d)*(-b*f+c*e))/(c*x^2+b*x+a)-(b^5*(-A*e+B*d)*f^2-2*b^4*f*(B*
c*d*e-A*(a*f^2-c*d*f+c*e^2))-4*c^2*(A*(c^3*d^3-3*a^3*f^3-a^2*c*f*(-7*d*f+e^
2)+a*c^2*d*(-5*d*f+3*e^2))-a*B*e*(c^2*d^2-3*a^2*f^2-a*c*(-2*d*f+e^2)))-4*b^
2*c*(B*c^2*d^2*e+A*f*(2*c^2*d^2+3*a^2*f^2+3*a*c*(-d*f+e^2)))+2*b*c*(B*(c^3*
d^3+3*a^3*f^3+a*c^2*d*(-7*d*f+e^2)+3*a^2*c*f*(d*f+e^2))+A*c*e*(3*c^2*d^2+3*
a^2*f^2+a*c*(2*d*f+3*e^2)))-b^3*(A*c*e*(-4*a*f^2-2*c*d*f+c*e^2)+B*(4*a*c*d*
f^2+a^2*f^3-c^2*d*(5*d*f+e^2))))*arctanh((2*c*x+b)/(-4*a*c+b^2)^(1/2))/(-4*
a*c+b^2)^(3/2)/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))^2+1
/2*(A*(-b*f+c*e)*(f*(-2*a*f+b*e)-c*(-2*d*f+e^2))-B*(2*c*d*f*(-a*f+b*e)-f^2*
(-a^2*f+b^2*d)-c^2*d*(-d*f+e^2)))*ln(c*x^2+b*x+a)/(c^2*d^2+f*(a^2*f-a*b*e+b
^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))^2-1/2*(A*(-b*f+c*e)*(f*(-2*a*f+b*e)-c*(-2*d
*f+e^2))-B*(2*c*d*f*(-a*f+b*e)-f^2*(-a^2*f+b^2*d)-c^2*d*(-d*f+e^2)))*ln(f*x
^2+e*x+d)/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))^2+(B*(c^
2*d*e*(-3*d*f+e^2)-2*c*d*f*(-a*e*f-2*b*d*f+b*e^2)+f^2*(a^2*e*f-4*a*b*d*f+b^
2*d*e))-A*(c^2*(2*d^2*f^2-4*d*e^2*f+e^4)-f^2*(2*a*b*e*f-2*a^2*f^2-b^2*(-2*d
*f+e^2))+2*c*f*(a*f*(-2*d*f+e^2)-b*(-3*d*e*f+e^3))))*arctanh((2*f*x+e)/(-4*
d*f+e^2)^(1/2))/(c^2*d^2+f*(a^2*f-a*b*e+b^2*d)-c*(b*d*e-a*(-2*d*f+e^2)))^2/
(-4*d*f+e^2)^(1/2)
```

**Rubi** [A]
time = 2.53, antiderivative size = 1067, normalized size of antiderivative = 0.99, number of steps used = 10, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {1030, 1086, 648, 632, 212, 642}

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)^2*(d + e*x + f*x^2)),x]
```

```
[Out] -((A*c*(2*a*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e +
2*a*f)) + c*(A*b^2*f + 2*c*(A*c*d + a*B*e - a*A*f) - b*(B*c*d + A*c*e + a*B
*f))*x)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(a + b*x +
 c*x^2))) - ((b^5*(B*d - A*e)*f^2 - 2*b^4*f*(B*c*d*e - a*A*f^2 - A*c*(e^2 -
 d*f)) - 4*c^2*(A*(c^3*d^3 - 3*a^3*f^3 - a^2*c*f*(e^2 - 7*d*f) + a*c^2*d*(3
```

```
*e^2 - 5*d*f)) - a*B*e*(c^2*d^2 - 3*a^2*f^2 - a*c*(e^2 - 2*d*f))) - 4*b^2*(
B*c^3*d^2*e + A*c*f*(2*c^2*d^2 + 3*a^2*f^2 + 3*a*c*(e^2 - d*f))) + 2*b*c*(B
*(c^3*d^3 + 3*a^3*f^3 + a*c^2*d*(e^2 - 7*d*f) + 3*a^2*c*f*(e^2 + d*f)) + A*
c*e*(3*c^2*d^2 + 3*a^2*f^2 + a*c*(3*e^2 + 2*d*f))) - b^3*(A*c*e*(c*e^2 - 2*
c*d*f - 4*a*f^2) + B*(4*a*c*d*f^2 + a^2*f^3 - c^2*d*(e^2 + 5*d*f))))*ArcTan
h[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])/((b^2 - 4*a*c)^(3/2)*(c^2*d^2 - b*c*d*e +
 f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))^2) + ((B*(c^2*d*e*(e^2 - 3*
d*f) - 2*c*d*f*(b*e^2 - 2*b*d*f - a*e*f) + f^2*(b^2*d*e - 4*a*b*d*f + a^2*e
*f)) - A*(c^2*(e^4 - 4*d*e^2*f + 2*d^2*f^2) - f^2*(2*a*b*e*f - 2*a^2*f^2 -
b^2*(e^2 - 2*d*f)) + 2*c*f*(a*f*(e^2 - 2*d*f) - b*(e^3 - 3*d*e*f))))*ArcTan
h[(e + 2*f*x)/Sqrt[e^2 - 4*d*f]])/(Sqrt[e^2 - 4*d*f]*(c^2*d^2 - b*c*d*e + f
*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))^2) + ((A*(c*e - b*f)*(f*(b*e
- 2*a*f) - c*(e^2 - 2*d*f)) - B*(2*c*d*f*(b*e - a*f) - f^2*(b^2*d - a^2*f)
- c^2*d*(e^2 - d*f)))*Log[a + b*x + c*x^2])/(2*(c^2*d^2 - b*c*d*e + f*(b^2*
d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))^2) - ((A*(c*e - b*f)*(f*(b*e - 2*a*
f) - c*(e^2 - 2*d*f)) - B*(2*c*d*f*(b*e - a*f) - f^2*(b^2*d - a^2*f) - c^2*
d*(e^2 - d*f)))*Log[d + e*x + f*x^2])/(2*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*
b*e + a^2*f) + a*c*(e^2 - 2*d*f))^2)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
```

```
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1086

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*((d_) + (e_.)*(x_) + (f_.)*(x_)^2)), x_Symbol] :> With[{q = c^2*d^2 - b*c*
d*e + a*c*e^2 + b^2*d*f - 2*a*c*d*f - a*b*e*f + a^2*f^2}, Dist[1/q, Int[(A*
c^2*d - a*c*C*d - A*b*c*e + a*B*c*e + A*b^2*f - a*b*B*f - a*A*c*f + a^2*C*f
 + c*(B*c*d - b*C*d - A*c*e + a*C*e + A*b*f - a*B*f)*x)/(a + b*x + c*x^2),
x], x] + Dist[1/q, Int[(c*C*d^2 - B*c*d*e + A*c*e^2 + b*B*d*f - A*c*d*f - a
*C*d*f - A*b*e*f + a*A*f^2 - f*(B*c*d - b*C*d - A*c*e + a*C*e + A*b*f - a*B
*f)*x)/(d + e*x + f*x^2), x], x] /; NeQ[q, 0]] /; FreeQ[{a, b, c, d, e, f,
A, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\int \frac{A+Bx}{(a+bx+cx^2)^2(d+ex+fx^2)}\,dx = -\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+}{(b^2-4ac)\left((cd-af)^2-(bd-ae\right.}$$

$$= -\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+}{(b^2-4ac)\left((cd-af)^2-(bd-ae\right.}$$

$$= -\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+}{(b^2-4ac)\left((cd-af)^2-(bd-ae\right.}$$

$$= -\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+}{(b^2-4ac)\left((cd-af)^2-(bd-ae\right.}$$

$$= -\frac{Ac(2ace-b(cd+af))+(Ab-aB)\left(2c^2d+b^2f-c(be+2af)\right)+}{(b^2-4ac)\left((cd-af)^2-(bd-ae\right.}$$

**Mathematica [A]**
time = 4.41, size = 952, normalized size = 0.89

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)^2*(d + e*x + f*x^2)),x]
```

```
[Out] ((-2*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))*(A
*(b^3*f + b^2*c*(-e + f*x) + b*c*(-3*a*f + c*(d - e*x)) + 2*c^2*(c*d*x + a*
(e - f*x))) + B*(2*a^2*c*f - b*c^2*d*x - a*(b^2*f + 2*c^2*(d - e*x) + b*c*(
-e + f*x)))))/((b^2 - 4*a*c)*(a + x*(b + c*x))) - (2*(b^5*(B*d - A*e)*f^2 +
 2*b^4*f*(-(B*c*d*e) + a*A*f^2 + A*c*(e^2 - d*f)) - 4*b^2*(B*c^3*d^2*e + A*
c*f*(2*c^2*d^2 + 3*a^2*f^2 + 3*a*c*(e^2 - d*f))) + 2*b*c*(B*(c^3*d^3 + 3*a^
3*f^3 + a*c^2*d*(e^2 - 7*d*f) + 3*a^2*c*f*(e^2 + d*f)) + A*c*e*(3*c^2*d^2 +
 3*a^2*f^2 + a*c*(3*e^2 + 2*d*f))) + 4*c^2*(a*B*e*(c^2*d^2 - 3*a^2*f^2 - a*
c*(e^2 - 2*d*f)) + A*(-(c^3*d^3) + 3*a^3*f^3 + a^2*c*f*(e^2 - 7*d*f) + a*c^
2*d*(-3*e^2 + 5*d*f))) + b^3*(A*c*e*(-(c*e^2) + 2*c*d*f + 4*a*f^2) + B*(-4*
a*c*d*f^2 - a^2*f^3 + c^2*d*(e^2 + 5*d*f))))*ArcTan[(b + 2*c*x)/Sqrt[-b^2 +
 4*a*c]])/(-b^2 + 4*a*c)^(3/2) + (2*(B*(c^2*d*e*(-e^2 + 3*d*f) - 2*c*d*f*(-
(b*e^2) + 2*b*d*f + a*e*f) + f^2*(-(b^2*d*e) + 4*a*b*d*f - a^2*e*f)) + A*(c
^2*(e^4 - 4*d*e^2*f + 2*d^2*f^2) + f^2*(-2*a*b*e*f + 2*a^2*f^2 + b^2*(e^2 -
 2*d*f)) + 2*c*f*(a*f*(e^2 - 2*d*f) - b*(e^3 - 3*d*e*f)))))*ArcTan[(e + 2*f*
x)/Sqrt[-e^2 + 4*d*f]])/Sqrt[-e^2 + 4*d*f] - (A*(c*e - b*f)*(f*(-(b*e) + 2*
a*f) + c*(e^2 - 2*d*f)) + B*(2*c*d*f*(b*e - a*f) + f^2*(-(b^2*d) + a^2*f) +
 c^2*d*(-e^2 + d*f)))*Log[a + x*(b + c*x)] + (A*(c*e - b*f)*(f*(-(b*e) + 2*
```

```
a*f) + c*(e^2 - 2*d*f)) + B*(2*c*d*f*(b*e - a*f) + f^2*(-(b^2*d) + a^2*f) +
 c^2*d*(-e^2 + d*f)))*Log[d + x*(e + f*x)])/(2*(c^2*d^2 - b*c*d*e + f*(b^2*
d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))^2)
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 51469 vs. $2(1065) = 2130$.
time = 0.04, size = 51470, normalized size = 47.88

output too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x)
```

```
[Out] result too large to display
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)**2/(f*x**2+e*x+d),x)
```

```
[Out] Timed out
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 3226 vs. $2(1095) = 2190$.
time = 2.30, size = 3226, normalized size = 3.00

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)^2/(f*x^2+e*x+d),x, algorithm="giac")
[Out] -1/2*(B*c^2*d^2*f - B*b^2*d*f^2 - 2*B*a*c*d*f^2 + 2*A*b*c*d*f^2 + B*a^2*f^3
 - 2*A*a*b*f^3 + 2*B*b*c*d*f*e - 2*A*c^2*d*f*e + A*b^2*f^2*e + 2*A*a*c*f^2*
e - B*c^2*d*e^2 - 2*A*b*c*f*e^2 + A*c^2*e^3)*log(c*x^2 + b*x + a)/(c^4*d^4
+ 2*b^2*c^2*d^3*f - 4*a*c^3*d^3*f + b^4*d^2*f^2 - 4*a*b^2*c*d^2*f^2 + 6*a^2
*c^2*d^2*f^2 + 2*a^2*b^2*d*f^3 - 4*a^3*c*d*f^3 + a^4*f^4 - 2*b*c^3*d^3*e -
2*b^3*c*d^2*f*e + 2*a*b*c^2*d^2*f*e - 2*a*b^3*d*f^2*e + 2*a^2*b*c*d*f^2*e -
 2*a^3*b*f^3*e + b^2*c^2*d^2*e^2 + 2*a*c^3*d^2*e^2 + 4*a*b^2*c*d*f*e^2 - 4*
a^2*c^2*d*f*e^2 + a^2*b^2*f^2*e^2 + 2*a^3*c*f^2*e^2 - 2*a*b*c^2*d*e^3 - 2*a
^2*b*c*f*e^3 + a^2*c^2*e^4) + 1/2*(B*c^2*d^2*f - B*b^2*d*f^2 - 2*B*a*c*d*f^
2 + 2*A*b*c*d*f^2 + B*a^2*f^3 - 2*A*a*b*f^3 + 2*B*b*c*d*f*e - 2*A*c^2*d*f*e
 + A*b^2*f^2*e + 2*A*a*c*f^2*e - B*c^2*d*e^2 - 2*A*b*c*f*e^2 + A*c^2*e^3)*l
og(f*x^2 + x*e + d)/(c^4*d^4 + 2*b^2*c^2*d^3*f - 4*a*c^3*d^3*f + b^4*d^2*f^
2 - 4*a*b^2*c*d^2*f^2 + 6*a^2*c^2*d^2*f^2 + 2*a^2*b^2*d*f^3 - 4*a^3*c*d*f^3
 + a^4*f^4 - 2*b*c^3*d^3*e - 2*b^3*c*d^2*f*e + 2*a*b*c^2*d^2*f*e - 2*a*b^3*
d*f^2*e + 2*a^2*b*c*d*f^2*e - 2*a^3*b*f^3*e + b^2*c^2*d^2*e^2 + 2*a*c^3*d^2
*e^2 + 4*a*b^2*c*d*f*e^2 - 4*a^2*c^2*d*f*e^2 + a^2*b^2*f^2*e^2 + 2*a^3*c*f^
2*e^2 - 2*a*b*c^2*d*e^3 - 2*a^2*b*c*f*e^3 + a^2*c^2*e^4) + (2*B*b*c^4*d^3 -
 4*A*c^5*d^3 + 5*B*b^3*c^2*d^2*f - 14*B*a*b*c^3*d^2*f - 8*A*b^2*c^3*d^2*f +
 20*A*a*c^4*d^2*f + B*b^5*d*f^2 - 4*B*a*b^3*c*d*f^2 - 2*A*b^4*c*d*f^2 + 6*B
*a^2*b*c^2*d*f^2 + 12*A*a*b^2*c^2*d*f^2 - 28*A*a^2*c^3*d*f^2 - B*a^2*b^3*f^
3 + 2*A*a*b^4*f^3 + 6*B*a^3*b*c*f^3 - 12*A*a^2*b^2*c*f^3 + 12*A*a^3*c^2*f^3
 - 4*B*b^2*c^3*d^2*e + 4*B*a*c^4*d^2*e + 6*A*b*c^4*d^2*e - 2*B*b^4*c*d*f*e
+ 2*A*b^3*c^2*d*f*e + 8*B*a^2*c^3*d*f*e + 4*A*a*b*c^3*d*f*e - A*b^5*f^2*e +
 4*A*a*b^3*c*f^2*e - 12*B*a^3*c^2*f^2*e + 6*A*a^2*b*c^2*f^2*e + B*b^3*c^2*d
*e^2 + 2*B*a*b*c^3*d*e^2 - 12*A*a*c^4*d*e^2 + 2*A*b^4*c*f*e^2 + 6*B*a^2*b*c
^2*f*e^2 - 12*A*a*b^2*c^2*f*e^2 + 4*A*a^2*c^3*f*e^2 - A*b^3*c^2*e^3 - 4*B*a
^2*c^3*e^3 + 6*A*a*b*c^3*e^3)*arctan((2*c*x + b)/sqrt(-b^2 + 4*a*c))/((b^2*
c^4*d^4 - 4*a*c^5*d^4 + 2*b^4*c^2*d^3*f - 12*a*b^2*c^3*d^3*f + 16*a^2*c^4*d
^3*f + b^6*d^2*f^2 - 8*a*b^4*c*d^2*f^2 + 22*a^2*b^2*c^2*d^2*f^2 - 24*a^3*c^
3*d^2*f^2 + 2*a^2*b^4*d*f^3 - 12*a^3*b^2*c*d*f^3 + 16*a^4*c^2*d*f^3 + a^4*b
^2*f^4 - 4*a^5*c*f^4 - 2*b^3*c^3*d^3*e + 8*a*b*c^4*d^3*e - 2*b^5*c*d^2*f*e
+ 10*a*b^3*c^2*d^2*f*e - 8*a^2*b*c^3*d^2*f*e - 2*a*b^5*d*f^2*e + 10*a^2*b^3
*c*d*f^2*e - 8*a^3*b*c^2*d*f^2*e - 2*a^3*b^3*f^3*e + 8*a^4*b*c*f^3*e + b^4*
c^2*d^2*e^2 - 2*a*b^2*c^3*d^2*e^2 - 8*a^2*c^4*d^2*e^2 + 4*a*b^4*c*d*f*e^2 -
```

```
 20*a^2*b^2*c^2*d*f*e^2 + 16*a^3*c^3*d*f*e^2 + a^2*b^4*f^2*e^2 - 2*a^3*b^2*
c*f^2*e^2 - 8*a^4*c^2*f^2*e^2 - 2*a*b^3*c^2*d*e^3 + 8*a^2*b*c^3*d*e^3 - 2*a
^2*b^3*c*f*e^3 + 8*a^3*b*c^2*f*e^3 + a^2*b^2*c^2*e^4 - 4*a^3*c^3*e^4)*sqrt(
-b^2 + 4*a*c)) - (4*B*b*c*d^2*f^2 - 2*A*c^2*d^2*f^2 - 4*B*a*b*d*f^3 + 2*A*b
^2*d*f^3 + 4*A*a*c*d*f^3 - 2*A*a^2*f^4 - 3*B*c^2*d^2*f*e + B*b^2*d*f^2*e +
2*B*a*c*d*f^2*e - 6*A*b*c*d*f^2*e + B*a^2*f^3*e + 2*A*a*b*f^3*e - 2*B*b*c*d
*f*e^2 + 4*A*c^2*d*f*e^2 - A*b^2*f^2*e^2 - 2*A*a*c*f^2*e^2 + B*c^2*d*e^3 +
2*A*b*c*f*e^3 - A*c^2*e^4)*arctan((2*f*x + e)/sqrt(4*d*f - e^2))/((c^4*d^4
+ 2*b^2*c^2*d^3*f - 4*a*c^3*d^3*f + b^4*d^2*f^2 - 4*a*b^2*c*d^2*f^2 + 6*a^2
*c^2*d^2*f^2 + 2*a^2*b^2*d*f^3 - 4*a^3*c*d*f^3 + a^4*f^4 - 2*b*c^3*d^3*e -
2*b^3*c*d^2*f*e + 2*a*b*c^2*d^2*f*e - 2*a*b^3*d*f^2*e + 2*a^2*b*c*d*f^2*e -
 2*a^3*b*f^3*e + b^2*c^2*d^2*e^2 + 2*a*c^3*d^2*e^2 + 4*a*b^2*c*d*f*e^2 - 4*
a^2*c^2*d*f*e^2 + a^2*b^2*f^2*e^2 + 2*a^3*c*f^2*e^2 - 2*a*b*c^2*d*e^3 - 2*a
^2*b*c*f*e^3 + a^2*c^2*e^4)*sqrt(4*d*f - e^2)) + (2*B*a*c^4*d^3 - A*b*c^4*d
^3 + 3*B*a*b^2*c^2*d^2*f - 2*A*b^3*c^2*d^2*f - 6*B*a^2*c^3*d^2*f + 5*A*a*b*
c^3*d^2*f + B*a*b^4*d*f^2 - A*b^5*d*f^2 - 4*B*a^2*b^2*c*d*f^2 + 5*A*a*b^3*c
*d*f^2 + 6*B*a^3*c^2*d*f^2 - 7*A*a^2*b*c^2*d*f^2 + B*a^3*b^2*f^3 - A*a^2*b^
3*f^3 - 2*B*a^4*c*f^3 + 3*A*a^3*b*c*f^3 - 3*B*a*b*c^3*d^2*e + 2*A*b^2*c^3*d
^2*e - 2*A*a*c^4*d^2*e - 2*B*a*b^3*c*d*f*e + 2*A*b^4*c*d*f*e + 2*B*a^2*b*c^
2*d*f*e - 6*A*a*b^2*c^2*d*f*e + 4*A*a^2*c^3*d*f*e - B*a^2*b^3*f^2*e + A*a*b
^4*f^2*e + B*a^3*b*c*f^2*e - 2*A*a^2*b^2*c*f^2*e - 2*A*a^3*c^2*f^2*e + B*a*
b^2*c^2*d*e^2 - A*b^3*c^2*d*e^2 + 2*B*a^2*c^3*d*e^2 + A*a*b*c^3*d*e^2 + 2*B
*a^2*b^2*c*f*e^2 - 2*A*a*b^3*c*f*e^2 - 2*B*a^3*c^2*f*e^2 + 5*A*a^2*b*c^2*f*
e^2 - B*a^2*b*c^2*e^3 + A*a*b^2*c^2*e^3 - 2*A*a^2*c^3*e^3 + (B*b*c^4*d^3 -
2*A*c^5*d^3 + B*b^3*c^2*d^2*f - B*a*b*c^3*d^2*f - 3*A*b^2*c^3*d^2*f + 6*A*a
*c^4*d^2*f + B*a*b^3*c*d*f^2 - A*b^4*c*d*f^2 - B*a^2*b*c^2*d*f^2 + 4*A*a*b^
2*c^2*d*f^2 - 6*A*a^2*c^3*d*f^2 + B*a^3*b*c*f^3 - A*a^2*b^2*c*f^3 + 2*A*a^3
*c^2*f^3 - B*b^2*c^3*d^2*e - 2*B*a*c^4*d^2*e + 3*A*b*c^4*d^2*e - 4*B*a*b^2*
c^2*d*f*e + 2*A*b^3*c^2*d*f*e + 4*B*a^2*c^3*d*f*e - 2*A*a*b*c^3*d*f*e - B*a
^2*b^2*c*f^2*e + A*a*b^3*c*f^2*e - 2*B*a^3*c^2*f^2*e - A*a^2*b*c^2*f^2*e +
3*B*a*b*c^3*d*e^2 - A*b^2*c^3*d*e^2 - 2*A*a*c^4*d*e^2 + 3*B*a^2*b*c^2*f*e^2
 - 2*A*a*b^2*c^2*f*e^2 + 2*A*a^2*c^3*f*e^2 - 2*B*a^2*c^3*e^3 + A*a*b*c^3*e^
3)*x)/((c^2*d^2 + b^2*d*f - 2*a*c*d*f + a^2*f^2...
```

**Mupad [B]**
time = 30.31, size = 2500, normalized size = 2.33

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a + b*x + c*x^2)^2*(d + e*x + f*x^2)),x)
```

```
[Out] symsum(log((x*(4*A^3*b^3*c^4*f^6 + 16*B^3*a^3*c^4*f^6 - 3*B^3*a^2*b^2*c^3*f
^6 + 4*B^3*a^2*c^5*e^2*f^4 + B^3*b^2*c^5*d^2*f^4 - 16*A^3*a*b*c^5*f^6 + 16*
A^3*a*c^6*e*f^5 + 20*A^2*B*a^2*c^5*f^6 - 3*A^2*B*b^4*c^3*f^6 + 4*A^2*B*c^7*
```

d^2*f^4 - 16*B^3*a^2*c^5*d*f^5 - 4*A^3*b^2*c^5*e*f^5 + 6*B^3*a*b^2*c^4*d*f^
5 - 4*B^3*a^2*b*c^4*e*f^5 + A^2*B*b^2*c^5*e^2*f^4 - 24*A^2*B*a*c^6*d*f^5 +
6*A*B^2*a*b^3*c^3*f^6 - 28*A*B^2*a^2*b*c^4*f^6 + 8*A^2*B*a*b^2*c^4*f^6 - 4*
A*B^2*b*c^6*d^2*f^4 + 8*A*B^2*a^2*c^5*e*f^5 - 6*A*B^2*b^3*c^4*d*f^5 + 8*A^2
*B*b^2*c^5*d*f^5 + 2*A^2*B*b^3*c^4*e*f^5 - 4*B^3*a*b*c^5*d*e*f^4 - 4*A*B^2*
a*b*c^5*e^2*f^4 + 2*A*B^2*a*b^2*c^4*e*f^5 + 2*A*B^2*b^2*c^5*d*e*f^4 + 16*A*
B^2*a*b*c^5*d*f^5 - 12*A^2*B*a*b*c^5*e*f^5 + 8*A*B^2*a*c^6*d*e*f^4 - 4*A^2*
B*b*c^6*d*e*f^4))/(16*a^2*c^6*d^4 + a^4*b^4*f^4 + 16*a^4*c^4*e^4 + b^4*c^4*
d^4 + 16*a^6*c^2*f^4 + b^8*d^2*f^2 - 8*a*b^2*c^5*d^4 - 8*a^5*b^2*c*f^4 + 2*
a^2*b^6*d*f^3 - 2*a^3*b^5*e*f^3 - 64*a^3*c^5*d^3*f - 64*a^5*c^3*d*f^3 - 2*b
^5*c^3*d^3*e + 2*b^6*c^2*d^3*f + a^2*b^4*c^2*e^4 - 8*a^3*b^2*c^3*e^4 + 32*a
^3*c^5*d^2*e^2 + a^2*b^6*e^2*f^2 + 96*a^4*c^4*d^2*f^2 + b^6*c^2*d^2*e^2 + 3
2*a^5*c^3*e^2*f^2 - 2*a*b^7*d*e*f^2 - 2*b^7*c*d^2*e*f + 54*a^2*b^4*c^2*d^2*
f^2 - 112*a^3*b^2*c^3*d^2*f^2 + 16*a*b^3*c^4*d^3*e - 2*a*b^5*c^2*d*e^3 - 32
*a^2*b*c^5*d^3*e - 32*a^3*b*c^4*d*e^3 - 20*a*b^4*c^3*d^3*f - 12*a*b^6*c*d^2
*f^2 - 20*a^3*b^4*c*d*f^3 - 2*a^2*b^5*c*e^3*f - 32*a^4*b*c^3*e^3*f + 16*a^4
*b^3*c*e*f^3 - 32*a^5*b*c^2*e*f^3 - 64*a^4*c^4*d*e^2*f - 6*a*b^4*c^3*d^2*e^
2 + 16*a^2*b^3*c^3*d*e^3 + 64*a^2*b^2*c^4*d^3*f + 64*a^4*b^2*c^2*d*f^3 + 16
*a^3*b^3*c^2*e^3*f - 6*a^3*b^4*c*e^2*f^2 - 48*a^2*b^3*c^3*d^2*e*f - 36*a^2*
b^4*c^2*d*e^2*f + 96*a^3*b^2*c^3*d*e^2*f - 48*a^3*b^3*c^2*d*e*f^2 + 4*a*b^6
*c*d*e^2*f + 18*a*b^5*c^2*d^2*e*f + 18*a^2*b^5*c*d*e*f^2 + 32*a^3*b*c^4*d^2
*e*f + 32*a^4*b*c^3*d*e*f^2) - root(48416*a^6*b^2*c^6*d^4*e^2*f^4*z^4 - 415
44*a^5*b^4*c^5*d^4*e^2*f^4*z^4 - 31872*a^7*b^2*c^5*d^3*e^2*f^5*z^4 - 31872*
a^5*b^2*c^7*d^5*e^2*f^3*z^4 - 29184*a^6*b^2*c^6*d^3*e^4*f^3*z^4 + 28800*a^5
*b^4*c^5*d^3*e^4*f^3*z^4 + 21510*a^4*b^6*c^4*d^4*e^2*f^4*z^4 + 21408*a^6*b^
4*c^4*d^3*e^2*f^5*z^4 + 21408*a^4*b^4*c^6*d^5*e^2*f^3*z^4 - 18112*a^7*b^3*c
^4*d^2*e^3*f^5*z^4 - 18112*a^4*b^3*c^7*d^5*e^3*f^2*z^4 - 15600*a^5*b^5*c^4*
d^3*e^3*f^4*z^4 - 15600*a^4*b^5*c^5*d^4*e^3*f^3*z^4 + 15296*a^6*b^3*c^5*d^3
*e^3*f^4*z^4 + 15296*a^5*b^3*c^6*d^4*e^3*f^3*z^4 + 14016*a^7*b^2*c^5*d^2*e^
4*f^4*z^4 + 14016*a^5*b^2*c^7*d^4*e^4*f^2*z^4 - 13920*a^4*b^6*c^4*d^3*e^4*f
^3*z^4 - 11648*a^6*b^3*c^5*d^2*e^5*f^3*z^4 - 11648*a^5*b^3*c^6*d^3*e^5*f^2*
z^4 + 10432*a^6*b^2*c^6*d^2*e^6*f^2*z^4 + 9008*a^6*b^5*c^3*d^2*e^3*f^5*z^4
+ 9008*a^3*b^5*c^6*d^5*e^3*f^2*z^4 + 8544*a^5*b^5*c^4*d^2*e^5*f^3*z^4 + 854
4*a^4*b^5*c^5*d^3*e^5*f^2*z^4 - 8496*a^5*b^4*c^5*d^2*e^6*f^2*z^4 + 7488*a^8
*b^2*c^4*d^2*e^2*f^6*z^4 + 7488*a^4*b^2*c^8*d^6*e^2*f^2*z^4 + 7380*a^4*b^7*
c^3*d^3*e^3*f^4*z^4 + 7380*a^3*b^7*c^4*d^4*e^3*f^3*z^4 - 6720*a^3*b^8*c^3*d
^4*e^2*f^4*z^4 - 5784*a^5*b^6*c^3*d^3*e^2*f^5*z^4 - 5784*a^3*b^6*c^5*d^5*e^
2*f^3*z^4 - 3440*a^6*b^4*c^4*d^2*e^4*f^4*z^4 - 3440*a^4*b^4*c^6*d^4*e^4*f^2
*z^4 + 3360*a^3*b^8*c^3*d^3*e^4*f^3*z^4 + 3140*a^4*b^6*c^4*d^2*e^6*f^2*z^4
- 2760*a^4*b^7*c^3*d^2*e^5*f^3*z^4 - 2760*a^3*b^7*c^4*d^3*e^5*f^2*z^4 - 176
4*a^5*b^7*c^2*d^2*e^3*f^5*z^4 - 1764*a^2*b^7*c^5*d^5*e^3*f^2*z^4 - 1640*a^3
*b^9*c^2*d^3*e^3*f^4*z^4 - 1640*a^2*b^9*c^3*d^4*e^3*f^3*z^4 - 1604*a^6*b^6*
c^2*d^2*e^2*f^6*z^4 - 1604*a^2*b^6*c^6*d^6*e^2*f^2*z^4 - 1500*a^5*b^6*c^3*d
^2*e^4*f^4*z^4 - 1500*a^3*b^6*c^5*d^4*e^4*f^2*z^4 + 1140*a^2*b^10*c^2*d^4*e
^2*f^4*z^4 + 810*a^4*b^8*c^2*d^2*e^4*f^4*z^4 + 810*a^2*b^8*c^4*d^4*e^4*f^2*

z^4 - 544*a^3*b^8*c^3*d^2*e^6*f^2*z^4 + 416*a^3*b^9*c^2*d^2*e^5*f^3*z^4 + 4
16*a^2*b^9*c^3*d^3*e^5*f^2*z^4 - 384*a^2*b^10*c^2*d^3*e^4*f^3*z^4 + 180*a^4
*b^8*c^2*d^3*e^2*f^5*z^4 + 180*a^2*b^8*c^4*d^5*e^2*f^3*z^4 + 48*a^7*b^4*c^3
*d^2*e^2*f^6*z^4 + 48*a^3*b^4*c^7*d^6*e^2*f^2*z^4 + 36*a^2*b^10*c^2*d^2*e^6
*f^2*z^4 - 1024*a^10*b*c^3*d*e*f^8*z^4 - 1024*a^3*b*c^10*d^8*e*f*z^4 - 192*
a^8*b^5*c*d*e*f^8*z^4 - 192*a*b^5*c^8*d^8*e*f*z^4 + 16128*a^7*b^3*c^4*d^3*e
*f^6*z^4 + 16128*a^4*b^3*c^7*d^6*e*f^3*z^4 - 11712*a^6*b^5*c^3*d^3*e*f^6*z^
4 - 11712*a^3*b^5*c^6*d^6*e*f^3*z^4 + 11520*a^8*b*c^5*d^2*e^3*f^5*z^4 + 115
20*a^5*b*c^8*d^5*e^3*f^2*z^4 - 9984*a^6*b^3*c^5*d^4*e*f^5*z^4 - 9984*a^5*b^
3*c^6*d^5*e*f^4*z^4 + 8640*a^5*b^5*c^4*d^4*e*f^5*z^4 + 8640*a^4*b^5*c^5*d^5
*e*f^4*z^4 - 7424*a^7*b*c^6*d^3*e^3*f^4*z^4 - 7424*a^6*b*c^7*d^4*e^3*f^3*z^
4 - 6912*a^8*b^3*c^3*d^2*e*f^7*z^4 - 6912*a^3*b^3*c^8*d^7*e*f^2*z^4 + 4800*
a^7*b^3*c^4*d*e^5*f^4*z^4 + 4800*a^4*b^3*c^7*d^4*e^5*f*z^4 + 4608*a^7*b*c^6
*d^2*e^5*f^3*z^4 + 4608*a^6*b*c^7*d^3*e^5*f^2*z^4 - 4560*a^4*b^7*c^3*d^4*e*
f^5*z^4 - 4560*a^3*b^7*c^4*d^5*e*f^4*z^4 + 4176*a^5*b^7*c^2*d^3*e*f^6*z^4 +
 4176*a^2*b^7*c^5*d^6*e*f^3*z^4 + 3264*a^7*b^5*c^2*d^2*e*f^7*z^4 + 3264*a^2
*b^5*c^7*d^7*e*f^2*z^4 + 3008*a^8*b^3*c^3*d*e^3*f^6*z^4 + 3008*a^3*b^3*c^8*
d^6*e^3*f*z^4 + 2880*a^6*b^3*c^5*d*e^7*f^2*z^4 + 2880*a^5*b^3*c^6*d^2*e^7*f
*z^4 - 2240*a^7*b^4*c^3*d*e^4*f^5*z^4 - 2240*a^...

## 3.17 $\int \frac{g+hx}{(a+bx+cx^2)(ad+bdx+cdx^2)^2}\,dx$

**Optimal**. Leaf size=140

$$-\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)\,d^2\,(a+bx+cx^2)^2}+\frac{3(2cg-bh)(b+2cx)}{2(b^2-4ac)^2\,d^2\,(a+bx+cx^2)}-\frac{6c(2cg-bh)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}\,d^2}$$

```
[Out] 1/2*(-b*g+2*a*h-(-b*h+2*c*g)*x)/(-4*a*c+b^2)/d^2/(c*x^2+b*x+a)^2+3/2*(-b*h+
2*c*g)*(2*c*x+b)/(-4*a*c+b^2)^2/d^2/(c*x^2+b*x+a)-6*c*(-b*h+2*c*g)*arctanh(
(2*c*x+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(5/2)/d^2
```

**Rubi** [A]
time = 0.09, antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}} = 0.147$, Rules used = {1012, 652, 628, 632, 212}

$$\frac{3(b+2cx)(2cg-bh)}{2d^2(b^2-4ac)^2(a+bx+cx^2)}-\frac{-2ah+x(2cg-bh)+bg}{2d^2(b^2-4ac)(a+bx+cx^2)^2}-\frac{6c(2cg-bh)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{d^2(b^2-4ac)^{5/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(g + h*x)/((a + b*x + c*x^2)*(a*d + b*d*x + c*d*x^2)^2),x]
```

```
[Out] -1/2*(b*g - 2*a*h + (2*c*g - b*h)*x)/((b^2 - 4*a*c)*d^2*(a + b*x + c*x^2)^2
) + (3*(2*c*g - b*h)*(b + 2*c*x))/(2*(b^2 - 4*a*c)^2*d^2*(a + b*x + c*x^2))
 - (6*c*(2*c*g - b*h)*ArcTanh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])/((b^2 - 4*a*c
)^(5/2)*d^2)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 628

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Dist[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c))), Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Free
Q[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && NeQ[p, -3/2] && Int
egerQ[4*p]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
```

```
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 652

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Dist[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c))), Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &&
 NeQ[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && NeQ[p, -3/2]
```

Rule 1012

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(c/f)^p, Int[(g + h*
x)^m*(d + e*x + f*x^2)^(p + q), x], x] /; FreeQ[{a, b, c, d, e, f, g, h, p,
 q}, x] && EqQ[c*d - a*f, 0] && EqQ[b*d - a*e, 0] && (IntegerQ[p] || GtQ[c/
f, 0]) && ( !IntegerQ[q] || LeafCount[d + e*x + f*x^2] <= LeafCount[a + b*x
 + c*x^2])
```

Rubi steps

$$\int \frac{g+hx}{(a+bx+cx^2)\left(ad+bdx+cdx^2\right)^2}\,dx = \frac{\int \frac{g+hx}{(a+bx+cx^2)^3}\,dx}{d^2}$$

$$= -\frac{bg-2ah+(2cg-bh)x}{2\left(b^2-4ac\right)d^2\left(a+bx+cx^2\right)^2} - \frac{(3(2cg-bh))\int \frac{1}{(a+bx+cx^2)^2}\,d}{2\left(b^2-4ac\right)d^2}$$

$$= -\frac{bg-2ah+(2cg-bh)x}{2\left(b^2-4ac\right)d^2\left(a+bx+cx^2\right)^2} + \frac{3(2cg-bh)(b+2cx)}{2\left(b^2-4ac\right)^2d^2\left(a+bx+cx\right.}$$

$$= -\frac{bg-2ah+(2cg-bh)x}{2\left(b^2-4ac\right)d^2\left(a+bx+cx^2\right)^2} + \frac{3(2cg-bh)(b+2cx)}{2\left(b^2-4ac\right)^2d^2\left(a+bx+cx\right.}$$

$$= -\frac{bg-2ah+(2cg-bh)x}{2\left(b^2-4ac\right)d^2\left(a+bx+cx^2\right)^2} + \frac{3(2cg-bh)(b+2cx)}{2\left(b^2-4ac\right)^2d^2\left(a+bx+cx\right.}$$

**Mathematica** [A]
time = 0.10, size = 131, normalized size = 0.94

$$\frac{\frac{(b^2-4ac)(-bg+2ah-2cgx+bhx)}{(a+x(b+cx))^2} + \frac{3(2cg-bh)(b+2cx)}{a+x(b+cx)} - \frac{12c(-2cg+bh)\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)}{\sqrt{-b^2+4ac}}}{2\left(b^2-4ac\right)^2d^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(g + h*x)/((a + b*x + c*x^2)*(a*d + b*d*x + c*d*x^2)^2),x]
```

```
[Out] (((b^2 - 4*a*c)*(-(b*g) + 2*a*h - 2*c*g*x + b*h*x))/(a + x*(b + c*x))^2 + (
3*(2*c*g - b*h)*(b + 2*c*x))/(a + x*(b + c*x)) - (12*c*(-2*c*g + b*h)*ArcTa
n[(b + 2*c*x)/Sqrt[-b^2 + 4*a*c]])/Sqrt[-b^2 + 4*a*c])/(2*(b^2 - 4*a*c)^2*d
^2)
```

**Maple [A]**
time = 0.18, size = 141, normalized size = 1.01

| method | result |
|---|---|
| default | $\frac{\frac{bg-2ah+(-bh+2cg)x}{2(4ac-b^2)(cx^2+bx+a)^2}+\frac{3(-bh+2cg)\left(\frac{2cx+b}{(4ac-b^2)(cx^2+bx+a)}+\frac{4c\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right)}{(4ac-b^2)^{\frac{3}{2}}}\right)}{2(4ac-b^2)}}{d^2}$ |
| risch | $\frac{-\frac{3c^2(bh-2cg)x^3}{16a^2c^2-8ab^2c+b^4}-\frac{9bc(bh-2cg)x^2}{2(16a^2c^2-8ab^2c+b^4)}-\frac{(5abch-10ac^2g+b^3h-2b^2cg)x}{16a^2c^2-8ab^2c+b^4}-\frac{8a^2ch+ab^2h-10abcg+b^3g}{2(16a^2c^2-8ab^2c+b^4)}}{(cx^2+bx+a)^2d^2} - \frac{3c\ln\left((32a^2c^3-16ab^2c^2+2b^4\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)/(c*x^2+b*x+a)/(c*d*x^2+b*d*x+a*d)^2,x,method=_RETURNVERBOSE)
```

```
[Out] 1/d^2*(1/2*(b*g-2*a*h+(-b*h+2*c*g)*x)/(4*a*c-b^2)/(c*x^2+b*x+a)^2+3/2*(-b*h
+2*c*g)/(4*a*c-b^2)*((2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)+4*c/(4*a*c-b^2)^(3
/2)*arctan((2*c*x+b)/(4*a*c-b^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="maxima"
)
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 565 vs. $2(132) = 264$.
time = 0.53, size = 1150, normalized size = 8.21

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="fricas"
)
```

```
[Out] [1/2*(6*(2*(b^2*c^3 - 4*a*c^4)*g - (b^3*c^2 - 4*a*b*c^3)*h)*x^3 + 9*(2*(b^3
*c^2 - 4*a*b*c^3)*g - (b^4*c - 4*a*b^2*c^2)*h)*x^2 - 6*(2*a^2*c^2*g - a^2*b
*c*h + (2*c^4*g - b*c^3*h)*x^4 + 2*(2*b*c^3*g - b^2*c^2*h)*x^3 + (2*(b^2*c^
2 + 2*a*c^3)*g - (b^3*c + 2*a*b*c^2)*h)*x^2 + 2*(2*a*b*c^2*g - a*b^2*c*h)*x
)*sqrt(b^2 - 4*a*c)*log((2*c^2*x^2 + 2*b*c*x + b^2 - 2*a*c + sqrt(b^2 - 4*a
*c)*(2*c*x + b))/(c*x^2 + b*x + a)) - (b^5 - 14*a*b^3*c + 40*a^2*b*c^2)*g -
 (a*b^4 + 4*a^2*b^2*c - 32*a^3*c^2)*h + 2*(2*(b^4*c + a*b^2*c^2 - 20*a^2*c^
3)*g - (b^5 + a*b^3*c - 20*a^2*b*c^2)*h)*x)/((b^6*c^2 - 12*a*b^4*c^3 + 48*a
^2*b^2*c^4 - 64*a^3*c^5)*d^2*x^4 + 2*(b^7*c - 12*a*b^5*c^2 + 48*a^2*b^3*c^3
 - 64*a^3*b*c^4)*d^2*x^3 + (b^8 - 10*a*b^6*c + 24*a^2*b^4*c^2 + 32*a^3*b^2*
c^3 - 128*a^4*c^4)*d^2*x^2 + 2*(a*b^7 - 12*a^2*b^5*c + 48*a^3*b^3*c^2 - 64*
a^4*b*c^3)*d^2*x + (a^2*b^6 - 12*a^3*b^4*c + 48*a^4*b^2*c^2 - 64*a^5*c^3)*d
^2), 1/2*(6*(2*(b^2*c^3 - 4*a*c^4)*g - (b^3*c^2 - 4*a*b*c^3)*h)*x^3 + 9*(2*
(b^3*c^2 - 4*a*b*c^3)*g - (b^4*c - 4*a*b^2*c^2)*h)*x^2 - 12*(2*a^2*c^2*g -
a^2*b*c*h + (2*c^4*g - b*c^3*h)*x^4 + 2*(2*b*c^3*g - b^2*c^2*h)*x^3 + (2*(b
^2*c^2 + 2*a*c^3)*g - (b^3*c + 2*a*b*c^2)*h)*x^2 + 2*(2*a*b*c^2*g - a*b^2*c
*h)*x)*sqrt(-b^2 + 4*a*c)*arctan(-sqrt(-b^2 + 4*a*c)*(2*c*x + b)/(b^2 - 4*a
*c)) - (b^5 - 14*a*b^3*c + 40*a^2*b*c^2)*g - (a*b^4 + 4*a^2*b^2*c - 32*a^3*
c^2)*h + 2*(2*(b^4*c + a*b^2*c^2 - 20*a^2*c^3)*g - (b^5 + a*b^3*c - 20*a^2*
b*c^2)*h)*x)/((b^6*c^2 - 12*a*b^4*c^3 + 48*a^2*b^2*c^4 - 64*a^3*c^5)*d^2*x^
4 + 2*(b^7*c - 12*a*b^5*c^2 + 48*a^2*b^3*c^3 - 64*a^3*b*c^4)*d^2*x^3 + (b^8
 - 10*a*b^6*c + 24*a^2*b^4*c^2 + 32*a^3*b^2*c^3 - 128*a^4*c^4)*d^2*x^2 + 2*
(a*b^7 - 12*a^2*b^5*c + 48*a^3*b^3*c^2 - 64*a^4*b*c^3)*d^2*x + (a^2*b^6 - 1
2*a^3*b^4*c + 48*a^4*b^2*c^2 - 64*a^5*c^3)*d^2)]
```

**Sympy** [B] Leaf count of result is larger than twice the leaf count of optimal. 709 vs. $2(133) = 266$.
time = 1.21, size = 709, normalized size = 5.06

$$\frac{3c\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)\log\left(x+\frac{-192a^3c^4\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)+144a^2b^2c^3\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)-36ab^4c^2\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)+3b^6c\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)+3b^2ch-6bc^2g}{6bc^2h-12c^3g}\right)}{d^2}-\frac{3c\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)\log\left(x+\frac{192a^3c^4\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)-144a^2b^2c^3\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)+36ab^4c^2\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)-3b^6c\sqrt{-\frac{1}{(4ac-b^2)^5}}\,(bh-2cg)+3b^2ch-6bc^2g}{6bc^2h-12c^3g}\right)}{d^2}+\frac{-8a^2ch-ab^2h+10abcg-b^3g+x^3(-6bc^2h+12c^3g)+x^2(-9b^2ch+18bc^2g)+x(-10abch+20ac^2g-2b^3h+4b^2cg)}{32a^4c^2d^2-16a^3b^2cd^2+2a^2b^4d^2+x^4\cdot(32a^2c^4d^2-16ab^2c^3d^2+2b^4c^2d^2)+x^3\cdot(64a^2bc^3d^2-32ab^3c^2d^2+4b^5cd^2)+x^2\cdot(64a^3c^3d^2-12ab^4cd^2+2b^6d^2)+x\,(64a^3bc^2d^2-32a^2b^3cd^2+4ab^5d^2)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x**2+b*x+a)/(c*d*x**2+b*d*x+a*d)**2,x)
```

```
[Out] 3*c*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g)*log(x + (-192*a**3*c**4*sqrt(-
1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 144*a**2*b**2*c**3*sqrt(-1/(4*a*c - b*
*2)**5)*(b*h - 2*c*g) - 36*a*b**4*c**2*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*
c*g) + 3*b**6*c*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 3*b**2*c*h - 6*b
*c**2*g)/(6*b*c**2*h - 12*c**3*g))/d**2 - 3*c*sqrt(-1/(4*a*c - b**2)**5)*(b
*h - 2*c*g)*log(x + (192*a**3*c**4*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g)
 - 144*a**2*b**2*c**3*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 36*a*b**4*
```

```
c**2*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) - 3*b**6*c*sqrt(-1/(4*a*c - b
**2)**5)*(b*h - 2*c*g) + 3*b**2*c*h - 6*b*c**2*g)/(6*b*c**2*h - 12*c**3*g))
/d**2 + (-8*a**2*c*h - a*b**2*h + 10*a*b*c*g - b**3*g + x**3*(-6*b*c**2*h +
 12*c**3*g) + x**2*(-9*b**2*c*h + 18*b*c**2*g) + x*(-10*a*b*c*h + 20*a*c**2
*g - 2*b**3*h + 4*b**2*c*g))/(32*a**4*c**2*d**2 - 16*a**3*b**2*c*d**2 + 2*a
**2*b**4*d**2 + x**4*(32*a**2*c**4*d**2 - 16*a*b**2*c**3*d**2 + 2*b**4*c**2
*d**2) + x**3*(64*a**2*b*c**3*d**2 - 32*a*b**3*c**2*d**2 + 4*b**5*c*d**2) +
 x**2*(64*a**3*c**3*d**2 - 12*a*b**4*c*d**2 + 2*b**6*d**2) + x*(64*a**3*b*c
**2*d**2 - 32*a**2*b**3*c*d**2 + 4*a*b**5*d**2))
```

**Giac** [A]
time = 3.47, size = 219, normalized size = 1.56

$$\frac{6\,(2\,c^2g-bch)\arctan\left(\frac{2\,cx+b}{\sqrt{-b^2+4\,ac}}\right)}{(b^4d^2-8\,ab^2cd^2+16\,a^2c^2d^2)\sqrt{-b^2+4\,ac}}+\frac{12\,c^3gx^3-6\,bc^2hx^3+18\,bc^2gx^2-9\,b^2chx^2+4\,b^2cgx+20\,ac^2gx-2\,b^3hx-10\,abchx-b^3g+10\,abcg-ab^2h-8\,a^2ch}{2\,(b^4d^2-8\,ab^2cd^2+16\,a^2c^2d^2)(cx^2+bx+a)^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="giac")
```

```
[Out] 6*(2*c^2*g - b*c*h)*arctan((2*c*x + b)/sqrt(-b^2 + 4*a*c))/((b^4*d^2 - 8*a*
b^2*c*d^2 + 16*a^2*c^2*d^2)*sqrt(-b^2 + 4*a*c)) + 1/2*(12*c^3*g*x^3 - 6*b*c
^2*h*x^3 + 18*b*c^2*g*x^2 - 9*b^2*c*h*x^2 + 4*b^2*c*g*x + 20*a*c^2*g*x - 2*
b^3*h*x - 10*a*b*c*h*x - b^3*g + 10*a*b*c*g - a*b^2*h - 8*a^2*c*h)/((b^4*d^
2 - 8*a*b^2*c*d^2 + 16*a^2*c^2*d^2)*(c*x^2 + b*x + a)^2)
```

**Mupad** [B]
time = 0.42, size = 395, normalized size = 2.82

$$\frac{6\,c\,\mathrm{atan}\left(\frac{d^2\,\left(\frac{6\,c^2\,x\,(b\,h-2\,c\,g)}{d^2\,{(4\,a\,c-b^2)}^{5/2}}+\frac{3\,c\,(b\,h-2\,c\,g)\,\left(16\,a^2\,b\,c^2\,d^2-8\,a\,b^3\,c\,d^2+b^5\,d^2\right)}{d^4\,{(4\,a\,c-b^2)}^{5/2}\,\left(16\,a^2\,c^2-8\,a\,b^2\,c+b^4\right)}\right)\,\left(16\,a^2\,c^2-8\,a\,b^2\,c+b^4\right)}{6\,c^2\,g-3\,b\,c\,h}\right)\,(b\,h-2\,c\,g)}{d^2\,{(4\,a\,c-b^2)}^{5/2}}-\frac{\frac{8\,c\,h\,a^2+h\,a\,b^2-10\,c\,g\,a\,b+g\,b^3}{2\,(16\,a^2\,c^2-8\,a\,b^2\,c+b^4)}+\frac{x\,(b^2+5\,a\,c)\,(b\,h-2\,c\,g)}{16\,a^2\,c^2-8\,a\,b^2\,c+b^4}+\frac{3\,c^2\,x^3\,(b\,h-2\,c\,g)}{16\,a^2\,c^2-8\,a\,b^2\,c+b^4}+\frac{9\,b\,c\,x^2\,(b\,h-2\,c\,g)}{2\,(16\,a^2\,c^2-8\,a\,b^2\,c+b^4)}}{x^2\,\left(b^2\,d^2+2\,a\,c\,d^2\right)+a^2\,d^2+c^2\,d^2\,x^4+2\,a\,b\,d^2\,x+2\,b\,c\,d^2\,x^3}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((g + h*x)/((a*d + b*d*x + c*d*x^2)^2*(a + b*x + c*x^2)),x)
```

```
[Out] (6*c*atan((d^2*((6*c^2*x*(b*h - 2*c*g))/(d^2*(4*a*c - b^2)^(5/2)) + (3*c*(b
*h - 2*c*g)*(b^5*d^2 + 16*a^2*b*c^2*d^2 - 8*a*b^3*c*d^2))/(d^4*(4*a*c - b^2
)^(5/2)*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)))*(b^4 + 16*a^2*c^2 - 8*a*b^2*c))/(6
*c^2*g - 3*b*c*h))*(b*h - 2*c*g))/(d^2*(4*a*c - b^2)^(5/2)) - ((b^3*g + a*b
^2*h + 8*a^2*c*h - 10*a*b*c*g)/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (x*(5*a
*c + b^2)*(b*h - 2*c*g))/(b^4 + 16*a^2*c^2 - 8*a*b^2*c) + (3*c^2*x^3*(b*h -
 2*c*g))/(b^4 + 16*a^2*c^2 - 8*a*b^2*c) + (9*b*c*x^2*(b*h - 2*c*g))/(2*(b^4
 + 16*a^2*c^2 - 8*a*b^2*c)))/(x^2*(b^2*d^2 + 2*a*c*d^2) + a^2*d^2 + c^2*d^2
*x^4 + 2*a*b*d^2*x + 2*b*c*d^2*x^3)
```

# 3.18 $\int \frac{g+hx}{(a+bx+cx^2)^2(ad+bdx+cdx^2)}\,dx$

**Optimal**. Leaf size=140

$$-\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)\,d\,(a+bx+cx^2)^2}+\frac{3(2cg-bh)(b+2cx)}{2(b^2-4ac)^2\,d\,(a+bx+cx^2)}-\frac{6c(2cg-bh)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{(b^2-4ac)^{5/2}\,d}$$

```
[Out] 1/2*(-b*g+2*a*h-(-b*h+2*c*g)*x)/(-4*a*c+b^2)/d/(c*x^2+b*x+a)^2+3/2*(-b*h+2*
c*g)*(2*c*x+b)/(-4*a*c+b^2)^2/d/(c*x^2+b*x+a)-6*c*(-b*h+2*c*g)*arctanh((2*c
*x+b)/(-4*a*c+b^2)^(1/2))/(-4*a*c+b^2)^(5/2)/d
```

**Rubi** [A]
time = 0.07, antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}} = 0.147$, Rules used = {1012, 652, 628, 632, 212}

$$\frac{3(b+2cx)(2cg-bh)}{2d(b^2-4ac)^2(a+bx+cx^2)}-\frac{-2ah+x(2cg-bh)+bg}{2d(b^2-4ac)(a+bx+cx^2)^2}-\frac{6c(2cg-bh)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{d(b^2-4ac)^{5/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(g + h*x)/((a + b*x + c*x^2)^2*(a*d + b*d*x + c*d*x^2)),x]
```

```
[Out] -1/2*(b*g - 2*a*h + (2*c*g - b*h)*x)/((b^2 - 4*a*c)*d*(a + b*x + c*x^2)^2)
+ (3*(2*c*g - b*h)*(b + 2*c*x))/(2*(b^2 - 4*a*c)^2*d*(a + b*x + c*x^2)) - (
6*c*(2*c*g - b*h)*ArcTanh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c]])/((b^2 - 4*a*c)^(5
/2)*d)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 628

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] - Dist[2*c*((2*p +
 3)/((p + 1)*(b^2 - 4*a*c))), Int[(a + b*x + c*x^2)^(p + 1), x], x] /; Free
Q[{a, b, c}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && NeQ[p, -3/2] && Int
egerQ[4*p]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 652

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[((b*d - 2*a*e + (2*c*d - b*e)*x)/((p + 1)*(b^2 - 4*a*c)))*(a + b*
x + c*x^2)^(p + 1), x] - Dist[(2*p + 3)*((2*c*d - b*e)/((p + 1)*(b^2 - 4*a*
c))), Int[(a + b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e}, x] &&
 NeQ[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && NeQ[p, -3/2]
```

Rule 1012

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(c/f)^p, Int[(g + h*
x)^m*(d + e*x + f*x^2)^(p + q), x], x] /; FreeQ[{a, b, c, d, e, f, g, h, p,
 q}, x] && EqQ[c*d - a*f, 0] && EqQ[b*d - a*e, 0] && (IntegerQ[p] || GtQ[c/
f, 0]) && ( !IntegerQ[q] || LeafCount[d + e*x + f*x^2] <= LeafCount[a + b*x
 + c*x^2])
```

Rubi steps

$$
\begin{aligned}
\int \frac{g+hx}{(a+bx+cx^2)^2(ad+bdx+cdx^2)}\,dx &= \frac{\int \frac{g+hx}{(a+bx+cx^2)^3}\,dx}{d} \\
&= -\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)d(a+bx+cx^2)^2} - \frac{(3(2cg-bh))\int \frac{1}{(a+bx+cx^2)^2}\,dx}{2(b^2-4ac)d} \\
&= -\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)d(a+bx+cx^2)^2} + \frac{3(2cg-bh)(b+2cx)}{2(b^2-4ac)^2d(a+bx+cx^2)} \\
&= -\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)d(a+bx+cx^2)^2} + \frac{3(2cg-bh)(b+2cx)}{2(b^2-4ac)^2d(a+bx+cx^2)} \\
&= -\frac{bg-2ah+(2cg-bh)x}{2(b^2-4ac)d(a+bx+cx^2)^2} + \frac{3(2cg-bh)(b+2cx)}{2(b^2-4ac)^2d(a+bx+cx^2)}
\end{aligned}
$$

**Mathematica [A]**
time = 0.02, size = 131, normalized size = 0.94

$$
\frac{\frac{(b^2-4ac)(-bg+2ah-2cgx+bhx)}{(a+x(b+cx))^2} + \frac{3(2cg-bh)(b+2cx)}{a+x(b+cx)} - \frac{12c(-2cg+bh)\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)}{\sqrt{-b^2+4ac}}}{2(b^2-4ac)^2d}
$$

Antiderivative was successfully verified.

```
[In] Integrate[(g + h*x)/((a + b*x + c*x^2)^2*(a*d + b*d*x + c*d*x^2)),x]
```

```
[Out] (((b^2 - 4*a*c)*(-(b*g) + 2*a*h - 2*c*g*x + b*h*x))/(a + x*(b + c*x))^2 + (
3*(2*c*g - b*h)*(b + 2*c*x))/(a + x*(b + c*x)) - (12*c*(-2*c*g + b*h)*ArcTa
n[(b + 2*c*x)/Sqrt[-b^2 + 4*a*c]])/Sqrt[-b^2 + 4*a*c])/(2*(b^2 - 4*a*c)^2*d
)
```

**Maple [A]**
time = 0.18, size = 141, normalized size = 1.01

| method | result |
|---|---|
| default | $\frac{\frac{bg-2ah+(-bh+2cg)x}{2\left(4ac-b^2\right)\left(cx^2+bx+a\right)^2}+\frac{3(-bh+2cg)\left(\frac{2cx+b}{\left(4ac-b^2\right)\left(cx^2+bx+a\right)}+\frac{4c\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^2}}\right)}{\left(4ac-b^2\right)^{\frac{3}{2}}}\right)}{2\left(4ac-b^2\right)}}{d}$ |
| risch | $\frac{-\frac{3c^2(bh-2cg)x^3}{16a^2c^2-8a\,b^2c+b^4}-\frac{9bc(bh-2cg)x^2}{2\left(16a^2c^2-8a\,b^2c+b^4\right)}-\frac{\left(5abch-10a\,c^2g+b^3h-2b^2cg\right)x}{16a^2c^2-8a\,b^2c+b^4}-\frac{8a^2ch+a\,b^2h-10abcg+b^3g}{2\left(16a^2c^2-8a\,b^2c+b^4\right)}}{\left(cx^2+bx+a\right)^2d}-\frac{3c\ln\left(\left(32a^2c^3-16a\,b^2c^2+2\right.\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)/(c*x^2+b*x+a)^2/(c*d*x^2+b*d*x+a*d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(1/2*(b*g-2*a*h+(-b*h+2*c*g)*x)/(4*a*c-b^2)/(c*x^2+b*x+a)^2+3/2*(-b*h+2
*c*g)/(4*a*c-b^2)*((2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)+4*c/(4*a*c-b^2)^(3/2
)*arctan((2*c*x+b)/(4*a*c-b^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)^2/(c*d*x^2+b*d*x+a*d),x, algorithm="maxima"
)
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 555 vs. $2(132) = 264$.
time = 0.38, size = 1130, normalized size = 8.07

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)^2/(c*d*x^2+b*d*x+a*d),x, algorithm="fricas"
)
```

```
[Out] [1/2*(6*(2*(b^2*c^3 - 4*a*c^4)*g - (b^3*c^2 - 4*a*b*c^3)*h)*x^3 + 9*(2*(b^3
*c^2 - 4*a*b*c^3)*g - (b^4*c - 4*a*b^2*c^2)*h)*x^2 - 6*(2*a^2*c^2*g - a^2*b
*c*h + (2*c^4*g - b*c^3*h)*x^4 + 2*(2*b*c^3*g - b^2*c^2*h)*x^3 + (2*(b^2*c^
2 + 2*a*c^3)*g - (b^3*c + 2*a*b*c^2)*h)*x^2 + 2*(2*a*b*c^2*g - a*b^2*c*h)*x
)*sqrt(b^2 - 4*a*c)*log((2*c^2*x^2 + 2*b*c*x + b^2 - 2*a*c + sqrt(b^2 - 4*a
*c)*(2*c*x + b))/(c*x^2 + b*x + a)) - (b^5 - 14*a*b^3*c + 40*a^2*b*c^2)*g -
 (a*b^4 + 4*a^2*b^2*c - 32*a^3*c^2)*h + 2*(2*(b^4*c + a*b^2*c^2 - 20*a^2*c^
3)*g - (b^5 + a*b^3*c - 20*a^2*b*c^2)*h)*x)/((b^6*c^2 - 12*a*b^4*c^3 + 48*a
^2*b^2*c^4 - 64*a^3*c^5)*d*x^4 + 2*(b^7*c - 12*a*b^5*c^2 + 48*a^2*b^3*c^3 -
 64*a^3*b*c^4)*d*x^3 + (b^8 - 10*a*b^6*c + 24*a^2*b^4*c^2 + 32*a^3*b^2*c^3
- 128*a^4*c^4)*d*x^2 + 2*(a*b^7 - 12*a^2*b^5*c + 48*a^3*b^3*c^2 - 64*a^4*b*
c^3)*d*x + (a^2*b^6 - 12*a^3*b^4*c + 48*a^4*b^2*c^2 - 64*a^5*c^3)*d), 1/2*(
6*(2*(b^2*c^3 - 4*a*c^4)*g - (b^3*c^2 - 4*a*b*c^3)*h)*x^3 + 9*(2*(b^3*c^2 -
 4*a*b*c^3)*g - (b^4*c - 4*a*b^2*c^2)*h)*x^2 - 12*(2*a^2*c^2*g - a^2*b*c*h
+ (2*c^4*g - b*c^3*h)*x^4 + 2*(2*b*c^3*g - b^2*c^2*h)*x^3 + (2*(b^2*c^2 + 2
*a*c^3)*g - (b^3*c + 2*a*b*c^2)*h)*x^2 + 2*(2*a*b*c^2*g - a*b^2*c*h)*x)*sqr
t(-b^2 + 4*a*c)*arctan(-sqrt(-b^2 + 4*a*c)*(2*c*x + b)/(b^2 - 4*a*c)) - (b^
5 - 14*a*b^3*c + 40*a^2*b*c^2)*g - (a*b^4 + 4*a^2*b^2*c - 32*a^3*c^2)*h + 2
*(2*(b^4*c + a*b^2*c^2 - 20*a^2*c^3)*g - (b^5 + a*b^3*c - 20*a^2*b*c^2)*h)*
x)/((b^6*c^2 - 12*a*b^4*c^3 + 48*a^2*b^2*c^4 - 64*a^3*c^5)*d*x^4 + 2*(b^7*c
 - 12*a*b^5*c^2 + 48*a^2*b^3*c^3 - 64*a^3*b*c^4)*d*x^3 + (b^8 - 10*a*b^6*c
+ 24*a^2*b^4*c^2 + 32*a^3*b^2*c^3 - 128*a^4*c^4)*d*x^2 + 2*(a*b^7 - 12*a^2*
b^5*c + 48*a^3*b^3*c^2 - 64*a^4*b*c^3)*d*x + (a^2*b^6 - 12*a^3*b^4*c + 48*a
^4*b^2*c^2 - 64*a^5*c^3)*d)]
```

**Sympy** [B] Leaf count of result is larger than twice the leaf count of optimal. 680 vs. $2(128) = 256$.
time = 1.18, size = 680, normalized size = 4.86

$$\frac{3c\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)\log\left(x+\frac{-192a^3c^4\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)+144a^2b^2c^3\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)-36ab^4c^2\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)+3b^6c\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)+3b^2ch-6bc^2g}{6bc^2h-12c^3g}\right)}{d}-\frac{3c\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)\log\left(x+\frac{192a^3c^4\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)-144a^2b^2c^3\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)+36ab^4c^2\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)-3b^6c\sqrt{-\frac{1}{(4ac-b^2)^5}}(bh-2cg)+3b^2ch-6bc^2g}{6bc^2h-12c^3g}\right)}{d}+\frac{-8a^2ch-ab^2h+10abcg-b^3g+x^3(-6bc^2h+12c^3g)+x^2(-9b^2ch+18bc^2g)+x(-10abch+20ac^2g-2b^3h+4b^2cg)}{32a^4c^2d-16a^3b^2cd+2a^2b^4d+x^4\cdot(32a^2c^4d-16ab^2c^3d+2b^4c^2d)+x^3\cdot(64a^2bc^3d-32ab^3c^2d+4b^5cd)+x^2\cdot(64a^3c^3d-12ab^4cd+2b^6d)+x\,(64a^3bc^2d-32a^2b^3cd+4ab^5d)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x**2+b*x+a)**2/(c*d*x**2+b*d*x+a*d),x)
```

```
[Out] 3*c*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g)*log(x + (-192*a**3*c**4*sqrt(-
1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 144*a**2*b**2*c**3*sqrt(-1/(4*a*c - b*
*2)**5)*(b*h - 2*c*g) - 36*a*b**4*c**2*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*
c*g) + 3*b**6*c*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 3*b**2*c*h - 6*b
*c**2*g)/(6*b*c**2*h - 12*c**3*g))/d - 3*c*sqrt(-1/(4*a*c - b**2)**5)*(b*h
- 2*c*g)*log(x + (192*a**3*c**4*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) -
144*a**2*b**2*c**3*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) + 36*a*b**4*c**
```

```
2*sqrt(-1/(4*a*c - b**2)**5)*(b*h - 2*c*g) - 3*b**6*c*sqrt(-1/(4*a*c - b**2
)**5)*(b*h - 2*c*g) + 3*b**2*c*h - 6*b*c**2*g)/(6*b*c**2*h - 12*c**3*g))/d
+ (-8*a**2*c*h - a*b**2*h + 10*a*b*c*g - b**3*g + x**3*(-6*b*c**2*h + 12*c*
*3*g) + x**2*(-9*b**2*c*h + 18*b*c**2*g) + x*(-10*a*b*c*h + 20*a*c**2*g - 2
*b**3*h + 4*b**2*c*g))/(32*a**4*c**2*d - 16*a**3*b**2*c*d + 2*a**2*b**4*d +
 x**4*(32*a**2*c**4*d - 16*a*b**2*c**3*d + 2*b**4*c**2*d) + x**3*(64*a**2*b
*c**3*d - 32*a*b**3*c**2*d + 4*b**5*c*d) + x**2*(64*a**3*c**3*d - 12*a*b**4
*c*d + 2*b**6*d) + x*(64*a**3*b*c**2*d - 32*a**2*b**3*c*d + 4*a*b**5*d))
```

**Giac** [A]
time = 5.03, size = 207, normalized size = 1.48

$$\frac{6\,(2\,c^2g-bch)\arctan\left(\frac{2\,cx+b}{\sqrt{-b^2+4\,ac}}\right)}{(b^4d-8\,ab^2cd+16\,a^2c^2d)\sqrt{-b^2+4\,ac}}+\frac{12\,c^3gx^3-6\,bc^2hx^3+18\,bc^2gx^2-9\,b^2chx^2+4\,b^2cgx+20\,ac^2gx-2\,b^3hx-10\,abchx-b^3g+10\,abcg-ab^2h-8\,a^2ch}{2\,(b^4d-8\,ab^2cd+16\,a^2c^2d)(cx^2+bx+a)^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(c*x^2+b*x+a)^2/(c*d*x^2+b*d*x+a*d),x, algorithm="giac")
```

```
[Out] 6*(2*c^2*g - b*c*h)*arctan((2*c*x + b)/sqrt(-b^2 + 4*a*c))/((b^4*d - 8*a*b^
2*c*d + 16*a^2*c^2*d)*sqrt(-b^2 + 4*a*c)) + 1/2*(12*c^3*g*x^3 - 6*b*c^2*h*x
^3 + 18*b*c^2*g*x^2 - 9*b^2*c*h*x^2 + 4*b^2*c*g*x + 20*a*c^2*g*x - 2*b^3*h*
x - 10*a*b*c*h*x - b^3*g + 10*a*b*c*g - a*b^2*h - 8*a^2*c*h)/((b^4*d - 8*a*
b^2*c*d + 16*a^2*c^2*d)*(c*x^2 + b*x + a)^2)
```

**Mupad** [B]
time = 4.02, size = 375, normalized size = 2.68

$$\frac{6\,c\,\mathrm{atan}\left(\frac{d\,\left(\frac{6\,c^2\,x\,(b\,h-2\,c\,g)}{d\,{\left(4\,a\,c-b^2\right)}^{5/2}}+\frac{3\,c\,(b\,h-2\,c\,g)\,\left(16\,d\,a^2\,b\,c^2-8\,d\,a\,b^3\,c+d\,b^5\right)}{d^2\,{\left(4\,a\,c-b^2\right)}^{5/2}\,\left(16\,a^2\,c^2-8\,a\,b^2\,c+b^4\right)}\right)\,\left(16\,a^2\,c^2-8\,a\,b^2\,c+b^4\right)}{6\,c^2\,g-3\,b\,c\,h}\right)\,(b\,h-2\,c\,g)}{d\,{(4\,a\,c-b^2)}^{5/2}}-\frac{\frac{8\,c\,h\,a^2+h\,a\,b^2-10\,c\,g\,a\,b+g\,b^3}{2\,(16\,a^2\,c^2-8\,a\,b^2\,c+b^4)}+\frac{x\,(b^2+5\,a\,c)\,(b\,h-2\,c\,g)}{16\,a^2\,c^2-8\,a\,b^2\,c+b^4}+\frac{3\,c^2\,x^3\,(b\,h-2\,c\,g)}{16\,a^2\,c^2-8\,a\,b^2\,c+b^4}+\frac{9\,b\,c\,x^2\,(b\,h-2\,c\,g)}{2\,(16\,a^2\,c^2-8\,a\,b^2\,c+b^4)}}{a^2\,d+x^2\,\left(d\,b^2+2\,a\,c\,d\right)+c^2\,d\,x^4+2\,b\,c\,d\,x^3+2\,a\,b\,d\,x}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((g + h*x)/((a*d + b*d*x + c*d*x^2)*(a + b*x + c*x^2)^2),x)
```

```
[Out] (6*c*atan((d*((6*c^2*x*(b*h - 2*c*g))/(d*(4*a*c - b^2)^(5/2)) + (3*c*(b*h -
 2*c*g)*(b^5*d - 8*a*b^3*c*d + 16*a^2*b*c^2*d))/(d^2*(4*a*c - b^2)^(5/2)*(b
^4 + 16*a^2*c^2 - 8*a*b^2*c)))*(b^4 + 16*a^2*c^2 - 8*a*b^2*c))/(6*c^2*g - 3
*b*c*h))*(b*h - 2*c*g))/(d*(4*a*c - b^2)^(5/2)) - ((b^3*g + a*b^2*h + 8*a^2
*c*h - 10*a*b*c*g)/(2*(b^4 + 16*a^2*c^2 - 8*a*b^2*c)) + (x*(5*a*c + b^2)*(b
*h - 2*c*g))/(b^4 + 16*a^2*c^2 - 8*a*b^2*c) + (3*c^2*x^3*(b*h - 2*c*g))/(b^
4 + 16*a^2*c^2 - 8*a*b^2*c) + (9*b*c*x^2*(b*h - 2*c*g))/(2*(b^4 + 16*a^2*c^
2 - 8*a*b^2*c)))/(a^2*d + x^2*(b^2*d + 2*a*c*d) + c^2*d*x^4 + 2*b*c*d*x^3 +
 2*a*b*d*x)
```

# 3.19 $\int \frac{(A+Bx)\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=617

$$\frac{B\sqrt{a+bx+cx^2}}{f}-\frac{(2Bce-bBf-2Acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2}+\frac{\left(2f(Af(cd-af)-Bd(ce-b\right.}{}$$

```
[Out] -1/2*(-2*A*c*f-B*b*f+2*B*c*e)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(
1/2))/f^2/c^(1/2)+B*(c*x^2+b*x+a)^(1/2)/f+1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c
*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1
/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*f
*(A*f*(-a*f+c*d)-B*d*(-b*f+c*e))-(A*f*(-b*f+c*e)+B*(f*(-a*f+b*e)-c*(-d*f+e^
2)))*(e-(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-
b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*arctanh(1/4*(4*a*f-b
*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+
b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1
/2))*(2*f*(A*f*(-a*f+c*d)-B*d*(-b*f+c*e))-(A*f*(-b*f+c*e)+B*(f*(-a*f+b*e)-c
*(-d*f+e^2)))*(e+(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2
-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 5.93, antiderivative size = 615, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.188$, Rules used = {1033, 1090, 635, 212, 1046, 738}

$$\frac{\left(2f(Af(cd-af)-Bd(ce-bf))-\left(e-\sqrt{e^2-4df}\right)(Bf(be-af)+Af(ce-bf)-Bc(e^2-df))\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f^2\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{\left(2f(Af(cd-af)-Bd(ce-bf))-\left(\sqrt{e^2-4df}+e\right)(Bf(be-af)+Af(ce-bf)-Bc(e^2-df))\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f^2\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)(-2Acf-bBf+2Bce)}{2\sqrt{c}\,f^2}+\frac{B\sqrt{a+bx+cx^2}}{f}$$

Warning: Unable to verify antiderivative.

```
[In] Int[((A + B*x)*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (B*Sqrt[a + b*x + c*x^2])/f - ((2*B*c*e - b*B*f - 2*A*c*f)*ArcTanh[(b + 2*c
*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[c]*f^2) + ((2*f*(A*f*(c*d -
 a*f) - B*d*(c*e - b*f)) - (e - Sqrt[e^2 - 4*d*f])*(B*f*(b*e - a*f) + A*f*(
c*e - b*f) - B*c*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) +
 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b
*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(S
qrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e
- b*f)*Sqrt[e^2 - 4*d*f]]) - ((2*f*(A*f*(c*d - a*f) - B*d*(c*e - b*f)) - (e
 + Sqrt[e^2 - 4*d*f])*(B*f*(b*e - a*f) + A*f*(c*e - b*f) - B*c*(e^2 - d*f))
)*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4
*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*
```

```
Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*
Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1033

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d
+ e*x + f*x^2)^(q + 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), I
nt[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[h*p*(b*d - a*e) + a*(
h*e - 2*g*f)*(p + q + 1) + (2*h*p*(c*d - a*f) + b*(h*e - 2*g*f)*(p + q + 1)
)*x + (h*p*(c*e - b*f) + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /; Fr
eeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d
*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
```

```
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\int \frac{(A+Bx)\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx = \frac{B\sqrt{a+bx+cx^2}}{f} - \frac{\int \frac{\frac{1}{2}(bBd-2aAf)-\frac{1}{2}(2Abf-B(2cd+be-2af))x+\frac{1}{2}(2Bce-bBf-2Acf)}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}}{f}$$

$$= \frac{B\sqrt{a+bx+cx^2}}{f} - \frac{\int \frac{\frac{1}{2}f(bBd-2aAf)-\frac{1}{2}d(2Bce-bBf-2Acf)+(-\frac{1}{2}e(2Bce-bBf-2Acf)}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}}{f^2}$$

$$= \frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(2Bce-bBf-2Acf)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2}{\sqrt{a+bx}}\right.}{f^2}$$

$$= \frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(2Bce-bBf-2Acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right.}{2\sqrt{c}\,f^2}$$

$$= \frac{B\sqrt{a+bx+cx^2}}{f} - \frac{(2Bce-bBf-2Acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right.}{2\sqrt{c}\,f^2}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.93, size = 889, normalized size = 1.44

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (2*B*f*Sqrt[a + x*(b + c*x)] + ((2*B*c*e - b*B*f - 2*A*c*f)*Log[b + 2*c*x -
 2*Sqrt[c]*Sqrt[a + x*(b + c*x)]])/Sqrt[c] + 2*RootSum[b^2*d - a*b*e + a^2*
f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^
2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(b*B*c*d*e*Log[-(Sqrt[c]*x) + Sqrt[a +
b*x + c*x^2] - #1]) + a*B*c*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
#1] + b^2*B*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + A*b*c*d*f*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*B*c*d*f*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1] - a*b*B*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
```

```
+ c*x^2] - #1] - a*A*c*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] +
 a^2*B*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*B*c^(3/2)*d*e
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*b*B*Sqrt[c]*d*f*Log[
-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*A*c^(3/2)*d*f*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*a*A*Sqrt[c]*f^2*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1 - B*c*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
 + c*x^2] - #1]*#1^2 + B*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #
1]*#1^2 + b*B*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + A*c
*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - A*b*f^2*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*B*f^2*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b
*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/(2*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1594 vs. $2(558) = 1116$.
time = 0.17, size = 1595, normalized size = 2.59

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1595 |
| risch | Expression too large to display | 11585 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(-2*A*f+B*(-4*d*f+e^2)^(1/2)+B*e)/(-4*d*f+e^2)^(1/2)/f*(1/2*(4*(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)
*ln((1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2
)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(
-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)
/f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f
-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x
+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d
*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(2*A
*f+B*(-4*d*f+e^2)^(1/2)-B*e)/(-4*d*f+e^2)^(1/2)/f*(1/2*(4*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*
f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e
*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(
```

```
c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2
)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+
e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/
((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2
)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e
*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))
^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2
*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2
)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(A+Bx)\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral((A + B*x)*sqrt(a + b*x + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument Value
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{(A+B\,x)\,\sqrt{c\,x^2+b\,x+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2),x)
```

```
[Out] int(((A + B*x)*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2), x)
```

## 3.20 $\int \frac{(A+Bx)\left(a+bx+cx^2\right)^{3/2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=1092

$$-\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf(2Bce-bBf-2Acf)x)\sqrt{a+bx+c}}{8cf^3}$$

```
[Out] 1/3*B*(c*x^2+b*x+a)^(3/2)/f+1/16*(2*A*c*f*(3*b^2*f^2-12*c*f*(-a*f+b*e)+8*c^
2*(-d*f+e^2))-B*(b^3*f^3+6*b*c*f^2*(-2*a*f+b*e)-24*c^2*f*(-a*e*f-b*d*f+b*e^
2)+16*c^3*(-2*d*e*f+e^3)))*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2
))/c^(3/2)/f^4-1/8*(2*A*c*f*(-5*b*f+4*c*e)-B*(b^2*f^2-2*c*f*(-4*a*f+5*b*e)+
8*c^2*(-d*f+e^2))+2*c*f*(-2*A*c*f-B*b*f+2*B*c*e)*x)*(c*x^2+b*x+a)^(1/2)/c/f
^3-1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e
^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+
c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*c*f*(B*d*(-b*f+c*e)*(2*a*f^2-b*e*f-2*c*d
*f+c*e^2)+A*f*(2*c*d*f*(-a*f+b*e)-f^2*(-a^2*f+b^2*d)-c^2*d*(-d*f+e^2)))-c*(
A*f*(-b*f+c*e)*(f*(-2*a*f+b*e)-c*(-2*d*f+e^2))+B*(c^2*(d^2*f^2-3*d*e^2*f+e^
4)-f^2*(2*a*b*e*f-a^2*f^2-b^2*(-d*f+e^2))+2*c*f*(a*f*(-d*f+e^2)-b*(-2*d*e*f
+e^3))))*(e-(-4*d*f+e^2)^(1/2)))/c/f^4*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*
c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)+1/2*arctanh(1/4*(4
*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(
c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/
2))^(1/2))*(2*f*(B*d*(-b*f+c*e)*(2*a*f^2-b*e*f-2*c*d*f+c*e^2)+A*f*(2*c*d*f*
(-a*f+b*e)-f^2*(-a^2*f+b^2*d)-c^2*d*(-d*f+e^2)))-(A*f*(-b*f+c*e)*(f*(-2*a*f
+b*e)-c*(-2*d*f+e^2))+B*(c^2*(d^2*f^2-3*d*e^2*f+e^4)-f^2*(2*a*b*e*f-a^2*f^2
-b^2*(-d*f+e^2))+2*c*f*(a*f*(-d*f+e^2)-b*(-2*d*e*f+e^3))))*(e+(-4*d*f+e^2)^
(1/2)))/f^4*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c
*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]
time = 16.59, antiderivative size = 1092, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.219$, Rules used = {1033, 1080, 1090, 635, 212, 1046, 738}

Warning: Unable to verify antiderivative.

```
[In] Int[((A + B*x)*(a + b*x + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] -1/8*((2*A*c*f*(4*c*e - 5*b*f) - B*(b^2*f^2 - 2*c*f*(5*b*e - 4*a*f) + 8*c^2
*(e^2 - d*f)) + 2*c*f*(2*B*c*e - b*B*f - 2*A*c*f)*x)*Sqrt[a + b*x + c*x^2])
```

```
/(c*f^3) + (B*(a + b*x + c*x^2)^(3/2))/(3*f) + ((2*A*c*f*(3*b^2*f^2 - 12*c*
f*(b*e - a*f) + 8*c^2*(e^2 - d*f)) - B*(b^3*f^3 + 6*b*c*f^2*(b*e - 2*a*f) -
 24*c^2*f*(b*e^2 - b*d*f - a*e*f) + 16*c^3*(e^3 - 2*d*e*f)))*ArcTanh[(b + 2
*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(16*c^(3/2)*f^4) - ((2*c*f*(B*d*(
c*e - b*f)*(c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2) + A*f*(2*c*d*f*(b*e - a*f) -
 f^2*(b^2*d - a^2*f) - c^2*d*(e^2 - d*f))) - c*(e - Sqrt[e^2 - 4*d*f])*(A*f
*(c*e - b*f)*(f*(b*e - 2*a*f) - c*(e^2 - 2*d*f)) + B*(c^2*(e^4 - 3*d*e^2*f
+ d^2*f^2) - f^2*(2*a*b*e*f - a^2*f^2 - b^2*(e^2 - d*f)) + 2*c*f*(a*f*(e^2
- d*f) - b*(e^3 - 2*d*e*f)))))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) +
 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b
*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(S
qrt[2]*c*f^4*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*
e - b*f)*Sqrt[e^2 - 4*d*f]]) + ((2*f*(B*d*(c*e - b*f)*(c*e^2 - 2*c*d*f - b*
e*f + 2*a*f^2) + A*f*(2*c*d*f*(b*e - a*f) - f^2*(b^2*d - a^2*f) - c^2*d*(e^
2 - d*f))) - (e + Sqrt[e^2 - 4*d*f])*(A*f*(c*e - b*f)*(f*(b*e - 2*a*f) - c*
(e^2 - 2*d*f)) + B*(c^2*(e^4 - 3*d*e^2*f + d^2*f^2) - f^2*(2*a*b*e*f - a^2*
f^2 - b^2*(e^2 - d*f)) + 2*c*f*(a*f*(e^2 - d*f) - b*(e^3 - 2*d*e*f)))))*Arc
Tanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]
))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[
e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f^4*Sqrt[e^2 - 4*d*f]*Sqrt[
c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1033

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d
+ e*x + f*x^2)^(q + 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), I
nt[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[h*p*(b*d - a*e) + a*(
```

```
h*e - 2*g*f)*(p + q + 1) + (2*h*p*(c*d - a*f) + b*(h*e - 2*g*f)*(p + q + 1)
)*x + (h*p*(c*e - b*f) + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /; Fr
eeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d
*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1080

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p
 + 2*q + 3) + C*(b*f*p - c*e*(2*p + q + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + b
*x + c*x^2)^p*((d + e*x + f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q +
3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + b*x + c*x
^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[p*(b*d - a*e)*(C*(c*e - b*f)*(q + 1) -
 c*(C*e - B*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*p + a*c*(C*(2*d*f
- e^2*(2*p + q + 2)) + f*(B*e - 2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)
*(C*(c*e - b*f)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*e
*f*p*(b^2 - 4*a*c) - b*c*(C*(e^2 - 4*d*f)*(2*p + q + 2) + f*(2*C*d - B*e +
2*A*f)*(2*p + 2*q + 3))))*x + (p*(c*e - b*f)*(C*(c*e - b*f)*(q + 1) - c*(C*
e - B*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2 - 4*a*c) - c^2*(C*(e^
2 - 4*d*f)*(2*p + q + 2) + f*(2*C*d - B*e + 2*A*f)*(2*p + 2*q + 3))))*x^2,
x], x], x] /; FreeQ[{a, b, c, d, e, f, A, B, C, q}, x] && NeQ[b^2 - 4*a*c,
0] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*
q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\int \frac{(A+Bx)\left(a+bx+cx^2\right)^{3/2}}{d+ex+fx^2}\,dx = \frac{B\left(a+bx+cx^2\right)^{3/2}}{3f} - \frac{\int \frac{\sqrt{a+bx+cx^2}\,\left(\frac{3}{2}(bBd-2aAf)-\frac{3}{2}(2Abf-B(2cd+be-2c\right.}{d+ex+fx^2}}{3f}$$

$$= -\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf}{8cf^3}$$

$$= -\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf}{8cf^3}$$

$$= -\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf}{8cf^3}$$

$$= -\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf}{8cf^3}$$

$$= -\frac{(2Acf(4ce-5bf)-B(b^2f^2-2cf(5be-4af)+8c^2(e^2-df))+2cf}{8cf^3}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 4.64, size = 2733, normalized size = 2.50

Result too large to show

Antiderivative was successfully verified.

```
[In] Integrate[((A + B*x)*(a + b*x + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] (2*Sqrt[c]*f*Sqrt[a + x*(b + c*x)]*(6*A*c*f*(-4*c*e + 5*b*f + 2*c*f*x) + B*
(3*b^2*f^2 + 2*c*f*(-15*b*e + 16*a*f + 7*b*f*x) + 4*c^2*(6*e^2 - 6*d*f - 3*
e*f*x + 2*f^2*x^2))) + 3*(2*A*c*f*(-3*b^2*f^2 + 12*c*f*(b*e - a*f) - 8*c^2*
(e^2 - d*f)) + B*(b^3*f^3 + 6*b*c*f^2*(b*e - 2*a*f) + 24*c^2*f*(-(b*e^2) +
b*d*f + a*e*f) + 16*c^3*(e^3 - 2*d*e*f)))*Log[c*(b + 2*c*x - 2*Sqrt[c]*Sqrt
[a + x*(b + c*x)])] - 48*c^(3/2)*RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c
]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*
e*#1^3 + f*#1^4 & , (b*B*c^2*d*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
 - #1] - a*B*c^2*e^4*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*B
*c^2*d^2*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b^2*B*c*d*e
```

^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - A*b*c^2*d*e^2*f*Log[-
(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 3*a*B*c^2*d*e^2*f*Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*a*b*B*c*e^3*f*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1] + a*A*c^2*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1] + 2*b^2*B*c*d^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
#1] + A*b*c^2*d^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*B*
c^2*d^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^3*B*d*e*f^2*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*A*b^2*c*d*e*f^2*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a*b*B*c*d*e*f^2*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1] - 2*a*A*c^2*d*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] - a*b^2*B*e^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1] - 2*a*A*b*c*e^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1] - 2*a^2*B*c*e^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - A
*b^3*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*b^2*B*d*f^3*L
og[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*a^2*B*c*d*f^3*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*A*b^2*e*f^3*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1] + 2*a^2*b*B*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1] + 2*a^2*A*c*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
#1] - a^2*A*b*f^4*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^3*B*f^
4*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*B*c^(5/2)*d*e^3*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 4*B*c^(5/2)*d^2*e*f*Log[-(Sqr
t[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 4*b*B*c^(3/2)*d*e^2*f*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*A*c^(5/2)*d*e^2*f*Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 4*b*B*c^(3/2)*d^2*f^2*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*A*c^(5/2)*d^2*f^2*Log[-(Sqrt[c]*x)
+ Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*b^2*B*Sqrt[c]*d*e*f^2*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1 - 4*A*b*c^(3/2)*d*e*f^2*Log[-(Sqrt[c]*x)
+ Sqrt[a + b*x + c*x^2] - #1]*#1 - 4*a*B*c^(3/2)*d*e*f^2*Log[-(Sqrt[c]*x) +
 Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*A*b^2*Sqrt[c]*d*f^3*Log[-(Sqrt[c]*x) +
Sqrt[a + b*x + c*x^2] - #1]*#1 + 4*a*b*B*Sqrt[c]*d*f^3*Log[-(Sqrt[c]*x) + S
qrt[a + b*x + c*x^2] - #1]*#1 + 4*a*A*c^(3/2)*d*f^3*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1]*#1 - 2*a^2*A*Sqrt[c]*f^4*Log[-(Sqrt[c]*x) + Sqrt[a
+ b*x + c*x^2] - #1]*#1 + B*c^2*e^4*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1]*#1^2 - 3*B*c^2*d*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #
1]*#1^2 - 2*b*B*c*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2
 - A*c^2*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + B*c^2*
d^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 4*b*B*c*d*e*f
^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*A*c^2*d*e*f^2*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + b^2*B*e^2*f^2*Log[-(Sqr
t[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*A*b*c*e^2*f^2*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*B*c*e^2*f^2*Log[-(Sqrt[c]*x) +
Sqrt[a + b*x + c*x^2] - #1]*#1^2 - b^2*B*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a +
b*x + c*x^2] - #1]*#1^2 - 2*A*b*c*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1]*#1^2 - 2*a*B*c*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1]*#1^2 - A*b^2*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2

```
 - 2*a*b*B*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - 2*a*
A*c*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*A*b*f^4
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a^2*B*f^4*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4
*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/(48*c^(3/2)
*f^4)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2907 vs. $2(1027) = 2054$.
time = 0.18, size = 2908, normalized size = 2.66

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2908 |
| risch | Expression too large to display | 32864 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)*(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(-2*A*f+B*(-4*d*f+e^2)^(1/2)+B*e)/(-4*d*f+e^2)^(1/2)/f*(1/3*((x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(3/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)
*(1/4*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-
c*e))/c*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*
f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*(-b*f*(-4
*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-1/f
^2*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2)/c^(3/2)*ln((1/2/f*(-c*(-4*d*f+e^2)^(1
/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(
1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/
2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*
d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2/f*(-c*(
-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+
1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/
2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4*d*f+e^2
)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-
b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/
f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e
```

```
+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f
)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+
2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e
^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))))+1/2*(2*A*f+B*(-4*d*f+e^
2)^(1/2)-B*e)/(-4*d*f+e^2)^(1/2)/f*(1/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^
2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*
(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)
/f^2)^(3/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(1/4*(2*c*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f)/c*((x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2
)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-
2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2/f^
2)/c^(3/2)*ln((1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2
)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^
(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*
(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)
/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+
b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2*(c*(-4*d*f+e
^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f+c*(x-1/2/f*(-
e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(
-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4
*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1
/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e
*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b
*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^
(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-
1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/
2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)
)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
```

```
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
[Out] Timed out
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)*(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{(A+B\,x)\,\left(c\,x^2+b\,x+a\right)^{3/2}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((A + B*x)*(a + b*x + c*x^2)^(3/2))/(d + e*x + f*x^2),x)
[Out] int(((A + B*x)*(a + b*x + c*x^2)^(3/2))/(d + e*x + f*x^2), x)
```

## 3.21 $\int \frac{A+Bx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}}\,dx$

**Optimal**. Leaf size=416

$$\frac{\left(bB-2Ac-B\sqrt{b^2-4ac}\right)\tanh^{-1}\left(\frac{4cd-\left(b-\sqrt{b^2-4ac}\right)e+2\left(ce-\left(b-\sqrt{b^2-4ac}\right)f\right)x}{2\sqrt{2}\sqrt{2c^2d-bce+b^2f-2acf+\sqrt{b^2-4ac}\,(ce-bf)}\sqrt{d+ex+f}}\right.}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{2c^2d-bce+b^2f-2acf+\sqrt{b^2-4ac}\,(ce-bf)}}$$

```
[Out] 1/2*arctanh(1/4*(4*c*d-e*(b+(-4*a*c+b^2)^(1/2))+2*x*(c*e-f*(b+(-4*a*c+b^2)^
(1/2))))*2^(1/2)/(f*x^2+e*x+d)^(1/2)/(2*c^2*d-b*c*e+b^2*f-2*a*c*f-(-b*f+c*e
)*(-4*a*c+b^2)^(1/2))^(1/2))*(2*A*c-B*(b+(-4*a*c+b^2)^(1/2)))*2^(1/2)/(-4*a
*c+b^2)^(1/2)/(2*c^2*d-b*c*e+b^2*f-2*a*c*f-(-b*f+c*e)*(-4*a*c+b^2)^(1/2))^(
1/2)+1/2*arctanh(1/4*(4*c*d+2*x*(c*e-f*(b-(-4*a*c+b^2)^(1/2)))-e*(b-(-4*a*c
+b^2)^(1/2)))*2^(1/2)/(f*x^2+e*x+d)^(1/2)/(2*c^2*d-b*c*e+b^2*f-2*a*c*f+(-b*
f+c*e)*(-4*a*c+b^2)^(1/2))^(1/2))*(b*B-2*A*c-B*(-4*a*c+b^2)^(1/2))*2^(1/2)/
(-4*a*c+b^2)^(1/2)/(2*c^2*d-b*c*e+b^2*f-2*a*c*f+(-b*f+c*e)*(-4*a*c+b^2)^(1/
2))^(1/2)
```

**Rubi** [A]
time = 1.79, antiderivative size = 416, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.094$, Rules used = {1046, 738, 212}

$$\frac{\left(-B\sqrt{b^2-4ac}-2Ac+bB\right)\tanh^{-1}\left(\frac{2x\left(ce-f\left(b-\sqrt{b^2-4ac}\right)\right)-e\left(b-\sqrt{b^2-4ac}\right)+4cd}{2\sqrt{2}\sqrt{d+ex+fx^2}\sqrt{\sqrt{b^2-4ac}\,(ce-bf)-2acf+b^2f-bce+2c^2d}}\right)}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{\sqrt{b^2-4ac}\,(ce-bf)-2acf+b^2f-bce+2c^2d}}+\frac{\left(2Ac-B\left(\sqrt{b^2-4ac}+b\right)\right)\tanh^{-1}\left(\frac{2x\left(ce-f\left(\sqrt{b^2-4ac}+b\right)\right)-e\left(\sqrt{b^2-4ac}+b\right)+4cd}{2\sqrt{2}\sqrt{d+ex+fx^2}\sqrt{-\sqrt{b^2-4ac}\,(ce-bf)-2acf+b^2f-bce+2c^2d}}\right)}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{-\sqrt{b^2-4ac}\,(ce-bf)-2acf+b^2f-bce+2c^2d}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]),x]
```

```
[Out] ((b*B - 2*A*c - B*Sqrt[b^2 - 4*a*c])*ArcTanh[(4*c*d - (b - Sqrt[b^2 - 4*a*c
])*e + 2*(c*e - (b - Sqrt[b^2 - 4*a*c])*f)*x)/(2*Sqrt[2]*Sqrt[2*c^2*d - b*c
*e + b^2*f - 2*a*c*f + Sqrt[b^2 - 4*a*c]*(c*e - b*f)]*Sqrt[d + e*x + f*x^2]
)])/(Sqrt[2]*Sqrt[b^2 - 4*a*c]*Sqrt[2*c^2*d - b*c*e + b^2*f - 2*a*c*f + Sqr
t[b^2 - 4*a*c]*(c*e - b*f)]) + ((2*A*c - B*(b + Sqrt[b^2 - 4*a*c]))*ArcTanh
[(4*c*d - (b + Sqrt[b^2 - 4*a*c])*e + 2*(c*e - (b + Sqrt[b^2 - 4*a*c])*f)*x
)/(2*Sqrt[2]*Sqrt[2*c^2*d - b*c*e + b^2*f - 2*a*c*f - Sqrt[b^2 - 4*a*c]*(c*
e - b*f)]*Sqrt[d + e*x + f*x^2])])/(Sqrt[2]*Sqrt[b^2 - 4*a*c]*Sqrt[2*c^2*d
- b*c*e + b^2*f - 2*a*c*f - Sqrt[b^2 - 4*a*c]*(c*e - b*f)])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{A+Bx}{\left(a+bx+cx^2\right)\sqrt{d+ex+fx^2}}\,dx = \frac{\left(2Ac-B\left(b-\sqrt{b^2-4ac}\right)\right)\int \frac{1}{\left(b-\sqrt{b^2-4ac}+2cx\right)\sqrt{d+ex+fx^2}} \cdots}{\sqrt{b^2-4ac}}$$

$$= \frac{\left(2\left(bB-2Ac-B\sqrt{b^2-4ac}\right)\right)\text{Subst}\left(\int \frac{}{16c^2d-8c\left(b-\sqrt{b^2-4ac}\right) \cdots} \cdots\right.}{}$$

$$= \frac{\left(bB-2Ac-B\sqrt{b^2-4ac}\right)\tanh^{-1}\left(\frac{4cd-\left(b-\sqrt{b^2-\cdots}\right.}{2\sqrt{2}\sqrt{2c^2d-bce+b^2f-\cdots}} \cdots\right.}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{2c^2d-bce+b^2f-2acf\cdots}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.45, size = 278, normalized size = 0.67

$$-\text{RootSum}\left[cd^2-bde+ae^2+2bd\sqrt{f}\#1-4ae\sqrt{f}\#1-2cd\#1^2+be\#1^2+4af\#1^2-2b\sqrt{f}\#1^3+c\#1^4\&,\frac{Bd\log\left(-\sqrt{f}x+\sqrt{d+ex+fx^2}-\#1\right)-Ae\log\left(-\sqrt{f}x+\sqrt{d+ex+fx^2}-\#1\right)+2A\sqrt{f}\log\left(-\sqrt{f}x+\sqrt{d+ex+fx^2}-\#1\right)\#1-B\log\left(-\sqrt{f}x+\sqrt{d+ex+fx^2}-\#1\right)\#1^2}{bd\sqrt{f}-2ae\sqrt{f}-2cd\#1+be\#1+4af\#1-3b\sqrt{f}\#1^2+2c\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]),x]
```

```
[Out] -RootSum[c*d^2 - b*d*e + a*e^2 + 2*b*d*Sqrt[f]*#1 - 4*a*e*Sqrt[f]*#1 - 2*c*
d*#1^2 + b*e*#1^2 + 4*a*f*#1^2 - 2*b*Sqrt[f]*#1^3 + c*#1^4 & , (B*d*Log[-(S
qrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1] - A*e*Log[-(Sqrt[f]*x) + Sqrt[d + e
*x + f*x^2] - #1] + 2*A*Sqrt[f]*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] -
#1]*#1 - B*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1^2)/(b*d*Sqrt[f
] - 2*a*e*Sqrt[f] - 2*c*d*#1 + b*e*#1 + 4*a*f*#1 - 3*b*Sqrt[f]*#1^2 + 2*c*#
1^3) & ]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 804 vs. $2(370) = 740$.
time = 0.20, size = 805, normalized size = 1.94

| method | result |
|---|---|
| default | $-\frac{\left(2Ac+B\sqrt{-4ac+b^2}-bB\right)\ln\left(\frac{-\frac{bf\sqrt{-4ac+b^2}-\sqrt{-4ac+b^2}\,ce+2acf-b^2f+bce-2c^2d}{c^2}-\frac{\left(-f\sqrt{-4ac+b^2}+bf\right.}{}}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -(2*A*c+B*(-4*a*c+b^2)^(1/2)-b*B)/(-4*a*c+b^2)^(1/2)/c/(-2*(b*f*(-4*a*c+b^2
)^(1/2)-(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f+b*c*e-2*c^2*d)/c^2)^(1/2)*ln((
-(b*f*(-4*a*c+b^2)^(1/2)-(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f+b*c*e-2*c^2*d
)/c^2-(-f*(-4*a*c+b^2)^(1/2)+b*f-c*e)/c*(x-1/2/c*(-b+(-4*a*c+b^2)^(1/2)))+1
/2*(-2*(b*f*(-4*a*c+b^2)^(1/2)-(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f+b*c*e-2
*c^2*d)/c^2)^(1/2)*(4*f*(x-1/2/c*(-b+(-4*a*c+b^2)^(1/2)))^2-4*(-f*(-4*a*c+b
^2)^(1/2)+b*f-c*e)/c*(x-1/2/c*(-b+(-4*a*c+b^2)^(1/2)))-2*(b*f*(-4*a*c+b^2)^
(1/2)-(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f+b*c*e-2*c^2*d)/c^2)^(1/2))/(x-1/
2/c*(-b+(-4*a*c+b^2)^(1/2))))-(-2*A*c+B*(-4*a*c+b^2)^(1/2)+b*B)/(-4*a*c+b^2
)^(1/2)/c/(-2*(-b*f*(-4*a*c+b^2)^(1/2)+(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f
+b*c*e-2*c^2*d)/c^2)^(1/2)*ln((-(-b*f*(-4*a*c+b^2)^(1/2)+(-4*a*c+b^2)^(1/2)
*c*e+2*a*c*f-b^2*f+b*c*e-2*c^2*d)/c^2-(f*(-4*a*c+b^2)^(1/2)+b*f-c*e)/c*(x+1
/2*(b+(-4*a*c+b^2)^(1/2))/c)+1/2*(-2*(-b*f*(-4*a*c+b^2)^(1/2)+(-4*a*c+b^2)^
(1/2)*c*e+2*a*c*f-b^2*f+b*c*e-2*c^2*d)/c^2)^(1/2)*(4*f*(x+1/2*(b+(-4*a*c+b^
2)^(1/2))/c)^2-4*(f*(-4*a*c+b^2)^(1/2)+b*f-c*e)/c*(x+1/2*(b+(-4*a*c+b^2)^(1
/2))/c)-2*(-b*f*(-4*a*c+b^2)^(1/2)+(-4*a*c+b^2)^(1/2)*c*e+2*a*c*f-b^2*f+b*c
*e-2*c^2*d)/c^2)^(1/2))/(x+1/2*(b+(-4*a*c+b^2)^(1/2))/c))
```

**Maxima** [F(-2)]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 21959 vs. $2(383) = 766$.
time = 169.42, size = 21959, normalized size = 52.79

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt(2)*sqrt(((B^2*b^2 + 2*A^2*c^2 - 2*(B^2*a + A*B*b)*c)*d + (2*B^2*a^
2 - 2*A*B*a*b + A^2*b^2 - 2*A^2*a*c)*f - (B^2*a*b - (4*A*B*a - A^2*b)*c)*e
+ ((b^2*c^2 - 4*a*c^3)*d^2 + (b^4 - 6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 -
 4*a^3*c)*f^2 + (a*b^2*c - 4*a^2*c^2)*e^2 - ((b^3*c - 4*a*b*c^2)*d + (a*b^3
 - 4*a^2*b*c)*f)*e)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(
2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a
^2 - 4*A^3*B*a*b + A^4*b^2)*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 -
 2*((B^4*a*b + 2*A^3*B*c^2 - (2*A*B^3*a + A^2*B^2*b)*c)*d + (2*A*B^3*a^2 -
A^2*B^2*a*b - (2*A^3*B*a - A^4*b)*c)*f)*e)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^
4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*b^2*c^2
- 24*a^3*c^3)*d^2*f^2 + 2*(a^2*b^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3 + (a^4*
b^2 - 4*a^5*c)*f^4 + (a^2*b^2*c^2 - 4*a^3*c^3)*e^4 - 2*((a*b^3*c^2 - 4*a^2*
b*c^3)*d + (a^2*b^3*c - 4*a^3*b*c^2)*f)*e^3 + ((b^4*c^2 - 2*a*b^2*c^3 - 8*a
^2*c^4)*d^2 + 4*(a*b^4*c - 5*a^2*b^2*c^2 + 4*a^3*c^3)*d*f + (a^2*b^4 - 2*a^
3*b^2*c - 8*a^4*c^2)*f^2)*e^2 - 2*((b^3*c^3 - 4*a*b*c^4)*d^3 + (b^5*c - 5*a
*b^3*c^2 + 4*a^2*b*c^3)*d^2*f + (a*b^5 - 5*a^2*b^3*c + 4*a^3*b*c^2)*d*f^2 +
 (a^3*b^3 - 4*a^4*b*c)*f^3)*e)))/((b^2*c^2 - 4*a*c^3)*d^2 + (b^4 - 6*a*b^2*
c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2 + (a*b^2*c - 4*a^2*c^2)*e^2 -
((b^3*c - 4*a*b*c^2)*d + (a*b^3 - 4*a^2*b*c)*f)*e))*log(-(2*(B^4*a*b^2 - A*
B^3*b^3 - 2*A^3*B*b*c^2 - (2*A*B^3*a*b - 3*A^2*B^2*b^2)*c)*d^2 + 2*(2*A*B^3
*a^2*b - 3*A^2*B^2*a*b^2 + A^3*B*b^3 + (2*A^3*B*a*b - A^4*b^2)*c)*d*f + sqr
t(2)*((B^3*b^4 - 8*A^2*B*a*c^3 + 2*(6*A*B^2*a*b + A^2*B*b^2)*c^2 - (4*B^3*a
*b^2 + 3*A*B^2*b^3)*c)*d^2 + (3*A*B^2*a*b^3 - A^2*B*b^4 + 4*(4*A^2*B*a^2 -
A^3*a*b)*c^2 - (12*A*B^2*a^2*b - A^3*b^3)*c)*d*f + (2*A^2*B*a^2*b^2 - A^3*a
```

```
*b^3 - 4*(2*A^2*B*a^3 - A^3*a^2*b)*c)*f^2 + (B^3*a^2*b^2 + 4*A^2*B*a^2*c^2
- (4*B^3*a^3 + A^2*B*a*b^2)*c)*e^2 - ((2*B^3*a*b^3 - 4*A^3*a*c^3 + (12*A*B^
2*a^2 + 4*A^2*B*a*b + A^3*b^2)*c^2 - (8*B^3*a^2*b + 3*A*B^2*a*b^2 + A^2*B*b
^3)*c)*d + (3*A*B^2*a^2*b^2 - A^2*B*a*b^3 + 4*A^3*a^2*c^2 - (12*A*B^2*a^3 -
 4*A^2*B*a^2*b + A^3*a*b^2)*c)*f)*e - ((B*b^4*c^2 + 4*(2*B*a^2 + A*a*b)*c^4
 - (6*B*a*b^2 + A*b^3)*c^3)*d^3 + (B*b^6 - 4*(6*B*a^3 + A*a^2*b)*c^3 + (22*
B*a^2*b^2 + 5*A*a*b^3)*c^2 - (8*B*a*b^4 + A*b^5)*c)*d^2*f + (3*B*a^2*b^4 -
A*a*b^5 + 4*(6*B*a^4 - A*a^3*b)*c^2 - (18*B*a^3*b^2 - 5*A*a^2*b^3)*c)*d*f^2
 + (2*B*a^4*b^2 - A*a^3*b^3 - 4*(2*B*a^5 - A*a^4*b)*c)*f^3 - (B*a^2*b^3*c +
 8*A*a^3*c^3 - 2*(2*B*a^3*b + A*a^2*b^2)*c^2)*e^3 + ((2*B*a*b^4*c + 4*(2*B*
a^3 + 3*A*a^2*b)*c^3 - (10*B*a^2*b^2 + 3*A*a*b^3)*c^2)*d + (B*a^2*b^4 - 4*(
2*B*a^4 - 3*A*a^3*b)*c^2 - (2*B*a^3*b^2 + 3*A*a^2*b^3)*c)*f)*e^2 - ((B*b^5*
c + 8*A*a^2*c^4 + 2*(2*B*a^2*b + A*a*b^2)*c^3 - (5*B*a*b^3 + A*b^4)*c^2)*d^
2 + 2*(B*a*b^5 - 8*A*a^3*c^3 + 2*(2*B*a^3*b + 5*A*a^2*b^2)*c^2 - (5*B*a^2*b
^3 + 2*A*a*b^4)*c)*d*f + (3*B*a^3*b^3 - A*a^2*b^4 + 8*A*a^4*c^2 - 2*(6*B*a^
4*b - A*a^3*b^2)*c)*f^2)*e)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d
^2 + 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A
^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^
2)*e^2 - 2*((B^4*a*b + 2*A^3*B*c^2 - (2*A*B^3*a + A^2*B^2*b)*c)*d + (2*A*B^
3*a^2 - A^2*B^2*a*b - (2*A^3*B*a - A^4*b)*c)*f)*e)/((b^2*c^4 - 4*a*c^5)*d^4
 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*
b^2*c^2 - 24*a^3*c^3)*d^2*f^2 + 2*(a^2*b^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3
 + (a^4*b^2 - 4*a^5*c)*f^4 + (a^2*b^2*c^2 - 4*a^3*c^3)*e^4 - 2*((a*b^3*c^2
- 4*a^2*b*c^3)*d + (a^2*b^3*c - 4*a^3*b*c^2)*f)*e^3 + ((b^4*c^2 - 2*a*b^2*c
^3 - 8*a^2*c^4)*d^2 + 4*(a*b^4*c - 5*a^2*b^2*c^2 + 4*a^3*c^3)*d*f + (a^2*b^
4 - 2*a^3*b^2*c - 8*a^4*c^2)*f^2)*e^2 - 2*((b^3*c^3 - 4*a*b*c^4)*d^3 + (b^5
*c - 5*a*b^3*c^2 + 4*a^2*b*c^3)*d^2*f + (a*b^5 - 5*a^2*b^3*c + 4*a^3*b*c^2)
*d*f^2 + (a^3*b^3 - 4*a^4*b*c)*f^3)*e)))*sqrt(f*x^2 + x*e + d)*sqrt(((B^2*b
^2 + 2*A^2*c^2 - 2*(B^2*a + A*B*b)*c)*d + (2*B^2*a^2 - 2*A*B*a*b + A^2*b^2
- 2*A^2*a*c)*f - (B^2*a*b - (4*A*B*a - A^2*b)*c)*e + ((b^2*c^2 - 4*a*c^3)*d
^2 + (b^4 - 6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2 + (a*b^2*c
 - 4*a^2*c^2)*e^2 - ((b^3*c - 4*a*b*c^2)*d + (a*b^3 - 4*a^2*b*c)*f)*e)*sqrt
(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(2*A*B^3*a*b - A^2*B^2*b^
2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b
^2)*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 - 2*((B^4*a*b + 2*A^3*B*c
^2 - (2*A*B^3*a + A^2*B^2*b)*c)*d + (2*A*B^3*a^2 - A^2*B^2*a*b - (2*A^3*B*a
 - A^4*b)*c)*f)*e)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*
a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*b^2*c^2 - 24*a^3*c^3)*d^2*f^2 +
2*(a^2*b^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3 + (a^4*b^2 - 4*a^5*c)*f^4 + (a^
2*b^2*c^2 - 4*a^3*c^3)*e^4 - 2*((a*b^3*c^2 - 4*a^2*b*c^3)*d + (a^2*b^3*c -
4*a^3*b*c^2)*f)*e^3 + ((b^4*c^2 - 2*a*b^2*c^3 - 8*a^2*c^4)*d^2 + 4*(a*b^4*c
 - 5*a^2*b^2*c^2 + 4*a^3*c^3)*d*f + (a^2*b^4 - 2*a^3*b^2*c - 8*a^4*c^2)*f^2
)*e^2 - 2*((b^3*c^3 - 4*a*b*c^4)*d^3 + (b^5*c -...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{A+Bx}{(a+bx+cx^2)\sqrt{d+ex+fx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)/(f*x**2+e*x+d)**(1/2),x)
```

```
[Out] Integral((A + B*x)/((a + b*x + c*x**2)*sqrt(d + e*x + f*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Unable to divide, perhaps due to roun
ding error%%%{%%{poly1[%%%{-4,[3,2,0]%%%}+%%%{16,[1,3,1]%%%},%%%{4,[4,2,0]%
%%}+%%
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{\left(c\,x^2+b\,x+a\right)\,\sqrt{f\,x^2+e\,x+d}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a + b*x + c*x^2)*(d + e*x + f*x^2)^(1/2)),x)
```

```
[Out] int((A + B*x)/((a + b*x + c*x^2)*(d + e*x + f*x^2)^(1/2)), x)
```

## 3.22 $\int \frac{A+Bx}{(a+cx^2)\sqrt{d+ex+fx^2}}\,dx$

**Optimal**. Leaf size=780

$$\sqrt{aBe+A\left(cd-af-\sqrt{c^2d^2+a^2f^2+ac\left(e^2-2df\right)}\right)}\sqrt{-Ace+B\left(cd-af+\sqrt{c^2d^2+a^2f^2+ac\left(e^2-\right.}\right.}$$

```
[Out] -1/2*arctanh(1/2*e^(1/2)*(a*(A*c*e-B*(c*d-a*f-(c^2*d^2+a^2*f^2+a*c*(-2*d*f+
e^2))^(1/2)))-c*x*(a*B*e+A*(c*d-a*f+(c^2*d^2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2
))))*2^(1/2)/a^(1/2)/c^(1/2)/(f*x^2+e*x+d)^(1/2)/(-A*c*e+B*(c*d-a*f-(c^2*d^
2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2)/(a*B*e+A*(c*d-a*f+(c^2*d^2+a^2*f^
2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2))*(-A*c*e+B*(c*d-a*f-(c^2*d^2+a^2*f^2+a*c*
(-2*d*f+e^2))^(1/2)))^(1/2)*(a*B*e+A*(c*d-a*f+(c^2*d^2+a^2*f^2+a*c*(-2*d*f+
e^2))^(1/2)))^(1/2)*2^(1/2)/a^(1/2)/c^(1/2)/e^(1/2)/(c^2*d^2+a^2*f^2+a*c*(-
2*d*f+e^2))^(1/2)+1/2*arctanh(1/2*e^(1/2)*(-c*x*(a*B*e+A*(c*d-a*f-(c^2*d^2+
a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)))+a*(A*c*e-B*(c*d-a*f+(c^2*d^2+a^2*f^2+a*c*
(-2*d*f+e^2))^(1/2))))*2^(1/2)/a^(1/2)/c^(1/2)/(f*x^2+e*x+d)^(1/2)/(a*B*e+A
*(c*d-a*f-(c^2*d^2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2)/(-A*c*e+B*(c*d-a
*f+(c^2*d^2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2))*(a*B*e+A*(c*d-a*f-(c^2
*d^2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2)*(-A*c*e+B*(c*d-a*f+(c^2*d^2+a^
2*f^2+a*c*(-2*d*f+e^2))^(1/2)))^(1/2)*2^(1/2)/a^(1/2)/c^(1/2)/e^(1/2)/(c^2*
d^2+a^2*f^2+a*c*(-2*d*f+e^2))^(1/2)
```

**Rubi** [A]
time = 3.39, antiderivative size = 780, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 29, $\frac{\text{number of rules}}{\text{integrand size}} = 0.103$, Rules used = {1050, 1044, 214}

$$\frac{\sqrt{A\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe}\sqrt{B\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)-Ace}\tanh^{-1}\left(\frac{\sqrt{e}\left(a\left(Ace-B\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)\right)-cx\left(A\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe\right)\right)}{\sqrt{2}\sqrt{a}\sqrt{c}\sqrt{d+ex+fx^2}\sqrt{A\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe}\sqrt{B\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)-Ace}}\right)}{\sqrt{2}\sqrt{a}\sqrt{c}\sqrt{e}\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}}-\frac{\sqrt{B\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)-Ace}\sqrt{A\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe}\tanh^{-1}\left(\frac{\sqrt{e}\left(a\left(Ace-B\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)\right)-cx\left(A\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe\right)\right)}{\sqrt{2}\sqrt{a}\sqrt{c}\sqrt{d+ex+fx^2}\sqrt{B\left(-\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)-Ace}\sqrt{A\left(\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}-af+cd\right)+aBe}}\right)}{\sqrt{2}\sqrt{a}\sqrt{c}\sqrt{e}\sqrt{a^2f^2+ac(e^2-2df)+c^2d^2}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + c*x^2)*Sqrt[d + e*x + f*x^2]),x]
```

```
[Out] (Sqrt[a*B*e + A*(c*d - a*f - Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])]*
Sqrt[-(A*c*e) + B*(c*d - a*f + Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])
]*ArcTanh[(Sqrt[e]*(a*(A*c*e - B*(c*d - a*f + Sqrt[c^2*d^2 + a^2*f^2 + a*c*
(e^2 - 2*d*f)])) - c*(a*B*e + A*(c*d - a*f - Sqrt[c^2*d^2 + a^2*f^2 + a*c*(
e^2 - 2*d*f)]))*x))/(Sqrt[2]*Sqrt[a]*Sqrt[c]*Sqrt[a*B*e + A*(c*d - a*f - Sq
rt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])]*Sqrt[-(A*c*e) + B*(c*d - a*f +
Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])]*Sqrt[d + e*x + f*x^2])])/(Sqr
t[2]*Sqrt[a]*Sqrt[c]*Sqrt[e]*Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)]) -
```

```
 (Sqrt[-(A*c*e) + B*(c*d - a*f - Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)
])]*Sqrt[a*B*e + A*(c*d - a*f + Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)]
)]*ArcTanh[(Sqrt[e]*(a*(A*c*e - B*(c*d - a*f - Sqrt[c^2*d^2 + a^2*f^2 + a*c
*(e^2 - 2*d*f)])) - c*(a*B*e + A*(c*d - a*f + Sqrt[c^2*d^2 + a^2*f^2 + a*c*
(e^2 - 2*d*f)]))*x))/(Sqrt[2]*Sqrt[a]*Sqrt[c]*Sqrt[-(A*c*e) + B*(c*d - a*f
- Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])]*Sqrt[a*B*e + A*(c*d - a*f +
 Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])]*Sqrt[d + e*x + f*x^2])])/(Sq
rt[2]*Sqrt[a]*Sqrt[c]*Sqrt[e]*Sqrt[c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 1044

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f
_.)*(x_)^2]), x_Symbol] :> Dist[-2*a*g*h, Subst[Int[1/Simp[2*a^2*g*h*c + a*
e*x^2, x], x], x, Simp[a*h - g*c*x, x]/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[
{a, c, d, e, f, g, h}, x] && EqQ[a*h^2*e + 2*g*h*(c*d - a*f) - g^2*c*e, 0]
```

Rule 1050

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 + a*c*e^2, 2]}, Dist
[1/(2*q), Int[Simp[(-a)*h*e - g*(c*d - a*f - q) + (h*(c*d - a*f + q) - g*c*
e)*x, x]/((a + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[Si
mp[(-a)*h*e - g*(c*d - a*f + q) + (h*(c*d - a*f - q) - g*c*e)*x, x]/((a + c
*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f, g, h}, x] &&
 NeQ[e^2 - 4*d*f, 0] && NegQ[(-a)*c]
```

Rubi steps

$$\int \frac{A+Bx}{(a+cx^2)\sqrt{d+ex+fx^2}}\,dx = -\frac{\int \frac{-aBe-A\left(cd-af+\sqrt{c^2d^2+a^2f^2+ac(e^2-2df)}\right)+\left(-Ace+B\left(cd-af-\sqrt{c^2d\right.\right.}{(a+cx^2)\sqrt{d+ex+fx^2}}}{2\sqrt{c^2d^2+a^2f^2+ac(e^2-2df)}}$$

$$= \frac{\left(a\left(Ace-B\left(cd-af-\sqrt{c^2d^2+a^2f^2+ac(e^2-2df)}\right)\right)\left(aBe+A\left(ca\right.\right.\right.}{}$$

$$= \frac{\sqrt{aBe+A\left(cd-af-\sqrt{c^2d^2+a^2f^2+ac(e^2-2df)}\right)}\sqrt{-Ace+B\left(c\right.}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.39, size = 218, normalized size = 0.28

$$\frac{1}{2}\text{RootSum}\left[cd^2+ae^2-4ae\sqrt{f}\,\#1-2cd\#1^2+4af\#1^2+c\#1^4\&,\frac{Bd\log\left(-\sqrt{f}\,x+\sqrt{d+ex+fx^2}-\#1\right)-Ae\log\left(-\sqrt{f}\,x+\sqrt{d+ex+fx^2}-\#1\right)+2A\sqrt{f}\,\log\left(-\sqrt{f}\,x+\sqrt{d+ex+fx^2}-\#1\right)\#1-B\log\left(-\sqrt{f}\,x+\sqrt{d+ex+fx^2}-\#1\right)\#1^2}{ae\sqrt{f}+cd\#1-2af\#1-c\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + c*x^2)*Sqrt[d + e*x + f*x^2]),x]
```

```
[Out] RootSum[c*d^2 + a*e^2 - 4*a*e*Sqrt[f]*#1 - 2*c*d*#1^2 + 4*a*f*#1^2 + c*#1^4
 & , (B*d*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1] - A*e*Log[-(Sqrt[f
]*x) + Sqrt[d + e*x + f*x^2] - #1] + 2*A*Sqrt[f]*Log[-(Sqrt[f]*x) + Sqrt[d
+ e*x + f*x^2] - #1]*#1 - B*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*
#1^2)/(a*e*Sqrt[f] + c*d*#1 - 2*a*f*#1 - c*#1^3) & ]/2
```

**Maple** [A]
time = 0.14, size = 425, normalized size = 0.54

| method | result |
|---|---|

| | |
|---|---|
| default | $-\frac{\left(Ac+B\sqrt{-ac}\right)\ln\left(\frac{-\frac{2\left(-\sqrt{-ac}\,e+fa-cd\right)}{c}+\frac{\left(2f\sqrt{-ac}+ce\right)\left(x-\frac{\sqrt{-ac}}{c}\right)}{c}+2\sqrt{-\frac{-\sqrt{-ac}\,e+fa-cd}{c}}\sqrt{f\left(x\right.}}{x-\frac{\sqrt{-ac}}{c}}\right.}{2\sqrt{-ac}\,c\sqrt{-\frac{-\sqrt{-ac}\,e+fa-cd}{c}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+a)/(f*x^2+e*x+d)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(A*c+B*(-a*c)^(1/2))/(-a*c)^(1/2)/c/(-(-(-a*c)^(1/2)*e+f*a-c*d)/c)^(1/
2)*ln((-2*(-(-a*c)^(1/2)*e+f*a-c*d)/c+(2*f*(-a*c)^(1/2)+c*e)/c*(x-(-a*c)^(1
/2)/c)+2*(-(-(-a*c)^(1/2)*e+f*a-c*d)/c)^(1/2)*(f*(x-(-a*c)^(1/2)/c)^2+(2*f*
(-a*c)^(1/2)+c*e)/c*(x-(-a*c)^(1/2)/c)-(-(-a*c)^(1/2)*e+f*a-c*d)/c)^(1/2))/
(x-(-a*c)^(1/2)/c))-1/2*(-A*c+B*(-a*c)^(1/2))/(-a*c)^(1/2)/c/(-((-a*c)^(1/2
)*e+f*a-c*d)/c)^(1/2)*ln((-2*((-a*c)^(1/2)*e+f*a-c*d)/c+1/c*(-2*f*(-a*c)^(1
/2)+c*e)*(x+(-a*c)^(1/2)/c)+2*(-((-a*c)^(1/2)*e+f*a-c*d)/c)^(1/2)*(f*(x+(-a
*c)^(1/2)/c)^2+1/c*(-2*f*(-a*c)^(1/2)+c*e)*(x+(-a*c)^(1/2)/c)-((-a*c)^(1/2)
*e+f*a-c*d)/c)^(1/2))/(x+(-a*c)^(1/2)/c))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((B*x + A)/((c*x^2 + a)*sqrt(f*x^2 + x*e + d)), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 6737 vs. $2(741) = 1482$.
time = 25.99, size = 6737, normalized size = 8.64

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt(-(2*A*B*a*c*e - (B^2*a*c - A^2*c^2)*d + (B^2*a^2 - A^2*a*c)*f + (
a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + a^2*c^2*e^2)*sqrt(-(4*A^2*B^2*c^2*d
^2 - 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4
```

```
*c^2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)*d - (A*B^3*a^2 - A^3*B*a*c)*f)*e)/(a
*c^5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3*d^2*f^2 - 4*a^4*c^2*d*f^3 + a^5*c*f^
4 + a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a^3*c^3*d*f + a^4*c^2*f^2)*e^2)))/(a*c
^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + a^2*c^2*e^2))*log(-(4*((A*B^3*a*c + A^
3*B*c^2)*d*f - (A*B^3*a^2 + A^3*B*a*c)*f^2)*x + 2*(2*A^2*B*c^3*d^2 - 4*A^2*
B*a*c^2*d*f + 2*A^2*B*a^2*c*f^2 + (B^3*a^2*c - A^2*B*a*c^2)*e^2 + ((3*A*B^2
*a*c^2 - A^3*c^3)*d - (3*A*B^2*a^2*c - A^3*a*c^2)*f)*e - (B*a*c^4*d^3 - 3*B
*a^2*c^3*d^2*f + 3*B*a^3*c^2*d*f^2 - B*a^4*c*f^3 - A*a^2*c^3*e^3 + (B*a^2*c
^3*d - B*a^3*c^2*f)*e^2 - (A*a*c^4*d^2 - 2*A*a^2*c^3*d*f + A*a^3*c^2*f^2)*e
)*sqrt(-(4*A^2*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2*f^2 + (B^4*a
^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)*d - (A*B^3*a
^2 - A^3*B*a*c)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3*d^2*f^2 - 4*
a^4*c^2*d*f^3 + a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a^3*c^3*d*f +
a^4*c^2*f^2)*e^2)))*sqrt(f*x^2 + x*e + d)*sqrt(-(2*A*B*a*c*e - (B^2*a*c - A
^2*c^2)*d + (B^2*a^2 - A^2*a*c)*f + (a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2
+ a^2*c^2*e^2)*sqrt(-(4*A^2*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2
*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)
*d - (A*B^3*a^2 - A^3*B*a*c)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3
*d^2*f^2 - 4*a^4*c^2*d*f^3 + a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a
^3*c^3*d*f + a^4*c^2*f^2)*e^2)))/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + a
^2*c^2*e^2)) + (B^4*a^2 - A^4*c^2)*e^2 + 2*((B^4*a^2 - A^4*c^2)*f*x + (A*B^
3*a*c + A^3*B*c^2)*d - (A*B^3*a^2 + A^3*B*a*c)*f)*e - (2*(B^2*a*c^3 + A^2*c
^4)*d^3 - 4*(B^2*a^2*c^2 + A^2*a*c^3)*d^2*f + 2*(B^2*a^3*c + A^2*a^2*c^2)*d
*f^2 + (B^2*a^2*c^2 + A^2*a*c^3)*x*e^3 + 2*(B^2*a^2*c^2 + A^2*a*c^3)*d*e^2
+ ((B^2*a*c^3 + A^2*c^4)*d^2 - 2*(B^2*a^2*c^2 + A^2*a*c^3)*d*f + (B^2*a^3*c
 + A^2*a^2*c^2)*f^2)*x*e)*sqrt(-(4*A^2*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*
A^2*B^2*a^2*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c -
 A^3*B*c^2)*d - (A*B^3*a^2 - A^3*B*a*c)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f
+ 6*a^3*c^3*d^2*f^2 - 4*a^4*c^2*d*f^3 + a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^
4*d^2 - 2*a^3*c^3*d*f + a^4*c^2*f^2)*e^2)))/x) + 1/4*sqrt(-(2*A*B*a*c*e - (
B^2*a*c - A^2*c^2)*d + (B^2*a^2 - A^2*a*c)*f + (a*c^3*d^2 - 2*a^2*c^2*d*f +
 a^3*c*f^2 + a^2*c^2*e^2)*sqrt(-(4*A^2*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*
A^2*B^2*a^2*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c -
 A^3*B*c^2)*d - (A*B^3*a^2 - A^3*B*a*c)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f
+ 6*a^3*c^3*d^2*f^2 - 4*a^4*c^2*d*f^3 + a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^
4*d^2 - 2*a^3*c^3*d*f + a^4*c^2*f^2)*e^2)))/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^
3*c*f^2 + a^2*c^2*e^2))*log(-(4*((A*B^3*a*c + A^3*B*c^2)*d*f - (A*B^3*a^2 +
 A^3*B*a*c)*f^2)*x - 2*(2*A^2*B*c^3*d^2 - 4*A^2*B*a*c^2*d*f + 2*A^2*B*a^2*c
*f^2 + (B^3*a^2*c - A^2*B*a*c^2)*e^2 + ((3*A*B^2*a*c^2 - A^3*c^3)*d - (3*A*
B^2*a^2*c - A^3*a*c^2)*f)*e - (B*a*c^4*d^3 - 3*B*a^2*c^3*d^2*f + 3*B*a^3*c^
2*d*f^2 - B*a^4*c*f^3 - A*a^2*c^3*e^3 + (B*a^2*c^3*d - B*a^3*c^2*f)*e^2 - (
A*a*c^4*d^2 - 2*A*a^2*c^3*d*f + A*a^3*c^2*f^2)*e)*sqrt(-(4*A^2*B^2*c^2*d^2
- 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2*f^2 + (B^4*a^2 - 2*A^2*B^2*a*c + A^4*c^
2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)*d - (A*B^3*a^2 - A^3*B*a*c)*f)*e)/(a*c^
5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3*d^2*f^2 - 4*a^4*c^2*d*f^3 + a^5*c*f^4 +
```

```
 a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a^3*c^3*d*f + a^4*c^2*f^2)*e^2)))*sqrt(f*
x^2 + x*e + d)*sqrt(-(2*A*B*a*c*e - (B^2*a*c - A^2*c^2)*d + (B^2*a^2 - A^2*
a*c)*f + (a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + a^2*c^2*e^2)*sqrt(-(4*A^2
*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2*f^2 + (B^4*a^2 - 2*A^2*B^2
*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)*d - (A*B^3*a^2 - A^3*B*a*c
)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3*d^2*f^2 - 4*a^4*c^2*d*f^3
+ a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a^3*c^3*d*f + a^4*c^2*f^2)*e
^2)))/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + a^2*c^2*e^2)) + (B^4*a^2 - A
^4*c^2)*e^2 + 2*((B^4*a^2 - A^4*c^2)*f*x + (A*B^3*a*c + A^3*B*c^2)*d - (A*B
^3*a^2 + A^3*B*a*c)*f)*e - (2*(B^2*a*c^3 + A^2*c^4)*d^3 - 4*(B^2*a^2*c^2 +
A^2*a*c^3)*d^2*f + 2*(B^2*a^3*c + A^2*a^2*c^2)*d*f^2 + (B^2*a^2*c^2 + A^2*a
*c^3)*x*e^3 + 2*(B^2*a^2*c^2 + A^2*a*c^3)*d*e^2 + ((B^2*a*c^3 + A^2*c^4)*d^
2 - 2*(B^2*a^2*c^2 + A^2*a*c^3)*d*f + (B^2*a^3*c + A^2*a^2*c^2)*f^2)*x*e)*s
qrt(-(4*A^2*B^2*c^2*d^2 - 8*A^2*B^2*a*c*d*f + 4*A^2*B^2*a^2*f^2 + (B^4*a^2
- 2*A^2*B^2*a*c + A^4*c^2)*e^2 + 4*((A*B^3*a*c - A^3*B*c^2)*d - (A*B^3*a^2
- A^3*B*a*c)*f)*e)/(a*c^5*d^4 - 4*a^2*c^4*d^3*f + 6*a^3*c^3*d^2*f^2 - 4*a^4
*c^2*d*f^3 + a^5*c*f^4 + a^3*c^3*e^4 + 2*(a^2*c^4*d^2 - 2*a^3*c^3*d*f + a^4
*c^2*f^2)*e^2)))/x) - 1/4*sqrt(-(2*A*B*a*c*e - (B^2*a*c - A^2*c^2)*d + (B^2
*a^2 - A^2*a*c)*f - (a*c^3*d^2 - 2*a^2*c^2*d*f ...
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{A+Bx}{(a+cx^2)\sqrt{d+ex+fx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+a)/(f*x**2+e*x+d)**(1/2),x)
```

```
[Out] Integral((A + B*x)/((a + c*x**2)*sqrt(d + e*x + f*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+e*x+d)^(1/2),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueWarning, integration
of abs
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{(c\,x^2+a)\,\sqrt{f\,x^2+e\,x+d}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a + c*x^2)*(d + e*x + f*x^2)^(1/2)),x)
[Out] int((A + B*x)/((a + c*x^2)*(d + e*x + f*x^2)^(1/2)), x)
```

## 3.23 $\int \frac{A+Bx}{(a+bx+cx^2)\sqrt{d+fx^2}}\,dx$

**Optimal**. Leaf size=302

$$\frac{\left(bB-2Ac-B\sqrt{b^2-4ac}\right)\tanh^{-1}\left(\frac{2cd-\left(b-\sqrt{b^2-4ac}\right)fx}{\sqrt{2}\sqrt{2c^2d-2acf+b\left(b-\sqrt{b^2-4ac}\right)f}\sqrt{d+fx^2}}\right)}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{2c^2d-2acf+b\left(b-\sqrt{b^2-4ac}\right)f}}+\frac{\left(2Ac-\right.}{}$$

```
[Out] 1/2*arctanh(1/2*(2*c*d-f*x*(b-(-4*a*c+b^2)^(1/2)))*2^(1/2)/(f*x^2+d)^(1/2)/
(2*c^2*d-2*a*c*f+b*f*(b-(-4*a*c+b^2)^(1/2)))^(1/2))*(b*B-2*A*c-B*(-4*a*c+b^
2)^(1/2))*2^(1/2)/(-4*a*c+b^2)^(1/2)/(2*c^2*d-2*a*c*f+b*f*(b-(-4*a*c+b^2)^(
1/2)))^(1/2)+1/2*arctanh(1/2*(2*c*d-f*x*(b+(-4*a*c+b^2)^(1/2)))*2^(1/2)/(f*
x^2+d)^(1/2)/(2*c^2*d-2*a*c*f+b*f*(b+(-4*a*c+b^2)^(1/2)))^(1/2))*(2*A*c-B*(
b+(-4*a*c+b^2)^(1/2)))*2^(1/2)/(-4*a*c+b^2)^(1/2)/(2*c^2*d-2*a*c*f+b*f*(b+(
-4*a*c+b^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 0.55, antiderivative size = 302, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 29, $\frac{\text{number of rules}}{\text{integrand size}} = 0.103$, Rules used = {1048, 739, 212}

$$\frac{\left(-B\sqrt{b^2-4ac}-2Ac+bB\right)\tanh^{-1}\left(\frac{2cd-fx\left(b-\sqrt{b^2-4ac}\right)}{\sqrt{2}\sqrt{d+fx^2}\sqrt{bf\left(b-\sqrt{b^2-4ac}\right)-2acf+2c^2d}}\right)}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{bf\left(b-\sqrt{b^2-4ac}\right)-2acf+2c^2d}}+\frac{\left(2Ac-B\left(\sqrt{b^2-4ac}+b\right)\right)\tanh^{-1}\left(\frac{2cd-fx\left(\sqrt{b^2-4ac}+b\right)}{\sqrt{2}\sqrt{d+fx^2}\sqrt{bf\left(\sqrt{b^2-4ac}+b\right)-2acf+2c^2d}}\right)}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{bf\left(\sqrt{b^2-4ac}+b\right)-2acf+2c^2d}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + b*x + c*x^2)*Sqrt[d + f*x^2]),x]
```

```
[Out] ((b*B - 2*A*c - B*Sqrt[b^2 - 4*a*c])*ArcTanh[(2*c*d - (b - Sqrt[b^2 - 4*a*c
])*f*x)/(Sqrt[2]*Sqrt[2*c^2*d - 2*a*c*f + b*(b - Sqrt[b^2 - 4*a*c])*f]*Sqrt
[d + f*x^2])])/(Sqrt[2]*Sqrt[b^2 - 4*a*c]*Sqrt[2*c^2*d - 2*a*c*f + b*(b - S
qrt[b^2 - 4*a*c])*f]) + ((2*A*c - B*(b + Sqrt[b^2 - 4*a*c]))*ArcTanh[(2*c*d
 - (b + Sqrt[b^2 - 4*a*c])*f*x)/(Sqrt[2]*Sqrt[2*c^2*d - 2*a*c*f + b*(b + Sq
rt[b^2 - 4*a*c])*f]*Sqrt[d + f*x^2])])/(Sqrt[2]*Sqrt[b^2 - 4*a*c]*Sqrt[2*c^
2*d - 2*a*c*f + b*(b + Sqrt[b^2 - 4*a*c])*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{A+Bx}{\left(a+bx+cx^2\right)\sqrt{d+fx^2}}\,dx = \frac{\left(2Ac-B\left(b-\sqrt{b^2-4ac}\right)\right)\int \frac{1}{\left(b-\sqrt{b^2-4ac}+2cx\right)\sqrt{d+fx^2}}\,dx}{\sqrt{b^2-4ac}} - \frac{(2\ldots}{\ldots}$$

$$= -\frac{\left(2Ac-B\left(b-\sqrt{b^2-4ac}\right)\right)\operatorname{Subst}\left(\int \frac{1}{4c^2d+\left(b-\sqrt{b^2-4ac}\right)^2f-x^2}\,dx, x\ldots\right.}{\sqrt{b^2-4ac}}$$

$$= \frac{\left(bB-2Ac-B\sqrt{b^2-4ac}\right)\tanh^{-1}\left(\frac{2cd-\left(b-\sqrt{b^2-4ac\ldots}\right.}{\sqrt{2}\sqrt{2c^2d-2acf+b\left(b-\sqrt{b^2-\ldots}\right.}}\right.}{\sqrt{2}\sqrt{b^2-4ac}\sqrt{2c^2d-2acf+b\left(b-\sqrt{b^2-4ac}\right)\ldots}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.39, size = 195, normalized size = 0.65

$$-\text{RootSum}\left[cd^2+2bd\sqrt{f}\,\#1-2cd\#1^2+4af\#1^2-2b\sqrt{f}\,\#1^3+c\#1^4\&, \frac{Bd\log\left(-\sqrt{f}\,x+\sqrt{d+fx^2}-\#1\right)+2A\sqrt{f}\,\log\left(-\sqrt{f}\,x+\sqrt{d+fx^2}-\#1\right)\#1-B\log\left(-\sqrt{f}\,x+\sqrt{d+fx^2}-\#1\right)\#1^2}{bd\sqrt{f}-2cd\#1+4af\#1-3b\sqrt{f}\,\#1^2+2c\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + b*x + c*x^2)*Sqrt[d + f*x^2]),x]
```

```
[Out] -RootSum[c*d^2 + 2*b*d*Sqrt[f]*#1 - 2*c*d*#1^2 + 4*a*f*#1^2 - 2*b*Sqrt[f]*#
1^3 + c*#1^4 & , (B*d*Log[-(Sqrt[f]*x) + Sqrt[d + f*x^2] - #1] + 2*A*Sqrt[f
```

```
]*Log[-(Sqrt[f]*x) + Sqrt[d + f*x^2] - #1]*#1 - B*Log[-(Sqrt[f]*x) + Sqrt[d
 + f*x^2] - #1]*#1^2)/(b*d*Sqrt[f] - 2*c*d*#1 + 4*a*f*#1 - 3*b*Sqrt[f]*#1^2
 + 2*c*#1^3) & ]
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 638 vs. $2(266) = 532$.

time = 0.16, size = 639, normalized size = 2.12

| method | result |
|---|---|
| default | $-\frac{\left(2Ac+B\sqrt{-4ac+b^2}-bB\right)\ln\left(\frac{-\frac{bf\sqrt{-4ac+b^2}+2acf-b^2f-2c^2d}{c^2}-\frac{f\left(b-\sqrt{-4ac+b^2}\right)\left(x-\frac{-b+\sqrt{-4ac+b^2}}{2c}\right.}{c}}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -(2*A*c+B*(-4*a*c+b^2)^(1/2)-b*B)/(-4*a*c+b^2)^(1/2)/c/(-2*(b*f*(-4*a*c+b^2
)^(1/2)+2*a*c*f-b^2*f-2*c^2*d)/c^2)^(1/2)*ln((-(b*f*(-4*a*c+b^2)^(1/2)+2*a*
c*f-b^2*f-2*c^2*d)/c^2-f*(b-(-4*a*c+b^2)^(1/2))/c*(x-1/2/c*(-b+(-4*a*c+b^2)
^(1/2)))+1/2*(-2*(b*f*(-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*c^2*d)/c^2)^(1/2)*
(4*f*(x-1/2/c*(-b+(-4*a*c+b^2)^(1/2)))^2-4*f*(b-(-4*a*c+b^2)^(1/2))/c*(x-1/
2/c*(-b+(-4*a*c+b^2)^(1/2)))-2*(b*f*(-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*c^2*
d)/c^2)^(1/2))/(x-1/2/c*(-b+(-4*a*c+b^2)^(1/2))))-(-2*A*c+B*(-4*a*c+b^2)^(1
/2)+b*B)/(-4*a*c+b^2)^(1/2)/c/(-2*(-b*f*(-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*
c^2*d)/c^2)^(1/2)*ln((-(-b*f*(-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*c^2*d)/c^2-
f*(b+(-4*a*c+b^2)^(1/2))/c*(x+1/2*(b+(-4*a*c+b^2)^(1/2))/c)+1/2*(-2*(-b*f*(
-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*c^2*d)/c^2)^(1/2)*(4*f*(x+1/2*(b+(-4*a*c+
b^2)^(1/2))/c)^2-4*f*(b+(-4*a*c+b^2)^(1/2))/c*(x+1/2*(b+(-4*a*c+b^2)^(1/2))
/c)-2*(-b*f*(-4*a*c+b^2)^(1/2)+2*a*c*f-b^2*f-2*c^2*d)/c^2)^(1/2))/(x+1/2*(b
+(-4*a*c+b^2)^(1/2))/c))
```

**Maxima** [**F(-2)**]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 8977 vs. $2(263) = 526$.
time = 33.48, size = 8977, normalized size = 29.73

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt(2)*sqrt(((B^2*b^2 + 2*A^2*c^2 - 2*(B^2*a + A*B*b)*c)*d + (2*B^2*a^
2 - 2*A*B*a*b + A^2*b^2 - 2*A^2*a*c)*f + ((b^2*c^2 - 4*a*c^3)*d^2 + (b^4 -
6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2)*sqrt(((B^4*b^2 - 4*A*
B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*
a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/((b^2*c^
4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a
*b^4*c + 22*a^2*b^2*c^2 - 24*a^3*c^3)*d^2*f^2 + 2*(a^2*b^4 - 6*a^3*b^2*c +
8*a^4*c^2)*d*f^3 + (a^4*b^2 - 4*a^5*c)*f^4)))/((b^2*c^2 - 4*a*c^3)*d^2 + (b
^4 - 6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2))*log((2*(B^4*a*b
^2 - A*B^3*b^3 - 2*A^3*B*b*c^2 - (2*A*B^3*a*b - 3*A^2*B^2*b^2)*c)*d^2 + 2*(
2*A*B^3*a^2*b - 3*A^2*B^2*a*b^2 + A^3*B*b^3 + (2*A^3*B*a*b - A^4*b^2)*c)*d*
f + sqrt(2)*((B^3*b^4 - 8*A^2*B*a*c^3 + 2*(6*A*B^2*a*b + A^2*B*b^2)*c^2 - (
4*B^3*a*b^2 + 3*A*B^2*b^3)*c)*d^2 + (3*A*B^2*a*b^3 - A^2*B*b^4 + 4*(4*A^2*B
*a^2 - A^3*a*b)*c^2 - (12*A*B^2*a^2*b - A^3*b^3)*c)*d*f + (2*A^2*B*a^2*b^2
- A^3*a*b^3 - 4*(2*A^2*B*a^3 - A^3*a^2*b)*c)*f^2 - ((B*b^4*c^2 + 4*(2*B*a^2
 + A*a*b)*c^4 - (6*B*a*b^2 + A*b^3)*c^3)*d^3 + (B*b^6 - 4*(6*B*a^3 + A*a^2*
b)*c^3 + (22*B*a^2*b^2 + 5*A*a*b^3)*c^2 - (8*B*a*b^4 + A*b^5)*c)*d^2*f + (3
*B*a^2*b^4 - A*a*b^5 + 4*(6*B*a^4 - A*a^3*b)*c^2 - (18*B*a^3*b^2 - 5*A*a^2*
b^3)*c)*d*f^2 + (2*B*a^4*b^2 - A*a^3*b^3 - 4*(2*B*a^5 - A*a^4*b)*c)*f^3)*sq
rt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(2*A*B^3*a*b - A^2*B^2*
b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4
*b^2)*f^2)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4)
*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*b^2*c^2 - 24*a^3*c^3)*d^2*f^2 + 2*(a^2*b
^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3 + (a^4*b^2 - 4*a^5*c)*f^4)))*sqrt(f*x^2
 + d)*sqrt(((B^2*b^2 + 2*A^2*c^2 - 2*(B^2*a + A*B*b)*c)*d + (2*B^2*a^2 - 2*
A*B*a*b + A^2*b^2 - 2*A^2*a*c)*f + ((b^2*c^2 - 4*a*c^3)*d^2 + (b^4 - 6*a*b^
2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2)*sqrt(((B^4*b^2 - 4*A*B^3*b*
c + 4*A^2*B^2*c^2)*d^2 + 2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^
3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/((b^2*c^4 - 4*
a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c
 + 22*a^2*b^2*c^2 - 24*a^3*c^3)*d^2*f^2 + 2*(a^2*b^4 - 6*a^3*b^2*c + 8*a^4*
```

```
c^2)*d*f^3 + (a^4*b^2 - 4*a^5*c)*f^4)))/((b^2*c^2 - 4*a*c^3)*d^2 + (b^4 - 6
*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2)) - 4*((B^4*a^2*b - A*B
^3*a*b^2 - 2*A^3*B*a*c^2 - (2*A*B^3*a^2 - 3*A^2*B^2*a*b)*c)*d*f + (2*A*B^3*
a^3 - 3*A^2*B^2*a^2*b + A^3*B*a*b^2 + (2*A^3*B*a^2 - A^4*a*b)*c)*f^2)*x + 2
*((4*A^2*a*c^4 + (4*B^2*a^2 - 4*A*B*a*b - A^2*b^2)*c^3 - (B^2*a*b^2 - A*B*b
^3)*c^2)*d^3 - (B^2*a*b^4 - A*B*b^5 + 8*A^2*a^2*c^3 + 2*(4*B^2*a^3 - 4*A*B*
a^2*b - 3*A^2*a*b^2)*c^2 - (6*B^2*a^2*b^2 - 6*A*B*a*b^3 - A^2*b^4)*c)*d^2*f
 - (B^2*a^3*b^2 - A*B*a^2*b^3 - 4*A^2*a^3*c^2 - (4*B^2*a^4 - 4*A*B*a^3*b -
A^2*a^2*b^2)*c)*d*f^2)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 +
2*(2*A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^
2*a^2 - 4*A^3*B*a*b + A^4*b^2)*f^2)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 -
 6*a*b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*b^2*c^2 - 24*a^
3*c^3)*d^2*f^2 + 2*(a^2*b^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3 + (a^4*b^2 - 4
*a^5*c)*f^4)))/x) - 1/4*sqrt(2)*sqrt(((B^2*b^2 + 2*A^2*c^2 - 2*(B^2*a + A*B
*b)*c)*d + (2*B^2*a^2 - 2*A*B*a*b + A^2*b^2 - 2*A^2*a*c)*f + ((b^2*c^2 - 4*
a*c^3)*d^2 + (b^4 - 6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f^2)*s
qrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(2*A*B^3*a*b - A^2*B^2
*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2 - 4*A^3*B*a*b + A^
4*b^2)*f^2)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*b^2*c^3 + 8*a^2*c^4
)*d^3*f + (b^6 - 8*a*b^4*c + 22*a^2*b^2*c^2 - 24*a^3*c^3)*d^2*f^2 + 2*(a^2*
b^4 - 6*a^3*b^2*c + 8*a^4*c^2)*d*f^3 + (a^4*b^2 - 4*a^5*c)*f^4)))/((b^2*c^2
 - 4*a*c^3)*d^2 + (b^4 - 6*a*b^2*c + 8*a^2*c^2)*d*f + (a^2*b^2 - 4*a^3*c)*f
^2))*log((2*(B^4*a*b^2 - A*B^3*b^3 - 2*A^3*B*b*c^2 - (2*A*B^3*a*b - 3*A^2*B
^2*b^2)*c)*d^2 + 2*(2*A*B^3*a^2*b - 3*A^2*B^2*a*b^2 + A^3*B*b^3 + (2*A^3*B*
a*b - A^4*b^2)*c)*d*f - sqrt(2)*((B^3*b^4 - 8*A^2*B*a*c^3 + 2*(6*A*B^2*a*b
+ A^2*B*b^2)*c^2 - (4*B^3*a*b^2 + 3*A*B^2*b^3)*c)*d^2 + (3*A*B^2*a*b^3 - A^
2*B*b^4 + 4*(4*A^2*B*a^2 - A^3*a*b)*c^2 - (12*A*B^2*a^2*b - A^3*b^3)*c)*d*f
 + (2*A^2*B*a^2*b^2 - A^3*a*b^3 - 4*(2*A^2*B*a^3 - A^3*a^2*b)*c)*f^2 - ((B*
b^4*c^2 + 4*(2*B*a^2 + A*a*b)*c^4 - (6*B*a*b^2 + A*b^3)*c^3)*d^3 + (B*b^6 -
 4*(6*B*a^3 + A*a^2*b)*c^3 + (22*B*a^2*b^2 + 5*A*a*b^3)*c^2 - (8*B*a*b^4 +
A*b^5)*c)*d^2*f + (3*B*a^2*b^4 - A*a*b^5 + 4*(6*B*a^4 - A*a^3*b)*c^2 - (18*
B*a^3*b^2 - 5*A*a^2*b^3)*c)*d*f^2 + (2*B*a^4*b^2 - A*a^3*b^3 - 4*(2*B*a^5 -
 A*a^4*b)*c)*f^3)*sqrt(((B^4*b^2 - 4*A*B^3*b*c + 4*A^2*B^2*c^2)*d^2 + 2*(2*
A*B^3*a*b - A^2*B^2*b^2 - 2*(2*A^2*B^2*a - A^3*B*b)*c)*d*f + (4*A^2*B^2*a^2
 - 4*A^3*B*a*b + A^4*b^2)*f^2)/((b^2*c^4 - 4*a*c^5)*d^4 + 2*(b^4*c^2 - 6*a*
b^2*c^3 + 8*a^2*c^4)*d^3*f + (b^6 - 8*a*b^4*c +...
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{A+Bx}{\sqrt{d+fx^2}\,(a+bx+cx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+b*x+a)/(f*x**2+d)**(1/2),x)
```

```
[Out] Integral((A + B*x)/(sqrt(d + f*x**2)*(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+b*x+a)/(f*x^2+d)^(1/2),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Unable to divide, perhaps due to roun
ding error%%%{%%{poly1[%%%{-4,[3,2,0]%%%}+%%%{16,[1,3,1]%%%},%%%{4,[4,2,0]%
%%}+%%
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{\sqrt{f\,x^2+d}\,\left(c\,x^2+b\,x+a\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((d + f*x^2)^(1/2)*(a + b*x + c*x^2)),x)
```

```
[Out] int((A + B*x)/((d + f*x^2)^(1/2)*(a + b*x + c*x^2)), x)
```

## 3.24 $\int \frac{A+Bx}{(a+cx^2)\sqrt{d+fx^2}}\,dx$

**Optimal**. Leaf size=101

$$\frac{A\tan^{-1}\left(\frac{\sqrt{cd-af}\,x}{\sqrt{a}\,\sqrt{d+fx^2}}\right)}{\sqrt{a}\,\sqrt{cd-af}}-\frac{B\tanh^{-1}\left(\frac{\sqrt{c}\,\sqrt{d+fx^2}}{\sqrt{cd-af}}\right)}{\sqrt{c}\,\sqrt{cd-af}}$$

```
[Out] A*arctan(x*(-a*f+c*d)^(1/2)/a^(1/2)/(f*x^2+d)^(1/2))/a^(1/2)/(-a*f+c*d)^(1/
2)-B*arctanh(c^(1/2)*(f*x^2+d)^(1/2)/(-a*f+c*d)^(1/2))/c^(1/2)/(-a*f+c*d)^(
1/2)
```

**Rubi** [A]
time = 0.08, antiderivative size = 101, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.231$, Rules used = {1024, 385, 211, 455, 65, 214}

$$\frac{A\text{ArcTan}\left(\frac{x\sqrt{cd-af}}{\sqrt{a}\,\sqrt{d+fx^2}}\right)}{\sqrt{a}\,\sqrt{cd-af}}-\frac{B\tanh^{-1}\left(\frac{\sqrt{c}\,\sqrt{d+fx^2}}{\sqrt{cd-af}}\right)}{\sqrt{c}\,\sqrt{cd-af}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/((a + c*x^2)*Sqrt[d + f*x^2]),x]
```

```
[Out] (A*ArcTan[(Sqrt[c*d - a*f]*x)/(Sqrt[a]*Sqrt[d + f*x^2])])/(Sqrt[a]*Sqrt[c*d
 - a*f]) - (B*ArcTanh[(Sqrt[c]*Sqrt[d + f*x^2])/Sqrt[c*d - a*f]])/(Sqrt[c]*
Sqrt[c*d - a*f])
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 211

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[a/b, 2]/a)*ArcTan[x/R
t[a/b, 2]], x] /; FreeQ[{a, b}, x] && PosQ[a/b]
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 385

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_)/((c_) + (d_.)*(x_)^(n_)), x_Symbol] :> Su
bst[Int[1/(c - (b*c - a*d)*x^n), x], x, x/(a + b*x^n)^(1/n)] /; FreeQ[{a, b
, c, d}, x] && NeQ[b*c - a*d, 0] && EqQ[n*p + 1, 0] && IntegerQ[n]
```

Rule 455

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_.)*((c_) + (d_.)*(x_)^(n_))^(q_.
), x_Symbol] :> Dist[1/n, Subst[Int[(a + b*x)^p*(c + d*x)^q, x], x, x^n], x
] /; FreeQ[{a, b, c, d, m, n, p, q}, x] && NeQ[b*c - a*d, 0] && EqQ[m - n +
 1, 0]
```

Rule 1024

```
Int[((g_) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (f_.)*(x_)^2)^(q
_), x_Symbol] :> Dist[g, Int[(a + c*x^2)^p*(d + f*x^2)^q, x], x] + Dist[h,
Int[x*(a + c*x^2)^p*(d + f*x^2)^q, x], x] /; FreeQ[{a, c, d, f, g, h, p, q}
, x]
```

Rubi steps

$$
\begin{aligned}
\int \frac{A+Bx}{(a+cx^2)\sqrt{d+fx^2}}\,dx &= A\int \frac{1}{(a+cx^2)\sqrt{d+fx^2}}\,dx + B\int \frac{x}{(a+cx^2)\sqrt{d+fx^2}}\,dx \\
&= A\text{Subst}\left(\int \frac{1}{a-(-cd+af)x^2}\,dx, x, \frac{x}{\sqrt{d+fx^2}}\right) + \frac{1}{2}B\text{Subst}\left(\int \frac{1}{(a+cx)\sqrt{\phantom{x}}}\right. \\
&= \frac{A\tan^{-1}\left(\frac{\sqrt{cd-af}\,x}{\sqrt{a}\sqrt{d+fx^2}}\right)}{\sqrt{a}\sqrt{cd-af}} + \frac{B\text{Subst}\left(\int \frac{1}{a-\frac{cd}{f}+\frac{cx^2}{f}}\,dx, x, \sqrt{d+fx^2}\right)}{f} \\
&= \frac{A\tan^{-1}\left(\frac{\sqrt{cd-af}\,x}{\sqrt{a}\sqrt{d+fx^2}}\right)}{\sqrt{a}\sqrt{cd-af}} - \frac{B\tanh^{-1}\left(\frac{\sqrt{c}\sqrt{d+fx^2}}{\sqrt{cd-af}}\right)}{\sqrt{c}\sqrt{cd-af}}
\end{aligned}
$$

**Mathematica** [B] Leaf count is larger than twice the leaf count of optimal. 349 vs. 2(101) = 202.
time = 1.77, size = 349, normalized size = 3.46

$$
\frac{\sqrt{a}\,B\left(\left(\sqrt{a}\sqrt{f}+\sqrt{-cd+af}\right)\sqrt{-cd+2af-2\sqrt{a}\sqrt{f}\sqrt{-cd+af}}\tan^{-1}\left(\frac{\sqrt{c}\left(\sqrt{f}\,x-\sqrt{d+fx^2}\right)}{\sqrt{-cd+2af-2\sqrt{a}\sqrt{f}\sqrt{-cd+af}}}\right)+\left(-\sqrt{a}\sqrt{f}+\sqrt{-cd+af}\right)\sqrt{-cd+2af+2\sqrt{a}\sqrt{f}\sqrt{-cd+af}}\tan^{-1}\left(\frac{\sqrt{c}\left(\sqrt{f}\,x-\sqrt{d+fx^2}\right)}{\sqrt{-cd+2af+2\sqrt{a}\sqrt{f}\sqrt{-cd+af}}}\right)\right)+Ac^{3/2}d\tanh^{-1}\left(\frac{a\sqrt{f}+cx\left(\sqrt{f}\,x-\sqrt{d+fx^2}\right)}{\sqrt{a}\sqrt{-cd+af}}\right)}{\sqrt{a}\,c^{3/2}d\sqrt{-cd+af}}
$$

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/((a + c*x^2)*Sqrt[d + f*x^2]),x]
```

```
[Out] (Sqrt[a]*B*((Sqrt[a]*Sqrt[f] + Sqrt[-(c*d) + a*f])*Sqrt[-(c*d) + 2*a*f - 2*
Sqrt[a]*Sqrt[f]*Sqrt[-(c*d) + a*f]]*ArcTan[(Sqrt[c]*(Sqrt[f]*x - Sqrt[d + f
*x^2]))/Sqrt[-(c*d) + 2*a*f - 2*Sqrt[a]*Sqrt[f]*Sqrt[-(c*d) + a*f]]] + (-(S
qrt[a]*Sqrt[f]) + Sqrt[-(c*d) + a*f])*Sqrt[-(c*d) + 2*a*f + 2*Sqrt[a]*Sqrt[
f]*Sqrt[-(c*d) + a*f]]*ArcTan[(Sqrt[c]*(Sqrt[f]*x - Sqrt[d + f*x^2]))/Sqrt[
-(c*d) + 2*a*f + 2*Sqrt[a]*Sqrt[f]*Sqrt[-(c*d) + a*f]]]) + A*c^(3/2)*d*ArcT
anh[(a*Sqrt[f] + c*x*(Sqrt[f]*x - Sqrt[d + f*x^2]))/(Sqrt[a]*Sqrt[-(c*d) +
a*f])]])/(Sqrt[a]*c^(3/2)*d*Sqrt[-(c*d) + a*f])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 336 vs. $2(81) = 162$.
time = 0.08, size = 337, normalized size = 3.34

| method | result |
|---|---|
| default | $-\frac{\left(Ac+B\sqrt{-ac}\right)\ln\left(\frac{-\frac{2(fa-cd)}{c}+\frac{2f\sqrt{-ac}\left(x-\frac{\sqrt{-ac}}{c}\right)}{c}+2\sqrt{-\frac{fa-cd}{c}}\sqrt{f\left(x-\frac{\sqrt{-ac}}{c}\right)^{2}+\frac{2f\sqrt{-ac}}{}}}{x-\frac{\sqrt{-ac}}{c}}\right)}{2\sqrt{-ac}\,c\sqrt{-\frac{fa-cd}{c}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(c*x^2+a)/(f*x^2+d)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(A*c+B*(-a*c)^(1/2))/(-a*c)^(1/2)/c/(-(a*f-c*d)/c)^(1/2)*ln((-2*(a*f-c
*d)/c+2*f*(-a*c)^(1/2)/c*(x-(-a*c)^(1/2)/c)+2*(-(a*f-c*d)/c)^(1/2)*(f*(x-(-
a*c)^(1/2)/c)^2+2*f*(-a*c)^(1/2)/c*(x-(-a*c)^(1/2)/c)-(a*f-c*d)/c)^(1/2))/(
x-(-a*c)^(1/2)/c))-1/2*(-A*c+B*(-a*c)^(1/2))/(-a*c)^(1/2)/c/(-(a*f-c*d)/c)^
(1/2)*ln((-2*(a*f-c*d)/c-2*f*(-a*c)^(1/2)/c*(x+(-a*c)^(1/2)/c)+2*(-(a*f-c*d
)/c)^(1/2)*(f*(x+(-a*c)^(1/2)/c)^2-2*f*(-a*c)^(1/2)/c*(x+(-a*c)^(1/2)/c)-(a
*f-c*d)/c)^(1/2))/(x+(-a*c)^(1/2)/c))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+d)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((B*x + A)/((c*x^2 + a)*sqrt(f*x^2 + d)), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 1515 vs. $2(81) = 162$.
time = 0.42, size = 1515, normalized size = 15.00

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+d)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt((B^2*a - A^2*c + 2*(a*c^2*d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 -
 2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^2*d - a^2*c*f))*log(((A*B^3*a + A^3*B*c)
*f*x + (A^2*B*c^2*d - A^2*B*a*c*f + (B*a*c^3*d^2 - 2*B*a^2*c^2*d*f + B*a^3*
c*f^2)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))*sqrt(f*x^2 +
 d)*sqrt((B^2*a - A^2*c + 2*(a*c^2*d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 -
2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^2*d - a^2*c*f)) + sqrt(-A^2*B^2/(a*c^3*d^
2 - 2*a^2*c^2*d*f + a^3*c*f^2))*((B^2*a*c^2 + A^2*c^3)*d^2 - (B^2*a^2*c + A
^2*a*c^2)*d*f))/x) + 1/4*sqrt((B^2*a - A^2*c + 2*(a*c^2*d - a^2*c*f)*sqrt(-
A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^2*d - a^2*c*f))*log(
((A*B^3*a + A^3*B*c)*f*x - (A^2*B*c^2*d - A^2*B*a*c*f + (B*a*c^3*d^2 - 2*B*
a^2*c^2*d*f + B*a^3*c*f^2)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c
*f^2)))*sqrt(f*x^2 + d)*sqrt((B^2*a - A^2*c + 2*(a*c^2*d - a^2*c*f)*sqrt(-A
^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^2*d - a^2*c*f)) + sqr
t(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2))*((B^2*a*c^2 + A^2*c^3)*
d^2 - (B^2*a^2*c + A^2*a*c^2)*d*f))/x) - 1/4*sqrt((B^2*a - A^2*c - 2*(a*c^2
*d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^
2*d - a^2*c*f))*log(((A*B^3*a + A^3*B*c)*f*x + (A^2*B*c^2*d - A^2*B*a*c*f -
 (B*a*c^3*d^2 - 2*B*a^2*c^2*d*f + B*a^3*c*f^2)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2
*a^2*c^2*d*f + a^3*c*f^2)))*sqrt(f*x^2 + d)*sqrt((B^2*a - A^2*c - 2*(a*c^2*
d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))/(a*c^2
*d - a^2*c*f)) - sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2))*((B
^2*a*c^2 + A^2*c^3)*d^2 - (B^2*a^2*c + A^2*a*c^2)*d*f))/x) + 1/4*sqrt((B^2*
a - A^2*c - 2*(a*c^2*d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f
+ a^3*c*f^2)))/(a*c^2*d - a^2*c*f))*log(((A*B^3*a + A^3*B*c)*f*x - (A^2*B*c
^2*d - A^2*B*a*c*f - (B*a*c^3*d^2 - 2*B*a^2*c^2*d*f + B*a^3*c*f^2)*sqrt(-A^
2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2)))*sqrt(f*x^2 + d)*sqrt((B^2*a
 - A^2*c - 2*(a*c^2*d - a^2*c*f)*sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d*f +
 a^3*c*f^2)))/(a*c^2*d - a^2*c*f)) - sqrt(-A^2*B^2/(a*c^3*d^2 - 2*a^2*c^2*d
*f + a^3*c*f^2))*((B^2*a*c^2 + A^2*c^3)*d^2 - (B^2*a^2*c + A^2*a*c^2)*d*f))
/x)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{A + Bx}{(a + cx^2)\sqrt{d + fx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x**2+a)/(f*x**2+d)**(1/2),x)
```

```
[Out] Integral((A + B*x)/((a + c*x**2)*sqrt(d + f*x**2)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(c*x^2+a)/(f*x^2+d)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\begin{cases} \frac{B\,\mathrm{atan}\left(\frac{c\,\sqrt{f\,x^2+d}}{\sqrt{a\,c\,f-c^2\,d}}\right)}{\sqrt{a\,c\,f-c^2\,d}}+\frac{A\,\mathrm{atan}\left(\frac{x\,\sqrt{c\,d-a\,f}}{\sqrt{a}\,\sqrt{f\,x^2+d}}\right)}{\sqrt{-a\,\left(a\,f-c\,d\right)}} & \text{if } 0<c\,d-a\,f \\ \frac{A\,\ln\left(\frac{\sqrt{a\,\left(f\,x^2+d\right)}+x\,\sqrt{a\,f-c\,d}}{\sqrt{a\,\left(f\,x^2+d\right)}-x\,\sqrt{a\,f-c\,d}}\right)}{2\,\sqrt{a\,\left(a\,f-c\,d\right)}}+\frac{B\,\mathrm{atan}\left(\frac{c\,\sqrt{f\,x^2+d}}{\sqrt{a\,c\,f-c^2\,d}}\right)}{\sqrt{a\,c\,f-c^2\,d}} & \text{if } c\,d-a\,f<0 \\ \int \frac{A+B\,x}{\left(c\,x^2+a\right)\,\sqrt{f\,x^2+d}}\,dx & \text{if } c\,d-a\,f\notin\mathbb{R}\vee a\,f=c\,d \end{cases}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a + c*x^2)*(d + f*x^2)^(1/2)),x)
```

```
[Out] piecewise(0 < - a*f + c*d, (B*atan((c*(d + f*x^2)^(1/2))/(- c^2*d + a*c*f)^
(1/2)))/(- c^2*d + a*c*f)^(1/2) + (A*atan((x*(- a*f + c*d)^(1/2))/(a^(1/2)*
(d + f*x^2)^(1/2))))/(-a*(a*f - c*d))^(1/2), - a*f + c*d < 0, (A*log(((a*(d
 + f*x^2))^(1/2) + x*(a*f - c*d)^(1/2))/((a*(d + f*x^2))^(1/2) - x*(a*f - c
*d)^(1/2))))/(2*(a*(a*f - c*d))^(1/2)) + (B*atan((c*(d + f*x^2)^(1/2))/(- c
^2*d + a*c*f)^(1/2)))/(- c^2*d + a*c*f)^(1/2), ~in(- a*f + c*d, 'real') | a
*f == c*d, int((A + B*x)/((a + c*x^2)*(d + f*x^2)^(1/2)), x))
```

# 3.25 $\int \frac{2+x}{(2+4x-3x^2)\sqrt{1+3x-2x^2}}\,dx$

**Optimal**. Leaf size=139

$$\frac{1}{2}\sqrt{-\frac{13}{5}+\sqrt{10}}\ \tan^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(1+4\sqrt{10}\right)x}{2\sqrt{1+\sqrt{10}}\ \sqrt{1+3x-2x^2}}\right)+\frac{1}{2}\sqrt{\frac{13}{5}+\sqrt{10}}\ \tanh^{-1}\left(\frac{3\left(4+\sqrt{10}\right)+\left(1\right.}{2\sqrt{-1+\sqrt{10}}\ \sqrt{1\right.}\right.$$

```
[Out] 1/10*arctan(1/2*(12-3*10^(1/2)+x*(1+4*10^(1/2)))/(-2*x^2+3*x+1)^(1/2)/(1+10
^(1/2))^(1/2))*(-65+25*10^(1/2))^(1/2)+1/10*arctanh(1/2*(x*(1-4*10^(1/2))+1
2+3*10^(1/2))/(-2*x^2+3*x+1)^(1/2)/(-1+10^(1/2))^(1/2))*(65+25*10^(1/2))^(1
/2)
```

**Rubi** [A]
time = 0.14, antiderivative size = 139, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {1046, 738, 210, 212}

$$\frac{1}{2}\sqrt{\sqrt{10}-\frac{13}{5}}\ \text{ArcTan}\left(\frac{\left(1+4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{1+\sqrt{10}}\ \sqrt{-2x^2+3x+1}}\right)+\frac{1}{2}\sqrt{\frac{13}{5}+\sqrt{10}}\ \tanh^{-1}\left(\frac{\left(1-4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{\sqrt{10}-1}\ \sqrt{-2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*Sqrt[1 + 3*x - 2*x^2]),x]
```

```
[Out] (Sqrt[-13/5 + Sqrt[10]]*ArcTan[(3*(4 - Sqrt[10]) + (1 + 4*Sqrt[10])*x)/(2*S
qrt[1 + Sqrt[10]]*Sqrt[1 + 3*x - 2*x^2])])/2 + (Sqrt[13/5 + Sqrt[10]]*ArcTa
nh[(3*(4 + Sqrt[10]) + (1 - 4*Sqrt[10])*x)/(2*Sqrt[-1 + Sqrt[10]]*Sqrt[1 +
3*x - 2*x^2])])/2
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{2+x}{(2+4x-3x^2)\sqrt{1+3x-2x^2}}\,dx = \frac{1}{5}\left(5-4\sqrt{10}\right)\int \frac{1}{\left(4-2\sqrt{10}-6x\right)\sqrt{1+3x-2x^2}}\,dx + \frac{1}{5}\left(5+\right.$$

$$= -\left(\frac{1}{5}\left(2\left(5-4\sqrt{10}\right)\right)\text{Subst}\left(\int \frac{1}{144+72\left(4-2\sqrt{10}\right)-8\left(4-\right.}\right.\right.$$

$$= \frac{1}{10}\sqrt{-65+25\sqrt{10}}\,\tan^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(1+4\sqrt{10}\right)x}{2\sqrt{1+\sqrt{10}}\,\sqrt{1+3x-2x^2}}\right)+$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.20, size = 149, normalized size = 1.07

$$-\frac{1}{2}\text{RootSum}\left[5+20\#1+8\#1^2-8\#1^3+2\#1^4\&,\frac{-7\log(x)+7\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)+2\log(x)\#1-2\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)\#1-2\log(x)\#1^2+2\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)\#1^2}{5+4\#1-6\#1^2+2\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*Sqrt[1 + 3*x - 2*x^2]),x]
```

```
[Out] -1/2*RootSum[5 + 20*#1 + 8*#1^2 - 8*#1^3 + 2*#1^4 & , (-7*Log[x] + 7*Log[-1
 + Sqrt[1 + 3*x - 2*x^2] - x*#1] + 2*Log[x]*#1 - 2*Log[-1 + Sqrt[1 + 3*x -
2*x^2] - x*#1]*#1 - 2*Log[x]*#1^2 + 2*Log[-1 + Sqrt[1 + 3*x - 2*x^2] - x*#1
]*#1^2)/(5 + 4*#1 - 6*#1^2 + 2*#1^3) & ]
```

**Maple [A]**
time = 0.70, size = 176, normalized size = 1.27

| method | result |
|---|---|
| default | $\frac{\left(8+\sqrt{10}\right)\sqrt{10}\,\operatorname{arctanh}\left(\frac{-1+\sqrt{10}+\frac{9\left(\frac{1}{3}-\frac{4\sqrt{10}}{3}\right)\left(x-\frac{2}{3}-\frac{\sqrt{10}}{3}\right)}{2}}{\sqrt{-1+\sqrt{10}}\,\sqrt{-18\left(x-\frac{2}{3}-\frac{\sqrt{10}}{3}\right)^{2}+9\left(\frac{1}{3}-\frac{4\sqrt{10}}{3}\right)\left(x-\frac{2}{3}-\frac{\sqrt{1}}{3}\right.}}\right.}{20\sqrt{-1+\sqrt{10}}}$ |
| trager | $\operatorname{RootOf}\left(\_Z^2+100\operatorname{RootOf}\left(400\_Z^4-520\_Z^2-81\right)^2-130\right)\ln\left(-\frac{129200\operatorname{RootOf}\left(\_Z^2+100\operatorname{RootOf}\left(400\_Z^4-520\_Z^2-81\right)^2-130\right.}{}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/20*(8+10^(1/2))*10^(1/2)/(-1+10^(1/2))^(1/2)*arctanh(9/2*(-2/9+2/9*10^(1/
2)+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2)))/(-1+10^(1/2))^(1/2)/(-18*(x-2/3
-1/3*10^(1/2))^2+9*(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))-1+10^(1/2))^(1/2
))-1/20*(-8+10^(1/2))*10^(1/2)/(1+10^(1/2))^(1/2)*arctan(9/2*(-2/9-2/9*10^(
1/2)+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2)))/(1+10^(1/2))^(1/2)/(-18*(x-2/
3+1/3*10^(1/2))^2+9*(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1-10^(1/2))^(1/
2))
```

**Maxima** [B] Leaf count of result is larger than twice the leaf count of optimal. 361 vs. $2(99) = 198$.
time = 0.51, size = 361, normalized size = 2.60

$$-\frac{1}{20}\sqrt{10}\left(\frac{\sqrt{10}\arcsin\left(\frac{8\sqrt{17}\sqrt{10}x}{17\left|6x+2\sqrt{10}-4\right|}+\frac{2\sqrt{17}x}{17\left|6x+2\sqrt{10}-4\right|}-\frac{6\sqrt{17}\sqrt{10}}{17\left|6x+2\sqrt{10}-4\right|}+\frac{24\sqrt{17}}{17\left|6x+2\sqrt{10}-4\right|}\right)}{\sqrt{\sqrt{10}+1}}-\frac{\sqrt{10}\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{-2x^2+3x+1}\sqrt{\sqrt{10}-1}}{3\left|6x-2\sqrt{10}-4\right|}+\frac{2\sqrt{10}}{9\left|6x-2\sqrt{10}-4\right|}-\frac{2}{9\left|6x-2\sqrt{10}-4\right|}+\frac{1}{18}\right)}{\sqrt{\sqrt{10}-1}}-\frac{8\arcsin\left(\frac{8\sqrt{17}\sqrt{10}x}{17\left|6x+2\sqrt{10}-4\right|}+\frac{2\sqrt{17}x}{17\left|6x+2\sqrt{10}-4\right|}-\frac{6\sqrt{17}\sqrt{10}}{17\left|6x+2\sqrt{10}-4\right|}+\frac{24\sqrt{17}}{17\left|6x+2\sqrt{10}-4\right|}\right)}{\sqrt{\sqrt{10}+1}}-\frac{8\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{-2x^2+3x+1}\sqrt{\sqrt{10}-1}}{3\left|6x-2\sqrt{10}-4\right|}+\frac{2\sqrt{10}}{9\left|6x-2\sqrt{10}-4\right|}-\frac{2}{9\left|6x-2\sqrt{10}-4\right|}+\frac{1}{18}\right)}{\sqrt{\sqrt{10}-1}}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(1/2),x, algorithm="maxima")
```

```
[Out] -1/20*sqrt(10)*(sqrt(10)*arcsin(8/17*sqrt(17)*sqrt(10)*x/abs(6*x + 2*sqrt(1
0) - 4) + 2/17*sqrt(17)*x/abs(6*x + 2*sqrt(10) - 4) - 6/17*sqrt(17)*sqrt(10
)/abs(6*x + 2*sqrt(10) - 4) + 24/17*sqrt(17)/abs(6*x + 2*sqrt(10) - 4))/sqr
t(sqrt(10) + 1) - sqrt(10)*log(-2/9*sqrt(10) + 2/3*sqrt(-2*x^2 + 3*x + 1)*s
qrt(sqrt(10) - 1)/abs(6*x - 2*sqrt(10) - 4) + 2/9*sqrt(10)/abs(6*x - 2*sqrt
(10) - 4) - 2/9/abs(6*x - 2*sqrt(10) - 4) + 1/18)/sqrt(sqrt(10) - 1) - 8*ar
csin(8/17*sqrt(17)*sqrt(10)*x/abs(6*x + 2*sqrt(10) - 4) + 2/17*sqrt(17)*x/a
bs(6*x + 2*sqrt(10) - 4) - 6/17*sqrt(17)*sqrt(10)/abs(6*x + 2*sqrt(10) - 4)
 + 24/17*sqrt(17)/abs(6*x + 2*sqrt(10) - 4))/sqrt(sqrt(10) + 1) - 8*log(-2/
9*sqrt(10) + 2/3*sqrt(-2*x^2 + 3*x + 1)*sqrt(sqrt(10) - 1)/abs(6*x - 2*sqrt
```

```
(10) - 4) + 2/9*sqrt(10)/abs(6*x - 2*sqrt(10) - 4) - 2/9/abs(6*x - 2*sqrt(1
0) - 4) + 1/18)/sqrt(sqrt(10) - 1))
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 322 vs. $2(99) = 198$.
time = 0.37, size = 322, normalized size = 2.32

$$\frac{2}{5}\sqrt{5}\sqrt{5\sqrt{5}\sqrt{2}-13}\arctan\left(\frac{\sqrt{2}\left(2\sqrt{5}x-\sqrt{2}x\right)\sqrt{5\sqrt{5}\sqrt{2}-13}\sqrt{\frac{\sqrt{5}\sqrt{2}(3x^2+2x)+6x^2-2\left(\sqrt{5}\sqrt{2}x+2x+2\right)\sqrt{-2x^2+3x+1}+10x+4}{x^2}}+2\left(\sqrt{2}(4x-1)+\sqrt{5}(x+2)-\sqrt{-2x^2+3x+1}\left(2\sqrt{5}-\sqrt{2}\right)\right)\sqrt{5\sqrt{5}\sqrt{2}-13}}{18x}\right)-\frac{1}{10}\sqrt{5}\sqrt{5\sqrt{5}\sqrt{2}+13}\log\left(\frac{9\sqrt{5}\sqrt{2}x+\left(4\sqrt{5}x-7\sqrt{2}x\right)\sqrt{5\sqrt{5}\sqrt{2}+13}-18x+18\sqrt{-2x^2+3x+1}-18}{x}\right)+\frac{1}{10}\sqrt{5}\sqrt{5\sqrt{5}\sqrt{2}+13}\log\left(\frac{9\sqrt{5}\sqrt{2}x-\left(4\sqrt{5}x-7\sqrt{2}x\right)\sqrt{5\sqrt{5}\sqrt{2}+13}-18x+18\sqrt{-2x^2+3x+1}-18}{x}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(1/2),x, algorithm="fricas")
```

```
[Out] 2/5*sqrt(5)*sqrt(5*sqrt(5)*sqrt(2) - 13)*arctan(1/18*(sqrt(2)*(2*sqrt(5)*x
- sqrt(2)*x)*sqrt(5*sqrt(5)*sqrt(2) - 13)*sqrt((sqrt(5)*sqrt(2)*(3*x^2 + 2*
x) + 6*x^2 - 2*(sqrt(5)*sqrt(2)*x + 2*x + 2)*sqrt(-2*x^2 + 3*x + 1) + 10*x
+ 4)/x^2) + 2*(sqrt(2)*(4*x - 1) + sqrt(5)*(x + 2) - sqrt(-2*x^2 + 3*x + 1)
*(2*sqrt(5) - sqrt(2)))*sqrt(5*sqrt(5)*sqrt(2) - 13))/x) - 1/10*sqrt(5)*sqr
t(5*sqrt(5)*sqrt(2) + 13)*log((9*sqrt(5)*sqrt(2)*x + (4*sqrt(5)*x - 7*sqrt(
2)*x)*sqrt(5*sqrt(5)*sqrt(2) + 13) - 18*x + 18*sqrt(-2*x^2 + 3*x + 1) - 18)
/x) + 1/10*sqrt(5)*sqrt(5*sqrt(5)*sqrt(2) + 13)*log((9*sqrt(5)*sqrt(2)*x -
(4*sqrt(5)*x - 7*sqrt(2)*x)*sqrt(5*sqrt(5)*sqrt(2) + 13) - 18*x + 18*sqrt(-
2*x^2 + 3*x + 1) - 18)/x)
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int\frac{x}{3x^2\sqrt{-2x^2+3x+1}-4x\sqrt{-2x^2+3x+1}-2\sqrt{-2x^2+3x+1}}\,dx-\int\frac{2}{3x^2\sqrt{-2x^2+3x+1}-4x\sqrt{-2x^2+3x+1}-2\sqrt{-2x^2+3x+1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(-2*x**2+3*x+1)**(1/2),x)
```

```
[Out] -Integral(x/(3*x**2*sqrt(-2*x**2 + 3*x + 1) - 4*x*sqrt(-2*x**2 + 3*x + 1) -
 2*sqrt(-2*x**2 + 3*x + 1)), x) - Integral(2/(3*x**2*sqrt(-2*x**2 + 3*x + 1
) - 4*x*sqrt(-2*x**2 + 3*x + 1) - 2*sqrt(-2*x**2 + 3*x + 1)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{\sqrt{-2\,x^2+3\,x+1}\,\left(-3\,x^2+4\,x+2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x - 2*x^2 + 1)^(1/2)*(4*x - 3*x^2 + 2)),x)
[Out] int((x + 2)/((3*x - 2*x^2 + 1)^(1/2)*(4*x - 3*x^2 + 2)), x)
```

# 3.26 $\int \frac{2+x}{(2+4x-3x^2)(1+3x-2x^2)^{3/2}}\,dx$

**Optimal**. Leaf size=166

$$-\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}}-\frac{9}{2}\sqrt{\frac{1}{5}\left(-3+\sqrt{10}\right)}\tan^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(1+4\sqrt{10}\right)x}{2\sqrt{1+\sqrt{10}}\sqrt{1+3x-2x^2}}\right)+\frac{9}{2}\sqrt{\frac{1}{5}\left(3+\sqrt{10}\right)}$$

```
[Out] -2/17*(15+14*x)/(-2*x^2+3*x+1)^(1/2)-9/10*arctan(1/2*(12-3*10^(1/2)+x*(1+4*
10^(1/2)))/(-2*x^2+3*x+1)^(1/2)/(1+10^(1/2))^(1/2))*(-15+5*10^(1/2))^(1/2)+
9/10*arctanh(1/2*(x*(1-4*10^(1/2))+12+3*10^(1/2))/(-2*x^2+3*x+1)^(1/2)/(-1+
10^(1/2))^(1/2))*(15+5*10^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.14, antiderivative size = 166, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {1030, 12, 1046, 738, 210, 212}

$$-\frac{9}{2}\sqrt{\frac{1}{5}\left(\sqrt{10}-3\right)}\text{ArcTan}\left(\frac{\left(1+4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{1+\sqrt{10}}\sqrt{-2x^2+3x+1}}\right)-\frac{2(14x+15)}{17\sqrt{-2x^2+3x+1}}+\frac{9}{2}\sqrt{\frac{1}{5}\left(3+\sqrt{10}\right)}\tanh^{-1}\left(\frac{\left(1-4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{\sqrt{10}-1}\sqrt{-2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x - 2*x^2)^(3/2)),x]
```

```
[Out] (-2*(15 + 14*x))/(17*Sqrt[1 + 3*x - 2*x^2]) - (9*Sqrt[(-3 + Sqrt[10])/5]*Ar
cTan[(3*(4 - Sqrt[10]) + (1 + 4*Sqrt[10])*x)/(2*Sqrt[1 + Sqrt[10]]*Sqrt[1 +
 3*x - 2*x^2])])/2 + (9*Sqrt[(3 + Sqrt[10])/5]*ArcTanh[(3*(4 + Sqrt[10]) +
(1 - 4*Sqrt[10])*x)/(2*Sqrt[-1 + Sqrt[10]]*Sqrt[1 + 3*x - 2*x^2])])/2
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{2+x}{(2+4x-3x^2)(1+3x-2x^2)^{3/2}}\,dx = -\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} + \frac{2}{17}\int \frac{153x}{2(2+4x-3x^2)\sqrt{1+3x-2x^2}}\,dx$$
$$= -\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} + 9\int \frac{x}{(2+4x-3x^2)\sqrt{1+3x-2x^2}}\,dx$$
$$= -\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} + \frac{1}{5}\left(9\left(5-\sqrt{10}\right)\right)\int \frac{1}{\left(4-2\sqrt{10}-6x\right)\cdots}$$
$$= -\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} - \frac{1}{5}\left(18\left(5-\sqrt{10}\right)\right)\text{Subst}\left(\int \frac{}{144+72\left(\cdots\right.}\right.$$
$$= -\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} - \frac{9}{2}\sqrt{\frac{1}{5}\left(-3+\sqrt{10}\right)}\tan^{-1}\left(\frac{3\left(4-\sqrt{10}\cdots\right.}{2\sqrt{1+\sqrt{1\cdots}}}\right.$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.32, size = 137, normalized size = 0.83

$$-\frac{2(15+14x)}{17\sqrt{1+3x-2x^2}} + \frac{9}{2}\text{RootSum}\left[5+20\#1+8\#1^2-8\#1^3+2\#1^4\&, \frac{3\log(x)-3\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)-2\log(x)\#1+2\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)\#1}{5+4\#1-6\#1^2+2\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x - 2*x^2)^(3/2)),x]
```

```
[Out] (-2*(15 + 14*x))/(17*Sqrt[1 + 3*x - 2*x^2]) + (9*RootSum[5 + 20*#1 + 8*#1^2
 - 8*#1^3 + 2*#1^4 & , (3*Log[x] - 3*Log[-1 + Sqrt[1 + 3*x - 2*x^2] - x*#1]
 - 2*Log[x]*#1 + 2*Log[-1 + Sqrt[1 + 3*x - 2*x^2] - x*#1]*#1)/(5 + 4*#1 - 6
*#1^2 + 2*#1^3) & ])/2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 454 vs. $2(118) = 236$.
time = 0.57, size = 455, normalized size = 2.74 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(3/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/20*(8+10^(1/2))*10^(1/2)*(1/3/(-1/9+1/9*10^(1/2))/(-2*(x-2/3-1/3*10^(1/2
))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))-1/9+1/9*10^(1/2))^(1/2)-1/3*(1
/3-4/3*10^(1/2))/(-1/9+1/9*10^(1/2))*(-4*x+3)/(8/9-8/9*10^(1/2)-(1/3-4/3*10
^(1/2))^2)/(-2*(x-2/3-1/3*10^(1/2))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2
))-1/9+1/9*10^(1/2))^(1/2)-1/(-1/9+1/9*10^(1/2))/(-1+10^(1/2))^(1/2)*arctan
h(9/2*(-2/9+2/9*10^(1/2)+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2)))/(-1+10^(1
```

```
/2))^(1/2)/(-18*(x-2/3-1/3*10^(1/2))^2+9*(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(
1/2))-1+10^(1/2))^(1/2)))-1/20*(-8+10^(1/2))*10^(1/2)*(1/3/(-1/9-1/9*10^(1/
2))/(-2*(x-2/3+1/3*10^(1/2))^2+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-
1/9*10^(1/2))^(1/2)-1/3*(1/3+4/3*10^(1/2))/(-1/9-1/9*10^(1/2))*(-4*x+3)/(8/
9+8/9*10^(1/2)-(1/3+4/3*10^(1/2))^2)/(-2*(x-2/3+1/3*10^(1/2))^2+(1/3+4/3*10
^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-1/9*10^(1/2))^(1/2)+1/(-1/9-1/9*10^(1/2))/
(1+10^(1/2))^(1/2)*arctan(9/2*(-2/9-2/9*10^(1/2)+(1/3+4/3*10^(1/2))*(x-2/3+
1/3*10^(1/2)))/(1+10^(1/2))^(1/2)/(-18*(x-2/3+1/3*10^(1/2))^2+9*(1/3+4/3*10
^(1/2))*(x-2/3+1/3*10^(1/2))-1-10^(1/2))^(1/2)))
```

**Maxima** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 678 vs. $2(118) = 236$.
time = 0.52, size = 678, normalized size = 4.08

$$\frac{1}{340}\sqrt{10}\left(\frac{124\sqrt{10}x}{\sqrt{10}\sqrt{-2x^2+3x+1}+\sqrt{-2x^2+3x+1}}-\frac{124\sqrt{10}x}{\sqrt{10}\sqrt{-2x^2+3x+1}-\sqrt{-2x^2+3x+1}}+\frac{153\sqrt{10}\arcsin\left(\frac{8\sqrt{17}\sqrt{10}x}{17|6x+2\sqrt{10}-4|}+\frac{2\sqrt{17}x}{17|6x+2\sqrt{10}-4|}-\frac{6\sqrt{17}\sqrt{10}}{17|6x+2\sqrt{10}-4|}+\frac{24\sqrt{17}}{17|6x+2\sqrt{10}-4|}\right)}{\sqrt{10}\sqrt{\sqrt{10}+1}+\sqrt{\sqrt{10}+1}}-\frac{128x}{\sqrt{10}\sqrt{-2x^2+3x+1}+\sqrt{-2x^2+3x+1}}-\frac{128x}{\sqrt{10}\sqrt{-2x^2+3x+1}-\sqrt{-2x^2+3x+1}}-\frac{1224\arcsin\left(\frac{8\sqrt{17}\sqrt{10}x}{17|6x+2\sqrt{10}-4|}+\frac{2\sqrt{17}x}{17|6x+2\sqrt{10}-4|}-\frac{6\sqrt{17}\sqrt{10}}{17|6x+2\sqrt{10}-4|}+\frac{24\sqrt{17}}{17|6x+2\sqrt{10}-4|}\right)}{\sqrt{10}\sqrt{\sqrt{10}+1}+\sqrt{\sqrt{10}+1}}+\frac{153\sqrt{10}\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{-2x^2+3x+1}\sqrt{\sqrt{10}-1}}{3|6x-2\sqrt{10}-4|}+\frac{2\sqrt{10}}{9|6x-2\sqrt{10}-4|}-\frac{2}{9|6x-2\sqrt{10}-4|}+\frac{1}{18}\right)}{\left(\sqrt{10}-1\right)^{\frac{3}{2}}}-\frac{42\sqrt{10}}{\sqrt{10}\sqrt{-2x^2+3x+1}+\sqrt{-2x^2+3x+1}}+\frac{42\sqrt{10}}{\sqrt{10}\sqrt{-2x^2+3x+1}-\sqrt{-2x^2+3x+1}}+\frac{1224\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{-2x^2+3x+1}\sqrt{\sqrt{10}-1}}{3|6x-2\sqrt{10}-4|}+\frac{2\sqrt{10}}{9|6x-2\sqrt{10}-4|}-\frac{2}{9|6x-2\sqrt{10}-4|}+\frac{1}{18}\right)}{\left(\sqrt{10}-1\right)^{\frac{3}{2}}}-\frac{312}{\sqrt{10}\sqrt{-2x^2+3x+1}+\sqrt{-2x^2+3x+1}}-\frac{312}{\sqrt{10}\sqrt{-2x^2+3x+1}-\sqrt{-2x^2+3x+1}}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(3/2),x, algorithm="maxima")
```

```
[Out] 1/340*sqrt(10)*(124*sqrt(10)*x/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) + sqrt(-2*x
^2 + 3*x + 1)) - 124*sqrt(10)*x/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) - sqrt(-2*
x^2 + 3*x + 1)) + 153*sqrt(10)*arcsin(8/17*sqrt(17)*sqrt(10)*x/abs(6*x + 2*
sqrt(10) - 4) + 2/17*sqrt(17)*x/abs(6*x + 2*sqrt(10) - 4) - 6/17*sqrt(17)*s
qrt(10)/abs(6*x + 2*sqrt(10) - 4) + 24/17*sqrt(17)/abs(6*x + 2*sqrt(10) - 4
))/(sqrt(10)*sqrt(sqrt(10) + 1) + sqrt(sqrt(10) + 1)) - 128*x/(sqrt(10)*sqr
t(-2*x^2 + 3*x + 1) + sqrt(-2*x^2 + 3*x + 1)) - 128*x/(sqrt(10)*sqrt(-2*x^2
 + 3*x + 1) - sqrt(-2*x^2 + 3*x + 1)) - 1224*arcsin(8/17*sqrt(17)*sqrt(10)*
x/abs(6*x + 2*sqrt(10) - 4) + 2/17*sqrt(17)*x/abs(6*x + 2*sqrt(10) - 4) - 6
/17*sqrt(17)*sqrt(10)/abs(6*x + 2*sqrt(10) - 4) + 24/17*sqrt(17)/abs(6*x +
2*sqrt(10) - 4))/(sqrt(10)*sqrt(sqrt(10) + 1) + sqrt(sqrt(10) + 1)) + 153*s
qrt(10)*log(-2/9*sqrt(10) + 2/3*sqrt(-2*x^2 + 3*x + 1)*sqrt(sqrt(10) - 1)/a
bs(6*x - 2*sqrt(10) - 4) + 2/9*sqrt(10)/abs(6*x - 2*sqrt(10) - 4) - 2/9/abs
(6*x - 2*sqrt(10) - 4) + 1/18)/(sqrt(10) - 1)^(3/2) - 42*sqrt(10)/(sqrt(10)
*sqrt(-2*x^2 + 3*x + 1) + sqrt(-2*x^2 + 3*x + 1)) + 42*sqrt(10)/(sqrt(10)*s
qrt(-2*x^2 + 3*x + 1) - sqrt(-2*x^2 + 3*x + 1)) + 1224*log(-2/9*sqrt(10) +
2/3*sqrt(-2*x^2 + 3*x + 1)*sqrt(sqrt(10) - 1)/abs(6*x - 2*sqrt(10) - 4) + 2
/9*sqrt(10)/abs(6*x - 2*sqrt(10) - 4) - 2/9/abs(6*x - 2*sqrt(10) - 4) + 1/1
8)/(sqrt(10) - 1)^(3/2) - 312/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) + sqrt(-2*x^
2 + 3*x + 1)) - 312/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) - sqrt(-2*x^2 + 3*x +
1)))
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 344 vs. $2(118) = 236$.
time = 0.40, size = 344, normalized size = 2.07

$$-\frac{612\sqrt{5}(2x^2-3x-1)\sqrt{\sqrt{10}-3}\arctan\left(\frac{\sqrt{10}\sqrt{5}\sqrt{2}x\sqrt{\sqrt{10}-3}\sqrt{\frac{6x^2+\sqrt{10}(3x^2+2x)-2\sqrt{-2x^2+3x+1}\left(\sqrt{10}x+2x+2\right)+10x+4}{x^2}}+2\left(\sqrt{10}\sqrt{5}(x+1)-\sqrt{10}\sqrt{5}\sqrt{-2x^2+3x+1}+5\sqrt{5}x\right)\sqrt{\sqrt{10}-3}}{10x}\right)+153\sqrt{5}(2x^2-3x-1)\sqrt{\sqrt{10}+3}\log\left(\frac{9\left(5\sqrt{10}x+\left(3\sqrt{10}\sqrt{5}x-10\sqrt{5}x\right)\sqrt{\sqrt{10}+3}-10x+10\sqrt{-2x^2+3x+1}-10\right)}{x}\right)-153\sqrt{5}(2x^2-3x-1)\sqrt{\sqrt{10}+3}\log\left(\frac{9\left(5\sqrt{10}x-\left(3\sqrt{10}\sqrt{5}x-10\sqrt{5}x\right)\sqrt{\sqrt{10}+3}-10x+10\sqrt{-2x^2+3x+1}-10\right)}{x}\right)+600x^2-20\sqrt{-2x^2+3x+1}(14x+15)-900x-300}{170(2x^2-3x-1)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(3/2),x, algorithm="fricas")
```

```
[Out] -1/170*(612*sqrt(5)*(2*x^2 - 3*x - 1)*sqrt(sqrt(10) - 3)*arctan(1/10*(sqrt(
10)*sqrt(5)*sqrt(2)*x*sqrt(sqrt(10) - 3)*sqrt((6*x^2 + sqrt(10)*(3*x^2 + 2*
x) - 2*sqrt(-2*x^2 + 3*x + 1)*(sqrt(10)*x + 2*x + 2) + 10*x + 4)/x^2) + 2*(
sqrt(10)*sqrt(5)*(x + 1) - sqrt(10)*sqrt(5)*sqrt(-2*x^2 + 3*x + 1) + 5*sqrt
(5)*x)*sqrt(sqrt(10) - 3))/x) + 153*sqrt(5)*(2*x^2 - 3*x - 1)*sqrt(sqrt(10)
 + 3)*log(9*(5*sqrt(10)*x + (3*sqrt(10)*sqrt(5)*x - 10*sqrt(5)*x)*sqrt(sqrt
(10) + 3) - 10*x + 10*sqrt(-2*x^2 + 3*x + 1) - 10)/x) - 153*sqrt(5)*(2*x^2
- 3*x - 1)*sqrt(sqrt(10) + 3)*log(9*(5*sqrt(10)*x - (3*sqrt(10)*sqrt(5)*x -
 10*sqrt(5)*x)*sqrt(sqrt(10) + 3) - 10*x + 10*sqrt(-2*x^2 + 3*x + 1) - 10)/
x) + 600*x^2 - 20*sqrt(-2*x^2 + 3*x + 1)*(14*x + 15) - 900*x - 300)/(2*x^2
- 3*x - 1)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$- \int \frac{x}{- 6 x^{4} \sqrt{- 2 x^{2} + 3 x + 1} + 17 x^{3} \sqrt{- 2 x^{2} + 3 x + 1} - 5 x^{2} \sqrt{- 2 x^{2} + 3 x + 1} - 10 x \sqrt{- 2 x^{2} + 3 x + 1} - 2 \sqrt{- 2 x^{2} + 3 x + 1}}\, dx - \int \frac{2}{- 6 x^{4} \sqrt{- 2 x^{2} + 3 x + 1} + 17 x^{3} \sqrt{- 2 x^{2} + 3 x + 1} - 5 x^{2} \sqrt{- 2 x^{2} + 3 x + 1} - 10 x \sqrt{- 2 x^{2} + 3 x + 1} - 2 \sqrt{- 2 x^{2} + 3 x + 1}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(-2*x**2+3*x+1)**(3/2),x)
```

```
[Out] -Integral(x/(-6*x**4*sqrt(-2*x**2 + 3*x + 1) + 17*x**3*sqrt(-2*x**2 + 3*x +
 1) - 5*x**2*sqrt(-2*x**2 + 3*x + 1) - 10*x*sqrt(-2*x**2 + 3*x + 1) - 2*sqr
t(-2*x**2 + 3*x + 1)), x) - Integral(2/(-6*x**4*sqrt(-2*x**2 + 3*x + 1) + 1
7*x**3*sqrt(-2*x**2 + 3*x + 1) - 5*x**2*sqrt(-2*x**2 + 3*x + 1) - 10*x*sqrt
(-2*x**2 + 3*x + 1) - 2*sqrt(-2*x**2 + 3*x + 1)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(3/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{{\left(-2\,x^2+3\,x+1\right)}^{3/2}\,\left(-3\,x^2+4\,x+2\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x - 2*x^2 + 1)^(3/2)*(4*x - 3*x^2 + 2)),x)
```

```
[Out] int((x + 2)/((3*x - 2*x^2 + 1)^(3/2)*(4*x - 3*x^2 + 2)), x)
```

# 3.27 $\int \frac{2+x}{(2+4x-3x^2)(1+3x-2x^2)^{5/2}}\,dx$

**Optimal**. Leaf size=193

$$-\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}-\frac{2(291+4814x)}{867\sqrt{1+3x-2x^2}}+\frac{9}{2}\sqrt{\frac{1}{5}\left(-53+17\sqrt{10}\right)}\tan^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(1+4\sqrt{10}\right)}{2\sqrt{1+\sqrt{10}}\sqrt{1+3x-2x}}\right.$$

```
[Out] -2/51*(15+14*x)/(-2*x^2+3*x+1)^(3/2)-2/867*(291+4814*x)/(-2*x^2+3*x+1)^(1/2
)+9/10*arctan(1/2*(12-3*10^(1/2)+x*(1+4*10^(1/2)))/(-2*x^2+3*x+1)^(1/2)/(1+
10^(1/2))^(1/2))*(-265+85*10^(1/2))^(1/2)+9/10*arctanh(1/2*(x*(1-4*10^(1/2)
)+12+3*10^(1/2))/(-2*x^2+3*x+1)^(1/2)/(-1+10^(1/2))^(1/2))*(265+85*10^(1/2)
)^(1/2)
```

**Rubi** [A]
time = 0.17, antiderivative size = 193, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.200, Rules used = {1030, 1074, 1046, 738, 210, 212}

$$\frac{9}{2}\sqrt{\frac{1}{5}\left(17\sqrt{10}-53\right)}\,\text{ArcTan}\left(\frac{\left(1+4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{1+\sqrt{10}}\sqrt{-2x^2+3x+1}}\right)-\frac{2(14x+15)}{51\left(-2x^2+3x+1\right)^{3/2}}-\frac{2(4814x+291)}{867\sqrt{-2x^2+3x+1}}+\frac{9}{2}\sqrt{\frac{1}{5}\left(53+17\sqrt{10}\right)}\tanh^{-1}\left(\frac{\left(1-4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{\sqrt{10}-1}\sqrt{-2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x - 2*x^2)^(5/2)),x]
```

```
[Out] (-2*(15 + 14*x))/(51*(1 + 3*x - 2*x^2)^(3/2)) - (2*(291 + 4814*x))/(867*Sqr
t[1 + 3*x - 2*x^2]) + (9*Sqrt[(-53 + 17*Sqrt[10])/5]*ArcTan[(3*(4 - Sqrt[10
]) + (1 + 4*Sqrt[10])*x)/(2*Sqrt[1 + Sqrt[10]]*Sqrt[1 + 3*x - 2*x^2])])/2 +
 (9*Sqrt[(53 + 17*Sqrt[10])/5]*ArcTanh[(3*(4 + Sqrt[10]) + (1 - 4*Sqrt[10])
*x)/(2*Sqrt[-1 + Sqrt[10]]*Sqrt[1 + 3*x - 2*x^2])])/2
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1074

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x +
c*x^2)^(p + 1)*((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (
b*d - a*e)*(c*e - b*f))*(p + 1)))*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) +
(A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) + c*(A*(2*c^2*d + b^2*f - c
*(b*e + 2*a*f)) - B*(b*c*d - 2*a*c*e + a*b*f) + C*(b^2*d - a*b*e - 2*a*(c*d
 - a*f)))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
- b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*x^2)^q*Simp[(b
*B - 2*A*c - 2*a*C)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1) + (b^
2*(C*d + A*f) - b*(B*c*d + A*c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*
(c*C*d - B*c*e - a*C*f)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((A*c - a*C)*(2*a
```

```
*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p
 + q + 2) - (2*f*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^
2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (b^2*(C*d + A*f) - b*(B*c*d +
 A*c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*
(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(C*d + A*f) - b*(B*c*d + A*
c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*(2*
p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, e, f, A, B, C, q}, x] &&
 NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f
)^2 - (b*d - a*e)*(c*e - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]) &&  !
IGtQ[q, 0]
```

Rubi steps

$$\int \frac{2+x}{(2+4x-3x^2)(1+3x-2x^2)^{5/2}}\,dx = -\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}+\frac{2}{51}\int \frac{-56+\frac{235x}{2}+84x^2}{(2+4x-3x^2)(1+3x-2x^2)^{3/2}}\,d$$

$$= -\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}-\frac{2(291+4814x)}{867\sqrt{1+3x-2x^2}}+\frac{4}{867}\int \frac{7}{(2+4x-}$$

$$= -\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}-\frac{2(291+4814x)}{867\sqrt{1+3x-2x^2}}+\frac{1}{5}\Big(27\Big(5-2\sqrt{10}$$

$$= -\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}-\frac{2(291+4814x)}{867\sqrt{1+3x-2x^2}}-\frac{1}{5}\Big(54\Big(5-2\sqrt{10}$$

$$= -\frac{2(15+14x)}{51(1+3x-2x^2)^{3/2}}-\frac{2(291+4814x)}{867\sqrt{1+3x-2x^2}}+\frac{9}{2}\sqrt{\frac{1}{5}\Big(-53+17$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.46, size = 183, normalized size = 0.95

$$-\frac{2(546+5925x+13860x^2-9628x^3)}{867(1+3x-2x^2)^{3/2}}-\frac{9}{2}\text{RootSum}\left[5+20\#1+8\#1^2-8\#1^3+2\#1^4\&,\frac{-13\log(x)+13\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)+6\log(x)\#1-6\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)\#1-2\log(x)\#1^2+2\log\left(-1+\sqrt{1+3x-2x^2}-x\#1\right)\#1^2}{5+4\#1-6\#1^2+2\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x - 2*x^2)^(5/2)),x]
```

```
[Out] (-2*(546 + 5925*x + 13860*x^2 - 9628*x^3))/(867*(1 + 3*x - 2*x^2)^(3/2)) -
(9*RootSum[5 + 20*#1 + 8*#1^2 - 8*#1^3 + 2*#1^4 & , (-13*Log[x] + 13*Log[-1
 + Sqrt[1 + 3*x - 2*x^2] - x*#1] + 6*Log[x]*#1 - 6*Log[-1 + Sqrt[1 + 3*x -
2*x^2] - x*#1]*#1 - 2*Log[x]*#1^2 + 2*Log[-1 + Sqrt[1 + 3*x - 2*x^2] - x*#1
]*#1^2)/(5 + 4*#1 - 6*#1^2 + 2*#1^3) & ])/2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 867 vs. $2(137) = 274$.

time = 0.61, size = 868, normalized size = 4.50

| method | result |
|---|---|
| trager | $\frac{2(9628x^3-13860x^2-5925x-546)\sqrt{-2x^2+3x+1}}{867(2x^2-3x-1)^2} - 18\,\mathrm{RootOf}\left(6400\_Z^4 - 8480\_Z^2 - 81\right)\ln\left(-\frac{-210}{}\right.$ |
| default | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(5/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/20*(8+10^(1/2))*10^(1/2)*(1/9/(-1/9+1/9*10^(1/2))/(-2*(x-2/3-1/3*10^(1/2
))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))-1/9+1/9*10^(1/2))^(3/2)-1/6*(1
/3-4/3*10^(1/2))/(-1/9+1/9*10^(1/2))*(2/3*(-4*x+3)/(8/9-8/9*10^(1/2)-(1/3-4
/3*10^(1/2))^2)/(-2*(x-2/3-1/3*10^(1/2))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10
^(1/2))-1/9+1/9*10^(1/2))^(3/2)-32/3/(8/9-8/9*10^(1/2)-(1/3-4/3*10^(1/2))^2
)^2*(-4*x+3)/(-2*(x-2/3-1/3*10^(1/2))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1
/2))-1/9+1/9*10^(1/2))^(1/2))+1/3/(-1/9+1/9*10^(1/2))*(1/(-1/9+1/9*10^(1/2)
)/(-2*(x-2/3-1/3*10^(1/2))^2+(1/3-4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))-1/9+1/
9*10^(1/2))^(1/2)-(1/3-4/3*10^(1/2))/(-1/9+1/9*10^(1/2))*(-4*x+3)/(8/9-8/9*
10^(1/2)-(1/3-4/3*10^(1/2))^2)/(-2*(x-2/3-1/3*10^(1/2))^2+(1/3-4/3*10^(1/2)
)*(x-2/3-1/3*10^(1/2))-1/9+1/9*10^(1/2))^(1/2)-3/(-1/9+1/9*10^(1/2))/(-1+10
^(1/2))^(1/2)*arctanh(9/2*(-2/9+2/9*10^(1/2)+(1/3-4/3*10^(1/2))*(x-2/3-1/3*
10^(1/2)))/(-1+10^(1/2))^(1/2)/(-18*(x-2/3-1/3*10^(1/2))^2+9*(1/3-4/3*10^(1
/2))*(x-2/3-1/3*10^(1/2))-1+10^(1/2))^(1/2))))-1/20*(-8+10^(1/2))*10^(1/2)*
(1/9/(-1/9-1/9*10^(1/2))/(-2*(x-2/3+1/3*10^(1/2))^2+(1/3+4/3*10^(1/2))*(x-2
/3+1/3*10^(1/2))-1/9-1/9*10^(1/2))^(3/2)-1/6*(1/3+4/3*10^(1/2))/(-1/9-1/9*1
0^(1/2))*(2/3*(-4*x+3)/(8/9+8/9*10^(1/2)-(1/3+4/3*10^(1/2))^2)/(-2*(x-2/3+1
/3*10^(1/2))^2+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-1/9*10^(1/2))^(3
/2)-32/3/(8/9+8/9*10^(1/2)-(1/3+4/3*10^(1/2))^2)^2*(-4*x+3)/(-2*(x-2/3+1/3*
10^(1/2))^2+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-1/9*10^(1/2))^(1/2)
)+1/3/(-1/9-1/9*10^(1/2))*(1/(-1/9-1/9*10^(1/2))/(-2*(x-2/3+1/3*10^(1/2))^2
+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-1/9*10^(1/2))^(1/2)-(1/3+4/3*1
0^(1/2))/(-1/9-1/9*10^(1/2))*(-4*x+3)/(8/9+8/9*10^(1/2)-(1/3+4/3*10^(1/2))^
2)/(-2*(x-2/3+1/3*10^(1/2))^2+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1/9-1
/9*10^(1/2))^(1/2)+3/(-1/9-1/9*10^(1/2))/(1+10^(1/2))^(1/2)*arctan(9/2*(-2/
9-2/9*10^(1/2)+(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2)))/(1+10^(1/2))^(1/2)/
(-18*(x-2/3+1/3*10^(1/2))^2+9*(1/3+4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))-1-10^
(1/2))^(1/2))))
```

**Maxima** [B] Leaf count of result is larger than twice the leaf count of optimal. 1276 vs. $2(137) = 274$.

time = 0.56, size = 1276, normalized size = 6.61

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(5/2),x, algorithm="maxima")
```

```
[Out] 1/17340*sqrt(10)*(2108*sqrt(10)*x/(sqrt(10)*(-2*x^2 + 3*x + 1)^(3/2) + (-2*
x^2 + 3*x + 1)^(3/2)) - 2108*sqrt(10)*x/(sqrt(10)*(-2*x^2 + 3*x + 1)^(3/2)
- (-2*x^2 + 3*x + 1)^(3/2)) - 56916*sqrt(10)*x/(2*sqrt(10)*sqrt(-2*x^2 + 3*
x + 1) + 11*sqrt(-2*x^2 + 3*x + 1)) + 56916*sqrt(10)*x/(2*sqrt(10)*sqrt(-2*
x^2 + 3*x + 1) - 11*sqrt(-2*x^2 + 3*x + 1)) + 1984*sqrt(10)*x/(sqrt(10)*sqr
t(-2*x^2 + 3*x + 1) + sqrt(-2*x^2 + 3*x + 1)) - 1984*sqrt(10)*x/(sqrt(10)*s
qrt(-2*x^2 + 3*x + 1) - sqrt(-2*x^2 + 3*x + 1)) - 70227*sqrt(10)*arcsin(8/1
7*sqrt(17)*sqrt(10)*x/abs(6*x + 2*sqrt(10) - 4) + 2/17*sqrt(17)*x/abs(6*x +
 2*sqrt(10) - 4) - 6/17*sqrt(17)*sqrt(10)/abs(6*x + 2*sqrt(10) - 4) + 24/17
*sqrt(17)/abs(6*x + 2*sqrt(10) - 4))/(2*sqrt(10)*sqrt(sqrt(10) + 1) + 11*sq
rt(sqrt(10) + 1)) - 2176*x/(sqrt(10)*(-2*x^2 + 3*x + 1)^(3/2) + (-2*x^2 + 3
*x + 1)^(3/2)) - 2176*x/(sqrt(10)*(-2*x^2 + 3*x + 1)^(3/2) - (-2*x^2 + 3*x
+ 1)^(3/2)) + 58752*x/(2*sqrt(10)*sqrt(-2*x^2 + 3*x + 1) + 11*sqrt(-2*x^2 +
 3*x + 1)) + 58752*x/(2*sqrt(10)*sqrt(-2*x^2 + 3*x + 1) - 11*sqrt(-2*x^2 +
3*x + 1)) - 2048*x/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) + sqrt(-2*x^2 + 3*x + 1
)) - 2048*x/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) - sqrt(-2*x^2 + 3*x + 1)) + 56
1816*arcsin(8/17*sqrt(17)*sqrt(10)*x/abs(6*x + 2*sqrt(10) - 4) + 2/17*sqrt(
17)*x/abs(6*x + 2*sqrt(10) - 4) - 6/17*sqrt(17)*sqrt(10)/abs(6*x + 2*sqrt(1
0) - 4) + 24/17*sqrt(17)/abs(6*x + 2*sqrt(10) - 4))/(2*sqrt(10)*sqrt(sqrt(1
0) + 1) + 11*sqrt(sqrt(10) + 1)) - 714*sqrt(10)/(sqrt(10)*(-2*x^2 + 3*x + 1
)^(3/2) + (-2*x^2 + 3*x + 1)^(3/2)) + 714*sqrt(10)/(sqrt(10)*(-2*x^2 + 3*x
+ 1)^(3/2) - (-2*x^2 + 3*x + 1)^(3/2)) + 19278*sqrt(10)/(2*sqrt(10)*sqrt(-2
*x^2 + 3*x + 1) + 11*sqrt(-2*x^2 + 3*x + 1)) - 19278*sqrt(10)/(2*sqrt(10)*s
qrt(-2*x^2 + 3*x + 1) - 11*sqrt(-2*x^2 + 3*x + 1)) - 1488*sqrt(10)/(sqrt(10
)*sqrt(-2*x^2 + 3*x + 1) + sqrt(-2*x^2 + 3*x + 1)) + 1488*sqrt(10)/(sqrt(10
)*sqrt(-2*x^2 + 3*x + 1) - sqrt(-2*x^2 + 3*x + 1)) - 5304/(sqrt(10)*(-2*x^2
 + 3*x + 1)^(3/2) + (-2*x^2 + 3*x + 1)^(3/2)) - 5304/(sqrt(10)*(-2*x^2 + 3*
x + 1)^(3/2) - (-2*x^2 + 3*x + 1)^(3/2)) + 143208/(2*sqrt(10)*sqrt(-2*x^2 +
 3*x + 1) + 11*sqrt(-2*x^2 + 3*x + 1)) + 143208/(2*sqrt(10)*sqrt(-2*x^2 + 3
*x + 1) - 11*sqrt(-2*x^2 + 3*x + 1)) + 1536/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1
) + sqrt(-2*x^2 + 3*x + 1)) + 1536/(sqrt(10)*sqrt(-2*x^2 + 3*x + 1) - sqrt(
-2*x^2 + 3*x + 1)) + 70227*sqrt(10)*log(-2/9*sqrt(10) + 2/3*sqrt(-2*x^2 + 3
*x + 1)*sqrt(sqrt(10) - 1)/abs(6*x - 2*sqrt(10) - 4) + 2/9*sqrt(10)/abs(6*x
 - 2*sqrt(10) - 4) - 2/9/abs(6*x - 2*sqrt(10) - 4) + 1/18)/(sqrt(10) - 1)^(
5/2) + 561816*log(-2/9*sqrt(10) + 2/3*sqrt(-2*x^2 + 3*x + 1)*sqrt(sqrt(10)
- 1)/abs(6*x - 2*sqrt(10) - 4) + 2/9*sqrt(10)/abs(6*x - 2*sqrt(10) - 4) - 2
/9/abs(6*x - 2*sqrt(10) - 4) + 1/18)/(sqrt(10) - 1)^(5/2))
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 439 vs. $2(137) = 274$.
time = 0.43, size = 439, normalized size = 2.27

$$-\frac{43680x^4 - 131040x^3 - 31212\sqrt{5}(4x^4-12x^3+5x^2+6x+1)\sqrt{17\sqrt{10}-53}\arctan\left(\frac{\sqrt{2}\left(\sqrt{10}\sqrt{5}x+10\sqrt{5}x\right)\sqrt{17\sqrt{10}-53}\sqrt{\frac{6x^2+\sqrt{10}(3x^2+2x)-2\sqrt{-2x^2+3x+1}\left(\sqrt{10}x+2x+2\right)+10x+4}{x^2}}+2\left(\sqrt{10}\sqrt{5}(6x+1)-\sqrt{-2x^2+3x+1}\left(\sqrt{10}\sqrt{5}+10\sqrt{5}\right)+5\sqrt{5}(3x+2)\right)\sqrt{17\sqrt{10}-53}}{90x}\right) - 7803\sqrt{5}(4x^4-12x^3+5x^2+6x+1)\sqrt{17\sqrt{10}+53}\log\left(\frac{9\left(45\sqrt{10}x+\left(13\sqrt{10}\sqrt{5}x-40\sqrt{5}x\right)\sqrt{17\sqrt{10}+53}-90x+90\sqrt{-2x^2+3x+1}-90\right)}{x}\right) + 7803\sqrt{5}(4x^4-12x^3+5x^2+6x+1)\sqrt{17\sqrt{10}+53}\log\left(\frac{9\left(45\sqrt{10}x-\left(13\sqrt{10}\sqrt{5}x-40\sqrt{5}x\right)\sqrt{17\sqrt{10}+53}-90x+90\sqrt{-2x^2+3x+1}-90\right)}{x}\right)+54600x^2-20(9628x^3-13860x^2-5925x-546)\sqrt{-2x^2+3x+1}+65520x+10920}{8670(4x^4-12x^3+5x^2+6x+1)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(5/2),x, algorithm="fricas")
```

```
[Out] -1/8670*(43680*x^4 - 131040*x^3 - 31212*sqrt(5)*(4*x^4 - 12*x^3 + 5*x^2 + 6
*x + 1)*sqrt(17*sqrt(10) - 53)*arctan(1/90*(sqrt(2)*(sqrt(10)*sqrt(5)*x + 1
0*sqrt(5)*x)*sqrt(17*sqrt(10) - 53)*sqrt((6*x^2 + sqrt(10)*(3*x^2 + 2*x) -
2*sqrt(-2*x^2 + 3*x + 1)*(sqrt(10)*x + 2*x + 2) + 10*x + 4)/x^2) + 2*(sqrt(
10)*sqrt(5)*(6*x + 1) - sqrt(-2*x^2 + 3*x + 1)*(sqrt(10)*sqrt(5) + 10*sqrt(
5)) + 5*sqrt(5)*(3*x + 2))*sqrt(17*sqrt(10) - 53))/x) - 7803*sqrt(5)*(4*x^4
 - 12*x^3 + 5*x^2 + 6*x + 1)*sqrt(17*sqrt(10) + 53)*log(9*(45*sqrt(10)*x +
(13*sqrt(10)*sqrt(5)*x - 40*sqrt(5)*x)*sqrt(17*sqrt(10) + 53) - 90*x + 90*s
qrt(-2*x^2 + 3*x + 1) - 90)/x) + 7803*sqrt(5)*(4*x^4 - 12*x^3 + 5*x^2 + 6*x
 + 1)*sqrt(17*sqrt(10) + 53)*log(9*(45*sqrt(10)*x - (13*sqrt(10)*sqrt(5)*x
- 40*sqrt(5)*x)*sqrt(17*sqrt(10) + 53) - 90*x + 90*sqrt(-2*x^2 + 3*x + 1) -
 90)/x) + 54600*x^2 - 20*(9628*x^3 - 13860*x^2 - 5925*x - 546)*sqrt(-2*x^2
+ 3*x + 1) + 65520*x + 10920)/(4*x^4 - 12*x^3 + 5*x^2 + 6*x + 1)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x}{12x^6\sqrt{-2x^2+3x+1}-52x^5\sqrt{-2x^2+3x+1}+55x^4\sqrt{-2x^2+3x+1}+22x^3\sqrt{-2x^2+3x+1}-31x^2\sqrt{-2x^2+3x+1}-16x\sqrt{-2x^2+3x+1}-2\sqrt{-2x^2+3x+1}}\,dx-\int \frac{2}{12x^6\sqrt{-2x^2+3x+1}-52x^5\sqrt{-2x^2+3x+1}+55x^4\sqrt{-2x^2+3x+1}+22x^3\sqrt{-2x^2+3x+1}-31x^2\sqrt{-2x^2+3x+1}-16x\sqrt{-2x^2+3x+1}-2\sqrt{-2x^2+3x+1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(-2*x**2+3*x+1)**(5/2),x)
```

```
[Out] -Integral(x/(12*x**6*sqrt(-2*x**2 + 3*x + 1) - 52*x**5*sqrt(-2*x**2 + 3*x +
 1) + 55*x**4*sqrt(-2*x**2 + 3*x + 1) + 22*x**3*sqrt(-2*x**2 + 3*x + 1) - 3
1*x**2*sqrt(-2*x**2 + 3*x + 1) - 16*x*sqrt(-2*x**2 + 3*x + 1) - 2*sqrt(-2*x
**2 + 3*x + 1)), x) - Integral(2/(12*x**6*sqrt(-2*x**2 + 3*x + 1) - 52*x**5
*sqrt(-2*x**2 + 3*x + 1) + 55*x**4*sqrt(-2*x**2 + 3*x + 1) + 22*x**3*sqrt(-
2*x**2 + 3*x + 1) - 31*x**2*sqrt(-2*x**2 + 3*x + 1) - 16*x*sqrt(-2*x**2 + 3
*x + 1) - 2*sqrt(-2*x**2 + 3*x + 1)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(-2*x^2+3*x+1)^(5/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{\left(-2\,x^2+3\,x+1\right)^{5/2}\,\left(-3\,x^2+4\,x+2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x - 2*x^2 + 1)^(5/2)*(4*x - 3*x^2 + 2)),x)
```

```
[Out] int((x + 2)/((3*x - 2*x^2 + 1)^(5/2)*(4*x - 3*x^2 + 2)), x)
```

## 3.28 $\int \frac{2+x}{(2+4x-3x^2)\sqrt{1+3x+2x^2}}\,dx$

**Optimal**. Leaf size=151

$$-\frac{1}{2}\sqrt{1+\frac{7\sqrt{\frac{2}{5}}}{5}}\tanh^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(17-4\sqrt{10}\right)x}{2\sqrt{55-17\sqrt{10}}\sqrt{1+3x+2x^2}}\right)+\frac{1}{2}\sqrt{1-\frac{7\sqrt{\frac{2}{5}}}{5}}\tanh^{-1}\left(\frac{3\left(4+\sqrt{10}\right)+}{2\sqrt{55+17\sqrt{10}}}\right.$$

```
[Out] 1/10*arctanh(1/2*(12+3*10^(1/2)+x*(17+4*10^(1/2)))/(2*x^2+3*x+1)^(1/2)/(55+
17*10^(1/2))^(1/2))*(25-7*10^(1/2))^(1/2)-1/10*arctanh(1/2*(x*(17-4*10^(1/2
))+12-3*10^(1/2))/(2*x^2+3*x+1)^(1/2)/(55-17*10^(1/2))^(1/2))*(25+7*10^(1/2
))^(1/2)
```

**Rubi** [A]

time = 0.14, antiderivative size = 151, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.100$, Rules used = {1046, 738, 212}

$$\frac{1}{2}\sqrt{1-\frac{7\sqrt{\frac{2}{5}}}{5}}\tanh^{-1}\left(\frac{\left(17+4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{55+17\sqrt{10}}\sqrt{2x^2+3x+1}}\right)-\frac{1}{2}\sqrt{1+\frac{7\sqrt{\frac{2}{5}}}{5}}\tanh^{-1}\left(\frac{\left(17-4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{55-17\sqrt{10}}\sqrt{2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*Sqrt[1 + 3*x + 2*x^2]),x]
```

```
[Out] -1/2*(Sqrt[1 + (7*Sqrt[2/5])/5]*ArcTanh[(3*(4 - Sqrt[10]) + (17 - 4*Sqrt[10
])*x)/(2*Sqrt[55 - 17*Sqrt[10]]*Sqrt[1 + 3*x + 2*x^2])]) + (Sqrt[1 - (7*Sqr
t[2/5])/5]*ArcTanh[(3*(4 + Sqrt[10]) + (17 + 4*Sqrt[10])*x)/(2*Sqrt[55 + 17
*Sqrt[10]]*Sqrt[1 + 3*x + 2*x^2])])/2
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{2+x}{(2+4x-3x^2)\sqrt{1+3x+2x^2}}\,dx = \frac{1}{5}\left(5-4\sqrt{10}\right)\int \frac{1}{\left(4-2\sqrt{10}-6x\right)\sqrt{1+3x+2x^2}}\,dx + \frac{1}{5}\left(5+4\right.$$

$$= -\left(\frac{1}{5}\left(2\left(5-4\sqrt{10}\right)\right)\text{Subst}\left(\int \frac{1}{144+72\left(4-2\sqrt{10}\right)+8\left(4-2\right.}\right.\right.$$

$$= -\frac{1}{10}\sqrt{25+7\sqrt{10}}\tanh^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(17-4\sqrt{10}\right)x}{2\sqrt{55-17\sqrt{10}}\sqrt{1+3x+2x^2}}\right)+$$

**Mathematica** [A]
time = 0.52, size = 109, normalized size = 0.72

$$-\frac{1}{5}\sqrt{25+7\sqrt{10}}\tanh^{-1}\left(\frac{\sqrt{1-\sqrt{\frac{2}{5}}}\sqrt{1+3x+2x^2}}{1+2x}\right)+\frac{1}{5}\sqrt{25-7\sqrt{10}}\tanh^{-1}\left(\frac{\sqrt{1+\sqrt{\frac{2}{5}}}\sqrt{1+3x+2x^2}}{1+2x}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*Sqrt[1 + 3*x + 2*x^2]),x]
```

```
[Out] -1/5*(Sqrt[25 + 7*Sqrt[10]]*ArcTanh[(Sqrt[1 - Sqrt[2/5]]*Sqrt[1 + 3*x + 2*x
^2])/(1 + 2*x)]) + (Sqrt[25 - 7*Sqrt[10]]*ArcTanh[(Sqrt[1 + Sqrt[2/5]]*Sqrt
[1 + 3*x + 2*x^2])/(1 + 2*x)])/5
```

**Maple** [A]
time = 0.62, size = 186, normalized size = 1.23

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{\left(8+\sqrt{10}\right)\sqrt{10}\,\operatorname{arctanh}\left(\frac{55+17\sqrt{10}+\frac{9\left(\frac{17}{3}+\frac{4\sqrt{10}}{3}\right)\left(x-\frac{2}{3}-\frac{\sqrt{10}}{3}\right)}{2}}{\sqrt{55+17\sqrt{10}}\,\sqrt{18\left(x-\frac{2}{3}-\frac{\sqrt{10}}{3}\right)^{2}+9\left(\frac{17}{3}+\frac{4\sqrt{10}}{3}\right)\left(x-\frac{2}{3}-\right.}}\right.}{20\sqrt{55+17\sqrt{10}}}$ |
| trager | $-\operatorname{RootOf}\left(2000\_Z^{4}-1000\_Z^{2}+27\right)\ln\left(\frac{10000x\operatorname{RootOf}\left(2000\_Z^{4}-1000\_Z^{2}+27\right)^{5}+12600\operatorname{RootOf}\left(2000\_Z\right.}{}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/20*(8+10^(1/2))*10^(1/2)/(55+17*10^(1/2))^(1/2)*arctanh(9/2*(110/9+34/9*1
0^(1/2)+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2)))/(55+17*10^(1/2))^(1/2)/(1
8*(x-2/3-1/3*10^(1/2))^2+9*(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))+55+17*1
0^(1/2))^(1/2))+1/20*(-8+10^(1/2))*10^(1/2)/(55-17*10^(1/2))^(1/2)*arctanh(
9/2*(110/9-34/9*10^(1/2)+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2)))/(55-17*1
0^(1/2))^(1/2)/(18*(x-2/3+1/3*10^(1/2))^2+9*(17/3-4/3*10^(1/2))*(x-2/3+1/3*
10^(1/2))+55-17*10^(1/2))^(1/2))
```

**Maxima** [B] Leaf count of result is larger than twice the leaf count of optimal. 363 vs. $2(103) = 206$.
time $= 0.51$, size $= 363$, normalized size $= 2.40$

$$\frac{1}{60}\sqrt{10}\left(\frac{3\sqrt{10}\log\left(\frac{2}{9}\sqrt{10}+\frac{2\sqrt{2x^{2}+3x+1}\sqrt{17\sqrt{10}+55}}{3\left|6x-2\sqrt{10}-4\right|}+\frac{34\sqrt{10}}{9\left|6x-2\sqrt{10}-4\right|}+\frac{110}{9\left|6x-2\sqrt{10}-4\right|}+\frac{17}{18}\right)}{\sqrt{17\sqrt{10}+55}}+\frac{\sqrt{10}\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{2x^{2}+3x+1}\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}{\left|6x+2\sqrt{10}-4\right|}-\frac{34\sqrt{10}}{9\left|6x+2\sqrt{10}-4\right|}+\frac{110}{9\left|6x+2\sqrt{10}-4\right|}+\frac{17}{18}\right)}{\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}+\frac{24\log\left(\frac{2}{9}\sqrt{10}+\frac{2\sqrt{2x^{2}+3x+1}\sqrt{17\sqrt{10}+55}}{3\left|6x-2\sqrt{10}-4\right|}+\frac{34\sqrt{10}}{9\left|6x-2\sqrt{10}-4\right|}+\frac{110}{9\left|6x-2\sqrt{10}-4\right|}+\frac{17}{18}\right)}{\sqrt{17\sqrt{10}+55}}-\frac{8\log\left(-\frac{2}{9}\sqrt{10}+\frac{2\sqrt{2x^{2}+3x+1}\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}{\left|6x+2\sqrt{10}-4\right|}-\frac{34\sqrt{10}}{9\left|6x+2\sqrt{10}-4\right|}+\frac{110}{9\left|6x+2\sqrt{10}-4\right|}+\frac{17}{18}\right)}{\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(1/2),x, algorithm="maxima")
```

```
[Out] 1/60*sqrt(10)*(3*sqrt(10)*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1)*sqrt
(17*sqrt(10) + 55)/abs(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x - 2*sq
rt(10) - 4) + 110/9/abs(6*x - 2*sqrt(10) - 4) + 17/18)/sqrt(17*sqrt(10) + 5
5) + sqrt(10)*log(-2/9*sqrt(10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-17/9*sqrt(1
0) + 55/9)/abs(6*x + 2*sqrt(10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*sqrt(10) -
 4) + 110/9/abs(6*x + 2*sqrt(10) - 4) + 17/18)/sqrt(-17/9*sqrt(10) + 55/9)
+ 24*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1)*sqrt(17*sqrt(10) + 55)/ab
s(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x - 2*sqrt(10) - 4) + 110/9/a
bs(6*x - 2*sqrt(10) - 4) + 17/18)/sqrt(17*sqrt(10) + 55) - 8*log(-2/9*sqrt(
10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-17/9*sqrt(10) + 55/9)/abs(6*x + 2*sqrt(
```

```
10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*sqrt(10) - 4) + 110/9/abs(6*x + 2*sqrt
(10) - 4) + 17/18)/sqrt(-17/9*sqrt(10) + 55/9))
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 245 vs. $2(103) = 206$.
time = 0.36, size = 245, normalized size = 1.62

$$\frac{1}{10}\sqrt{7\sqrt{10}+25}\log\left(-\frac{3\sqrt{10}x+\left(\sqrt{10}x-4x\right)\sqrt{7\sqrt{10}+25}+6x-6\sqrt{2x^2+3x+1}+6}{x}\right)-\frac{1}{10}\sqrt{7\sqrt{10}+25}\log\left(-\frac{3\sqrt{10}x-\left(\sqrt{10}x-4x\right)\sqrt{7\sqrt{10}+25}+6x-6\sqrt{2x^2+3x+1}+6}{x}\right)+\frac{1}{10}\sqrt{-7\sqrt{10}+25}\log\left(\frac{3\sqrt{10}x+\left(\sqrt{10}x+4x\right)\sqrt{-7\sqrt{10}+25}-6x+6\sqrt{2x^2+3x+1}-6}{x}\right)-\frac{1}{10}\sqrt{-7\sqrt{10}+25}\log\left(\frac{3\sqrt{10}x-\left(\sqrt{10}x+4x\right)\sqrt{-7\sqrt{10}+25}-6x+6\sqrt{2x^2+3x+1}-6}{x}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/10*sqrt(7*sqrt(10) + 25)*log(-(3*sqrt(10)*x + (sqrt(10)*x - 4*x)*sqrt(7*s
qrt(10) + 25) + 6*x - 6*sqrt(2*x^2 + 3*x + 1) + 6)/x) - 1/10*sqrt(7*sqrt(10
) + 25)*log(-(3*sqrt(10)*x - (sqrt(10)*x - 4*x)*sqrt(7*sqrt(10) + 25) + 6*x
 - 6*sqrt(2*x^2 + 3*x + 1) + 6)/x) + 1/10*sqrt(-7*sqrt(10) + 25)*log((3*sqr
t(10)*x + (sqrt(10)*x + 4*x)*sqrt(-7*sqrt(10) + 25) - 6*x + 6*sqrt(2*x^2 +
3*x + 1) - 6)/x) - 1/10*sqrt(-7*sqrt(10) + 25)*log((3*sqrt(10)*x - (sqrt(10
)*x + 4*x)*sqrt(-7*sqrt(10) + 25) - 6*x + 6*sqrt(2*x^2 + 3*x + 1) - 6)/x)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int\frac{x}{3x^2\sqrt{2x^2+3x+1}-4x\sqrt{2x^2+3x+1}-2\sqrt{2x^2+3x+1}}\,dx-\int\frac{2}{3x^2\sqrt{2x^2+3x+1}-4x\sqrt{2x^2+3x+1}-2\sqrt{2x^2+3x+1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(2*x**2+3*x+1)**(1/2),x)
```

```
[Out] -Integral(x/(3*x**2*sqrt(2*x**2 + 3*x + 1) - 4*x*sqrt(2*x**2 + 3*x + 1) - 2
*sqrt(2*x**2 + 3*x + 1)), x) - Integral(2/(3*x**2*sqrt(2*x**2 + 3*x + 1) -
4*x*sqrt(2*x**2 + 3*x + 1) - 2*sqrt(2*x**2 + 3*x + 1)), x)
```

**Giac** [A]
time = 7.02, size = 93, normalized size = 0.62

$$0.169235232112667\log\left(-\sqrt{2}x+\sqrt{2x^2+3x+1}+5.90976932712000\right)-0.686556214893333\log\left(-\sqrt{2}x+\sqrt{2x^2+3x+1}-0.176527156327000\right)+0.686556214893333\log\left(-\sqrt{2}x+\sqrt{2x^2+3x+1}-0.919278730509000\right)-0.169235232112667\log\left(-\sqrt{2}x+\sqrt{2x^2+3x+1}-1.04272727395000\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(1/2),x, algorithm="giac")
```

```
[Out] 0.169235232112667*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) + 5.90976932712000
) - 0.686556214893333*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 0.1765271563
27000) + 0.686556214893333*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 0.91927
8730509000) - 0.169235232112667*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 1.
04272727395000)
```

**Mupad [F]**

time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{\sqrt{2\,x^2+3\,x+1}\,\left(-3\,x^2+4\,x+2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x + 2*x^2 + 1)^(1/2)*(4*x - 3*x^2 + 2)),x)
[Out] int((x + 2)/((3*x + 2*x^2 + 1)^(1/2)*(4*x - 3*x^2 + 2)), x)
```

## 3.29 $\int \frac{2+x}{(2+4x-3x^2)(1+3x+2x^2)^{3/2}}\,dx$

**Optimal**. Leaf size=174

$$\frac{2(21+22x)}{5\sqrt{1+3x+2x^2}}-\frac{1}{10}\sqrt{\frac{3}{5}\left(2065+653\sqrt{10}\right)}\,\tanh^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+\left(17-4\sqrt{10}\right)x}{2\sqrt{55-17\sqrt{10}}\,\sqrt{1+3x+2x^2}}\right)+\frac{1}{10}\sqrt{\frac{3}{5}\left(2065\right.}$$

```
[Out] 2/5*(21+22*x)/(2*x^2+3*x+1)^(1/2)+1/50*arctanh(1/2*(12+3*10^(1/2)+x*(17+4*1
0^(1/2)))/(2*x^2+3*x+1)^(1/2)/(55+17*10^(1/2))^(1/2))*(30975-9795*10^(1/2))
^(1/2)-1/50*arctanh(1/2*(x*(17-4*10^(1/2))+12-3*10^(1/2))/(2*x^2+3*x+1)^(1/
2)/(55-17*10^(1/2))^(1/2))*(30975+9795*10^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.16, antiderivative size = 174, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {1030, 1046, 738, 212}

$$\frac{2(22x+21)}{5\sqrt{2x^2+3x+1}}-\frac{1}{10}\sqrt{\frac{3}{5}\left(2065+653\sqrt{10}\right)}\,\tanh^{-1}\left(\frac{\left(17-4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{55-17\sqrt{10}}\,\sqrt{2x^2+3x+1}}\right)+\frac{1}{10}\sqrt{\frac{3}{5}\left(2065-653\sqrt{10}\right)}\,\tanh^{-1}\left(\frac{\left(17+4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{55+17\sqrt{10}}\,\sqrt{2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x + 2*x^2)^(3/2)),x]
```

```
[Out] (2*(21 + 22*x))/(5*Sqrt[1 + 3*x + 2*x^2]) - (Sqrt[(3*(2065 + 653*Sqrt[10]))
/5]*ArcTanh[(3*(4 - Sqrt[10]) + (17 - 4*Sqrt[10])*x)/(2*Sqrt[55 - 17*Sqrt[1
0]]*Sqrt[1 + 3*x + 2*x^2])])/10 + (Sqrt[(3*(2065 - 653*Sqrt[10]))/5]*ArcTan
h[(3*(4 + Sqrt[10]) + (17 + 4*Sqrt[10])*x)/(2*Sqrt[55 + 17*Sqrt[10]]*Sqrt[1
 + 3*x + 2*x^2])])/10
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
```

```
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{2+x}{(2+4x-3x^2)(1+3x+2x^2)^{3/2}}\,dx = \frac{2(21+22x)}{5\sqrt{1+3x+2x^2}} - \frac{2}{15}\int \frac{-72+\frac{81x}{2}}{(2+4x-3x^2)\sqrt{1+3x+2x^2}}\,dx$$

$$= \frac{2(21+22x)}{5\sqrt{1+3x+2x^2}} - \frac{1}{5}\left(9\left(3-\sqrt{10}\right)\right)\int \frac{1}{\left(4+2\sqrt{10}-6x\right)\sqrt{1}}$$

$$= \frac{2(21+22x)}{5\sqrt{1+3x+2x^2}} + \frac{1}{5}\left(18\left(3-\sqrt{10}\right)\right)\text{Subst}\left(\int \frac{}{144+72\left(4+\right.}\right.$$

$$= \frac{2(21+22x)}{5\sqrt{1+3x+2x^2}} - \frac{1}{10}\sqrt{\frac{3}{5}\left(2065+653\sqrt{10}\right)}\tanh^{-1}\left(\frac{3\left(4-\right.}{2\sqrt{55}}\right.$$

**Mathematica** [A]

time = 0.80, size = 130, normalized size = 0.75

$$\frac{1}{25}\left(\frac{5(42+44x)}{\sqrt{1+3x+2x^2}}-\sqrt{30975+9795\sqrt{10}}\tanh^{-1}\left(\frac{\sqrt{1-\sqrt{\frac{2}{5}}}\sqrt{1+3x+2x^2}}{1+2x}\right)+\frac{45\tanh^{-1}\left(\frac{\sqrt{1+\sqrt{\frac{2}{5}}}\sqrt{1+3x+2x^2}}{1+2x}\right)}{\sqrt{2065+653\sqrt{10}}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x + 2*x^2)^(3/2)),x]
```

```
[Out] ((5*(42 + 44*x))/Sqrt[1 + 3*x + 2*x^2] - Sqrt[30975 + 9795*Sqrt[10]]*ArcTan
h[(Sqrt[1 - Sqrt[2/5]]*Sqrt[1 + 3*x + 2*x^2])/(1 + 2*x)] + (45*ArcTanh[(Sqr
t[1 + Sqrt[2/5]]*Sqrt[1 + 3*x + 2*x^2])/(1 + 2*x)])/Sqrt[2065 + 653*Sqrt[10
]])/25
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 465 vs. $2(122) = 244$.

time = 0.62, size = 466, normalized size = 2.68 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(3/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/20*(8+10^(1/2))*10^(1/2)*(1/3/(55/9+17/9*10^(1/2))/(2*(x-2/3-1/3*10^(1/2
))^2+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))+55/9+17/9*10^(1/2))^(1/2)-1/3
*(17/3+4/3*10^(1/2))/(55/9+17/9*10^(1/2))*(3+4*x)/(440/9+136/9*10^(1/2)-(17
/3+4/3*10^(1/2))^2)/(2*(x-2/3-1/3*10^(1/2))^2+(17/3+4/3*10^(1/2))*(x-2/3-1/
3*10^(1/2))+55/9+17/9*10^(1/2))^(1/2)-1/(55/9+17/9*10^(1/2))/(55+17*10^(1/2
))^(1/2)*arctanh(9/2*(110/9+34/9*10^(1/2)+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10
^(1/2)))/(55+17*10^(1/2))^(1/2)/(18*(x-2/3-1/3*10^(1/2))^2+9*(17/3+4/3*10^(
1/2))*(x-2/3-1/3*10^(1/2))+55+17*10^(1/2))^(1/2)))-1/20*(-8+10^(1/2))*10^(1
/2)*(1/3/(55/9-17/9*10^(1/2))/(2*(x-2/3+1/3*10^(1/2))^2+(17/3-4/3*10^(1/2))
*(x-2/3+1/3*10^(1/2))+55/9-17/9*10^(1/2))^(1/2)-1/3*(17/3-4/3*10^(1/2))/(55
/9-17/9*10^(1/2))*(3+4*x)/(440/9-136/9*10^(1/2)-(17/3-4/3*10^(1/2))^2)/(2*(
x-2/3+1/3*10^(1/2))^2+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*10
^(1/2))^(1/2)-1/(55/9-17/9*10^(1/2))/(55-17*10^(1/2))^(1/2)*arctanh(9/2*(11
0/9-34/9*10^(1/2)+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2)))/(55-17*10^(1/2)
)^(1/2)/(18*(x-2/3+1/3*10^(1/2))^2+9*(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2
))+55-17*10^(1/2))^(1/2)))
```

**Maxima** [B] Leaf count of result is larger than twice the leaf count of optimal. 668 vs. $2(122) = 244$.

time = 0.53, size = 668, normalized size = 3.84

$$-\frac{1}{60}\sqrt{10}\left(\frac{588\sqrt{10}x}{17\sqrt{10}\sqrt{2x^2+3x+1}+55\sqrt{2x^2+3x+1}}-\frac{588\sqrt{10}x}{17\sqrt{10}\sqrt{2x^2+3x+1}-55\sqrt{2x^2+3x+1}}+\frac{2112x}{17\sqrt{10}\sqrt{2x^2+3x+1}+55\sqrt{2x^2+3x+1}}+\frac{2112x}{17\sqrt{10}\sqrt{2x^2+3x+1}-55\sqrt{2x^2+3x+1}}-\frac{27\sqrt{10}\log\left(\frac{2}{3}\sqrt{10}+\frac{2\sqrt{2x^2+3x+1}\sqrt{17\sqrt{10}+55}}{3|6x-2\sqrt{10}-4|}+\frac{34\sqrt{10}}{9|6x-2\sqrt{10}-4|}+\frac{110}{9|6x-2\sqrt{10}-4|}+\frac{17}{18}\right)}{\left(17\sqrt{10}+55\right)^{\frac{3}{2}}}-\frac{\sqrt{10}\log\left(-\frac{2}{3}\sqrt{10}+\frac{2\sqrt{2x^2+3x+1}\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}{|6x+2\sqrt{10}-4|}-\frac{34\sqrt{10}}{9|6x+2\sqrt{10}-4|}+\frac{110}{9|6x+2\sqrt{10}-4|}+\frac{17}{18}\right)}{\left(-\frac{17}{9}\sqrt{10}+\frac{55}{9}\right)^{\frac{3}{2}}}+\frac{450\sqrt{10}}{17\sqrt{10}\sqrt{2x^2+3x+1}+55\sqrt{2x^2+3x+1}}-\frac{450\sqrt{10}}{17\sqrt{10}\sqrt{2x^2+3x+1}-55\sqrt{2x^2+3x+1}}-\frac{216\log\left(\frac{2}{3}\sqrt{10}+\frac{2\sqrt{2x^2+3x+1}\sqrt{17\sqrt{10}+55}}{3|6x-2\sqrt{10}-4|}+\frac{34\sqrt{10}}{9|6x-2\sqrt{10}-4|}+\frac{110}{9|6x-2\sqrt{10}-4|}+\frac{17}{18}\right)}{\left(17\sqrt{10}+55\right)^{\frac{3}{2}}}+\frac{8\log\left(-\frac{2}{3}\sqrt{10}+\frac{2\sqrt{2x^2+3x+1}\sqrt{-\frac{17}{9}\sqrt{10}+\frac{55}{9}}}{|6x+2\sqrt{10}-4|}-\frac{34\sqrt{10}}{9|6x+2\sqrt{10}-4|}+\frac{110}{9|6x+2\sqrt{10}-4|}+\frac{17}{18}\right)}{\left(-\frac{17}{9}\sqrt{10}+\frac{55}{9}\right)^{\frac{3}{2}}}+\frac{1656}{17\sqrt{10}\sqrt{2x^2+3x+1}+55\sqrt{2x^2+3x+1}}+\frac{1656}{17\sqrt{10}\sqrt{2x^2+3x+1}-55\sqrt{2x^2+3x+1}}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(3/2),x, algorithm="maxima")
```

```
[Out] -1/60*sqrt(10)*(588*sqrt(10)*x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqrt
(2*x^2 + 3*x + 1)) - 588*sqrt(10)*x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55
*sqrt(2*x^2 + 3*x + 1)) + 2112*x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sq
rt(2*x^2 + 3*x + 1)) + 2112*x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(
2*x^2 + 3*x + 1)) - 27*sqrt(10)*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1
)*sqrt(17*sqrt(10) + 55)/abs(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x
- 2*sqrt(10) - 4) + 110/9/abs(6*x - 2*sqrt(10) - 4) + 17/18)/(17*sqrt(10) +
 55)^(3/2) - sqrt(10)*log(-2/9*sqrt(10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-17/
9*sqrt(10) + 55/9)/abs(6*x + 2*sqrt(10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*sq
rt(10) - 4) + 110/9/abs(6*x + 2*sqrt(10) - 4) + 17/18)/(-17/9*sqrt(10) + 55
/9)^(3/2) + 450*sqrt(10)/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqrt(2*x^2
 + 3*x + 1)) - 450*sqrt(10)/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(2*
x^2 + 3*x + 1)) - 216*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1)*sqrt(17*
sqrt(10) + 55)/abs(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x - 2*sqrt(1
0) - 4) + 110/9/abs(6*x - 2*sqrt(10) - 4) + 17/18)/(17*sqrt(10) + 55)^(3/2)
 + 8*log(-2/9*sqrt(10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-17/9*sqrt(10) + 55/9
)/abs(6*x + 2*sqrt(10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*sqrt(10) - 4) + 110
/9/abs(6*x + 2*sqrt(10) - 4) + 17/18)/(-17/9*sqrt(10) + 55/9)^(3/2) + 1656/
(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqrt(2*x^2 + 3*x + 1)) + 1656/(17*s
qrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(2*x^2 + 3*x + 1)))
```

**<u>Fricas</u>** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 365 vs. $2(122) = 244$.
time = 0.36, size = 365, normalized size = 2.10

$$\frac{\sqrt{5}\,(2x^2+3x+1)\sqrt{1959\sqrt{10}+6195}\,\log\left(-\frac{45\sqrt{10}\,x+\left(41\sqrt{10}\,\sqrt{5}\,x-130\sqrt{5}\,x\right)\sqrt{1959\sqrt{10}+6195}+90x-90\sqrt{2x^2+3x+1}+90}{x}\right)-\sqrt{5}\,(2x^2+3x+1)\sqrt{1959\sqrt{10}+6195}\,\log\left(-\frac{45\sqrt{10}\,x-\left(41\sqrt{10}\,\sqrt{5}\,x-130\sqrt{5}\,x\right)\sqrt{1959\sqrt{10}+6195}+90x-90\sqrt{2x^2+3x+1}+90}{x}\right)+\sqrt{5}\,(2x^2+3x+1)\sqrt{-1959\sqrt{10}+6195}\,\log\left(\frac{45\sqrt{10}\,x+\left(41\sqrt{10}\,\sqrt{5}\,x+130\sqrt{5}\,x\right)\sqrt{-1959\sqrt{10}+6195}-90x+90\sqrt{2x^2+3x+1}-90}{x}\right)-\sqrt{5}\,(2x^2+3x+1)\sqrt{-1959\sqrt{10}+6195}\,\log\left(\frac{45\sqrt{10}\,x-\left(41\sqrt{10}\,\sqrt{5}\,x+130\sqrt{5}\,x\right)\sqrt{-1959\sqrt{10}+6195}-90x+90\sqrt{2x^2+3x+1}-90}{x}\right)+840x^2+20\sqrt{2x^2+3x+1}\,(22x+21)+1260x+420}{50\,(2x^2+3x+1)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(3/2),x, algorithm="fricas")
```

```
[Out] 1/50*(sqrt(5)*(2*x^2 + 3*x + 1)*sqrt(1959*sqrt(10) + 6195)*log(-(45*sqrt(10
)*x + (41*sqrt(10)*sqrt(5)*x - 130*sqrt(5)*x)*sqrt(1959*sqrt(10) + 6195) +
90*x - 90*sqrt(2*x^2 + 3*x + 1) + 90)/x) - sqrt(5)*(2*x^2 + 3*x + 1)*sqrt(1
959*sqrt(10) + 6195)*log(-(45*sqrt(10)*x - (41*sqrt(10)*sqrt(5)*x - 130*sqr
t(5)*x)*sqrt(1959*sqrt(10) + 6195) + 90*x - 90*sqrt(2*x^2 + 3*x + 1) + 90)/
x) + sqrt(5)*(2*x^2 + 3*x + 1)*sqrt(-1959*sqrt(10) + 6195)*log((45*sqrt(10)
*x + (41*sqrt(10)*sqrt(5)*x + 130*sqrt(5)*x)*sqrt(-1959*sqrt(10) + 6195) -
90*x + 90*sqrt(2*x^2 + 3*x + 1) - 90)/x) - sqrt(5)*(2*x^2 + 3*x + 1)*sqrt(-
1959*sqrt(10) + 6195)*log((45*sqrt(10)*x - (41*sqrt(10)*sqrt(5)*x + 130*sqr
t(5)*x)*sqrt(-1959*sqrt(10) + 6195) - 90*x + 90*sqrt(2*x^2 + 3*x + 1) - 90)
/x) + 840*x^2 + 20*sqrt(2*x^2 + 3*x + 1)*(22*x + 21) + 1260*x + 420)/(2*x^2
 + 3*x + 1)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x}{6x^4\sqrt{2x^2+3x+1}+x^3\sqrt{2x^2+3x+1}-13x^2\sqrt{2x^2+3x+1}-10x\sqrt{2x^2+3x+1}-2\sqrt{2x^2+3x+1}}\,dx-\int \frac{2}{6x^4\sqrt{2x^2+3x+1}+x^3\sqrt{2x^2+3x+1}-13x^2\sqrt{2x^2+3x+1}-10x\sqrt{2x^2+3x+1}-2\sqrt{2x^2+3x+1}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(2*x**2+3*x+1)**(3/2),x)
```

```
[Out] -Integral(x/(6*x**4*sqrt(2*x**2 + 3*x + 1) + x**3*sqrt(2*x**2 + 3*x + 1) -
13*x**2*sqrt(2*x**2 + 3*x + 1) - 10*x*sqrt(2*x**2 + 3*x + 1) - 2*sqrt(2*x**
2 + 3*x + 1)), x) - Integral(2/(6*x**4*sqrt(2*x**2 + 3*x + 1) + x**3*sqrt(2
*x**2 + 3*x + 1) - 13*x**2*sqrt(2*x**2 + 3*x + 1) - 10*x*sqrt(2*x**2 + 3*x
+ 1) - 2*sqrt(2*x**2 + 3*x + 1)), x)
```

**Giac [A]**
time = 4.42, size = 112, normalized size = 0.64

$$\frac{2\,(22\,x+21)}{5\,\sqrt{2\,x^2+3\,x+1}}+0.0140045514133333\,\log\left(-\sqrt{2}\,x+\sqrt{2\,x^2+3\,x+1}+5.90976932712000\right)-4.97793168620000\,\log\left(-\sqrt{2}\,x+\sqrt{2\,x^2+3\,x+1}-0.176527156327000\right)+4.97793168620000\,\log\left(-\sqrt{2}\,x+\sqrt{2\,x^2+3\,x+1}-0.919278730509000\right)-0.0140045514125333\,\log\left(-\sqrt{2}\,x+\sqrt{2\,x^2+3\,x+1}-1.04272727395000\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(3/2),x, algorithm="giac")
```

```
[Out] 2/5*(22*x + 21)/sqrt(2*x^2 + 3*x + 1) + 0.0140045514133333*log(-sqrt(2)*x +
 sqrt(2*x^2 + 3*x + 1) + 5.90976932712000) - 4.97793168620000*log(-sqrt(2)*
x + sqrt(2*x^2 + 3*x + 1) - 0.176527156327000) + 4.97793168620000*log(-sqrt
(2)*x + sqrt(2*x^2 + 3*x + 1) - 0.919278730509000) - 0.0140045514125333*log
(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 1.04272727395000)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{{\left(2\,x^2+3\,x+1\right)}^{3/2}\,\left(-3\,x^2+4\,x+2\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x + 2*x^2 + 1)^(3/2)*(4*x - 3*x^2 + 2)),x)
```

```
[Out] int((x + 2)/((3*x + 2*x^2 + 1)^(3/2)*(4*x - 3*x^2 + 2)), x)
```

# 3.30 $\int \frac{2+x}{(2+4x-3x^2)(1+3x+2x^2)^{5/2}}\,dx$

**Optimal**. Leaf size=197

$$\frac{2(21+22x)}{15(1+3x+2x^2)^{3/2}}+\frac{2(273+230x)}{15\sqrt{1+3x+2x^2}}-\frac{1}{50}\sqrt{\frac{1}{3}\left(4885115+1544809\sqrt{10}\right)}\tanh^{-1}\left(\frac{3\left(4-\sqrt{10}\right)+(}{2\sqrt{55-17\sqrt{10}}}\right.$$

```
[Out] 2/15*(21+22*x)/(2*x^2+3*x+1)^(3/2)+2/15*(273+230*x)/(2*x^2+3*x+1)^(1/2)+1/1
50*arctanh(1/2*(12+3*10^(1/2)+x*(17+4*10^(1/2)))/(2*x^2+3*x+1)^(1/2)/(55+17
*10^(1/2))^(1/2))*(14655345-4634427*10^(1/2))^(1/2)-1/150*arctanh(1/2*(x*(1
7-4*10^(1/2))+12-3*10^(1/2))/(2*x^2+3*x+1)^(1/2)/(55-17*10^(1/2))^(1/2))*(1
4655345+4634427*10^(1/2))^(1/2)
```

**Rubi** [A]

time = 0.18, antiderivative size = 197, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {1030, 1074, 1046, 738, 212}

$$\frac{2(22x+21)}{15(2x^2+3x+1)^{3/2}}+\frac{2(230x+273)}{15\sqrt{2x^2+3x+1}}-\frac{1}{50}\sqrt{\frac{1}{3}\left(4885115+1544809\sqrt{10}\right)}\tanh^{-1}\left(\frac{\left(17-4\sqrt{10}\right)x+3\left(4-\sqrt{10}\right)}{2\sqrt{55-17\sqrt{10}}\sqrt{2x^2+3x+1}}\right)+\frac{1}{50}\sqrt{\frac{1}{3}\left(4885115-1544809\sqrt{10}\right)}\tanh^{-1}\left(\frac{\left(17+4\sqrt{10}\right)x+3\left(4+\sqrt{10}\right)}{2\sqrt{55+17\sqrt{10}}\sqrt{2x^2+3x+1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x + 2*x^2)^(5/2)),x]
```

```
[Out] (2*(21 + 22*x))/(15*(1 + 3*x + 2*x^2)^(3/2)) + (2*(273 + 230*x))/(15*Sqrt[1
 + 3*x + 2*x^2]) - (Sqrt[(4885115 + 1544809*Sqrt[10])/3]*ArcTanh[(3*(4 - Sq
rt[10]) + (17 - 4*Sqrt[10])*x)/(2*Sqrt[55 - 17*Sqrt[10]]*Sqrt[1 + 3*x + 2*x
^2])])/50 + (Sqrt[(4885115 - 1544809*Sqrt[10])/3]*ArcTanh[(3*(4 + Sqrt[10])
 + (17 + 4*Sqrt[10])*x)/(2*Sqrt[55 + 17*Sqrt[10]]*Sqrt[1 + 3*x + 2*x^2])])/
50
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1074

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x +
c*x^2)^(p + 1)*((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (
b*d - a*e)*(c*e - b*f))*(p + 1)))*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) +
(A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) + c*(A*(2*c^2*d + b^2*f - c
*(b*e + 2*a*f)) - B*(b*c*d - 2*a*c*e + a*b*f) + C*(b^2*d - a*b*e - 2*a*(c*d
 - a*f)))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
- b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*x^2)^q*Simp[(b
*B - 2*A*c - 2*a*C)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1) + (b^
2*(C*d + A*f) - b*(B*c*d + A*c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*
(c*C*d - B*c*e - a*C*f)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((A*c - a*C)*(2*a
*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p
 + q + 2) - (2*f*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b - a*B)*(2*c^
2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (b^2*(C*d + A*f) - b*(B*c*d +
 A*c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*
```

```
(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(C*d + A*f) - b*(B*c*d + A*
c*e + a*C*e + a*B*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - B*c*e - a*C*f)))*(2*
p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, e, f, A, B, C, q}, x] &&
 NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f
)^2 - (b*d - a*e)*(c*e - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]) &&  !
IGtQ[q, 0]
```

Rubi steps

$$\int \frac{2+x}{\left(2+4x-3x^2\right)\left(1+3x+2x^2\right)^{5/2}}\,dx = \frac{2(21+22x)}{15\left(1+3x+2x^2\right)^{3/2}} - \frac{2}{45}\int \frac{-480-\frac{813x}{2}+396x^2}{\left(2+4x-3x^2\right)\left(1+3x+2x^2\right)^{3/2}}\,dx$$

$$= \frac{2(21+22x)}{15\left(1+3x+2x^2\right)^{3/2}} + \frac{2(273+230x)}{15\sqrt{1+3x+2x^2}} + \frac{4}{675}\int \frac{\frac{2335}{2}\cdots}{(2+4x-3x\cdots}$$

$$= \frac{2(21+22x)}{15\left(1+3x+2x^2\right)^{3/2}} + \frac{2(273+230x)}{15\sqrt{1+3x+2x^2}} - \frac{1}{25}\Big(3\Big(335-106\sqrt{1\cdots}$$

$$= \frac{2(21+22x)}{15\left(1+3x+2x^2\right)^{3/2}} + \frac{2(273+230x)}{15\sqrt{1+3x+2x^2}} + \frac{1}{25}\Big(6\Big(335-106\sqrt{1\cdots}$$

$$= \frac{2(21+22x)}{15\left(1+3x+2x^2\right)^{3/2}} + \frac{2(273+230x)}{15\sqrt{1+3x+2x^2}} - \frac{1}{50}\sqrt{\frac{1}{3}\Big(4885115+\cdots}$$

**Mathematica** [**A**]
time = 0.89, size = 154, normalized size = 0.78

$$\frac{2\sqrt{1+3x+2x^2}\,(294+1071x+1236x^2+460x^3)}{15(1+x)^2(1+2x)^2} - \frac{1}{75}\sqrt{14655345+4634427\sqrt{10}}\,\tanh^{-1}\left(\frac{\sqrt{1-\sqrt{\frac{2}{5}}}\,\sqrt{1+3x+2x^2}}{1+2x}\right) + \frac{81\tanh^{-1}\left(\frac{\sqrt{1+\sqrt{\frac{2}{5}}}\,\sqrt{1+3x+2x^2}}{1+2x}\right)}{5\sqrt{24425575+7724045\sqrt{10}}}$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + x)/((2 + 4*x - 3*x^2)*(1 + 3*x + 2*x^2)^(5/2)),x]
```

```
[Out] (2*Sqrt[1 + 3*x + 2*x^2]*(294 + 1071*x + 1236*x^2 + 460*x^3))/(15*(1 + x)^2
*(1 + 2*x)^2) - (Sqrt[14655345 + 4634427*Sqrt[10]]*ArcTanh[(Sqrt[1 - Sqrt[2
/5]]*Sqrt[1 + 3*x + 2*x^2])/(1 + 2*x)])/75 + (81*ArcTanh[(Sqrt[1 + Sqrt[2/5
]]*Sqrt[1 + 3*x + 2*x^2])/(1 + 2*x)])/(5*Sqrt[24425575 + 7724045*Sqrt[10]])
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 877 vs. $2(141) = 282$.
time = 0.60, size = 878, normalized size = 4.46

| method | result |
|---|---|
| trager | $\frac{\frac{184}{3}x^3+\frac{824}{5}x^2+\frac{714}{5}x+\frac{196}{5}}{(2x^2+3x+1)^{\frac{3}{2}}}+\frac{2\,\mathrm{RootOf}\left(96000\_Z^4-781618400\_Z^2+6561\right)\ln\left(-\frac{22695840000\,\mathrm{RootOf}\left(96000\_Z^4-781618400\_Z^2+\right.}{}\right.}{}$ |
| default | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+2)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(5/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/20*(8+10^(1/2))*10^(1/2)*(1/9/(55/9+17/9*10^(1/2))/(2*(x-2/3-1/3*10^(1/2
))^2+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))+55/9+17/9*10^(1/2))^(3/2)-1/6
*(17/3+4/3*10^(1/2))/(55/9+17/9*10^(1/2))*(2/3*(3+4*x)/(440/9+136/9*10^(1/2
)-(17/3+4/3*10^(1/2))^2)/(2*(x-2/3-1/3*10^(1/2))^2+(17/3+4/3*10^(1/2))*(x-2
/3-1/3*10^(1/2))+55/9+17/9*10^(1/2))^(3/2)+32/3/(440/9+136/9*10^(1/2)-(17/3
+4/3*10^(1/2))^2)^2*(3+4*x)/(2*(x-2/3-1/3*10^(1/2))^2+(17/3+4/3*10^(1/2))*(
x-2/3-1/3*10^(1/2))+55/9+17/9*10^(1/2))^(1/2))+1/3/(55/9+17/9*10^(1/2))*(1/
(55/9+17/9*10^(1/2))/(2*(x-2/3-1/3*10^(1/2))^2+(17/3+4/3*10^(1/2))*(x-2/3-1
/3*10^(1/2))+55/9+17/9*10^(1/2))^(1/2)-(17/3+4/3*10^(1/2))/(55/9+17/9*10^(1
/2))*(3+4*x)/(440/9+136/9*10^(1/2)-(17/3+4/3*10^(1/2))^2)/(2*(x-2/3-1/3*10^
(1/2))^2+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))+55/9+17/9*10^(1/2))^(1/2)
-3/(55/9+17/9*10^(1/2))/(55+17*10^(1/2))^(1/2)*arctanh(9/2*(110/9+34/9*10^(
1/2)+(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2)))/(55+17*10^(1/2))^(1/2)/(18*(
x-2/3-1/3*10^(1/2))^2+9*(17/3+4/3*10^(1/2))*(x-2/3-1/3*10^(1/2))+55+17*10^(
1/2))^(1/2)))-1/20*(-8+10^(1/2))*10^(1/2)*(1/9/(55/9-17/9*10^(1/2))/(2*(x-
2/3+1/3*10^(1/2))^2+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*10^(
1/2))^(3/2)-1/6*(17/3-4/3*10^(1/2))/(55/9-17/9*10^(1/2))*(2/3*(3+4*x)/(440/
9-136/9*10^(1/2)-(17/3-4/3*10^(1/2))^2)/(2*(x-2/3+1/3*10^(1/2))^2+(17/3-4/3
*10^(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*10^(1/2))^(3/2)+32/3/(440/9-136/9
*10^(1/2)-(17/3-4/3*10^(1/2))^2)^2*(3+4*x)/(2*(x-2/3+1/3*10^(1/2))^2+(17/3-
4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*10^(1/2))^(1/2))+1/3/(55/9-17/
9*10^(1/2))*(1/(55/9-17/9*10^(1/2))/(2*(x-2/3+1/3*10^(1/2))^2+(17/3-4/3*10^
(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*10^(1/2))^(1/2)-(17/3-4/3*10^(1/2))/(
55/9-17/9*10^(1/2))*(3+4*x)/(440/9-136/9*10^(1/2)-(17/3-4/3*10^(1/2))^2)/(2
*(x-2/3+1/3*10^(1/2))^2+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2))+55/9-17/9*
10^(1/2))^(1/2)-3/(55/9-17/9*10^(1/2))/(55-17*10^(1/2))^(1/2)*arctanh(9/2*(
110/9-34/9*10^(1/2)+(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1/2)))/(55-17*10^(1/
2))^(1/2)/(18*(x-2/3+1/3*10^(1/2))^2+9*(17/3-4/3*10^(1/2))*(x-2/3+1/3*10^(1
/2))+55-17*10^(1/2))^(1/2))))
```

**Maxima** [B] Leaf count of result is larger than twice the leaf count of optimal. 1276 vs. $2(141) = 282$.

time = 0.54, size = 1276, normalized size = 6.48

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(5/2),x, algorithm="maxima")
```

```
[Out] -1/300*sqrt(10)*(980*sqrt(10)*x/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) + 55*(
2*x^2 + 3*x + 1)^(3/2)) - 980*sqrt(10)*x/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/
2) - 55*(2*x^2 + 3*x + 1)^(3/2)) + 5292*sqrt(10)*x/(374*sqrt(10)*sqrt(2*x^2
 + 3*x + 1) + 1183*sqrt(2*x^2 + 3*x + 1)) - 5292*sqrt(10)*x/(374*sqrt(10)*s
qrt(2*x^2 + 3*x + 1) - 1183*sqrt(2*x^2 + 3*x + 1)) - 15680*sqrt(10)*x/(17*s
qrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqrt(2*x^2 + 3*x + 1)) + 15680*sqrt(10)*
x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(2*x^2 + 3*x + 1)) + 3520*x/(
17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) + 55*(2*x^2 + 3*x + 1)^(3/2)) + 3520*x/
(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) - 55*(2*x^2 + 3*x + 1)^(3/2)) + 19008*
x/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 1183*sqrt(2*x^2 + 3*x + 1)) + 19008
*x/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 1183*sqrt(2*x^2 + 3*x + 1)) - 5632
0*x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqrt(2*x^2 + 3*x + 1)) - 56320*
x/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(2*x^2 + 3*x + 1)) + 750*sqrt
(10)/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) + 55*(2*x^2 + 3*x + 1)^(3/2)) - 7
50*sqrt(10)/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) - 55*(2*x^2 + 3*x + 1)^(3/
2)) + 4050*sqrt(10)/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 1183*sqrt(2*x^2 +
 3*x + 1)) - 4050*sqrt(10)/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 1183*sqrt(
2*x^2 + 3*x + 1)) - 11760*sqrt(10)/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*
sqrt(2*x^2 + 3*x + 1)) + 11760*sqrt(10)/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1)
 - 55*sqrt(2*x^2 + 3*x + 1)) + 2760/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) + 5
5*(2*x^2 + 3*x + 1)^(3/2)) + 2760/(17*sqrt(10)*(2*x^2 + 3*x + 1)^(3/2) - 55
*(2*x^2 + 3*x + 1)^(3/2)) + 14904/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 118
3*sqrt(2*x^2 + 3*x + 1)) + 14904/(374*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 1183
*sqrt(2*x^2 + 3*x + 1)) - 42240/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) + 55*sqr
t(2*x^2 + 3*x + 1)) - 42240/(17*sqrt(10)*sqrt(2*x^2 + 3*x + 1) - 55*sqrt(2*
x^2 + 3*x + 1)) - 1215*sqrt(10)*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1
)*sqrt(17*sqrt(10) + 55)/abs(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x
 - 2*sqrt(10) - 4) + 110/9/abs(6*x - 2*sqrt(10) - 4) + 17/18)/(17*sqrt(10) +
 55)^(5/2) - 5*sqrt(10)*log(-2/9*sqrt(10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-1
7/9*sqrt(10) + 55/9)/abs(6*x + 2*sqrt(10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*
sqrt(10) - 4) + 110/9/abs(6*x + 2*sqrt(10) - 4) + 17/18)/(-17/9*sqrt(10) +
55/9)^(5/2) - 9720*log(2/9*sqrt(10) + 2/3*sqrt(2*x^2 + 3*x + 1)*sqrt(17*sqr
t(10) + 55)/abs(6*x - 2*sqrt(10) - 4) + 34/9*sqrt(10)/abs(6*x - 2*sqrt(10)
 - 4) + 110/9/abs(6*x - 2*sqrt(10) - 4) + 17/18)/(17*sqrt(10) + 55)^(5/2) +
40*log(-2/9*sqrt(10) + 2*sqrt(2*x^2 + 3*x + 1)*sqrt(-17/9*sqrt(10) + 55/9)/
abs(6*x + 2*sqrt(10) - 4) - 34/9*sqrt(10)/abs(6*x + 2*sqrt(10) - 4) + 110/9
/abs(6*x + 2*sqrt(10) - 4) + 17/18)/(-17/9*sqrt(10) + 55/9)^(5/2))
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 435 vs. $2(141) = 282$.
time $= 0.42$, size $= 435$, normalized size $= 2.21$

$$\frac{23520 x^{4}+70560 x^{3}+\sqrt{3}\left(4 x^{4}+12 x^{3}+13 x^{2}+6 x+1\right) \sqrt{1544809 \sqrt{10}+4885115} \log \left(-\frac{243 \sqrt{10} x+\left(893 \sqrt{10} \sqrt{3} x-2824 \sqrt{3} x\right) \sqrt{1544809 \sqrt{10}+4885115}+486 x-486 \sqrt{2 x^{2}+3 x+1}+486}{x}\right)-\sqrt{3}\left(4 x^{4}+12 x^{3}+13 x^{2}+6 x+1\right) \sqrt{1544809 \sqrt{10}+4885115} \log \left(-\frac{243 \sqrt{10} x-\left(893 \sqrt{10} \sqrt{3} x-2824 \sqrt{3} x\right) \sqrt{1544809 \sqrt{10}+4885115}+486 x-486 \sqrt{2 x^{2}+3 x+1}+486}{x}\right)+\sqrt{3}\left(4 x^{4}+12 x^{3}+13 x^{2}+6 x+1\right) \sqrt{-1544809 \sqrt{10}+4885115} \log \left(\frac{243 \sqrt{10} x+\left(893 \sqrt{10} \sqrt{3} x+2824 \sqrt{3} x\right) \sqrt{-1544809 \sqrt{10}+4885115}-486 x+486 \sqrt{2 x^{2}+3 x+1}-486}{x}\right)-\sqrt{3}\left(4 x^{4}+12 x^{3}+13 x^{2}+6 x+1\right) \sqrt{-1544809 \sqrt{10}+4885115} \log \left(\frac{243 \sqrt{10} x-\left(893 \sqrt{10} \sqrt{3} x+2824 \sqrt{3} x\right) \sqrt{-1544809 \sqrt{10}+4885115}-486 x+486 \sqrt{2 x^{2}+3 x+1}-486}{x}\right)+76440 x^{2}+20\left(460 x^{3}+1236 x^{2}+1071 x+294\right) \sqrt{2 x^{2}+3 x+1}+35280 x+5880}{150\left(4 x^{4}+12 x^{3}+13 x^{2}+6 x+1\right)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(5/2),x, algorithm="fricas")
```

```
[Out] 1/150*(23520*x^4 + 70560*x^3 + sqrt(3)*(4*x^4 + 12*x^3 + 13*x^2 + 6*x + 1)*
sqrt(1544809*sqrt(10) + 4885115)*log(-(243*sqrt(10)*x + (893*sqrt(10)*sqrt(
3)*x - 2824*sqrt(3)*x)*sqrt(1544809*sqrt(10) + 4885115) + 486*x - 486*sqrt(
2*x^2 + 3*x + 1) + 486)/x) - sqrt(3)*(4*x^4 + 12*x^3 + 13*x^2 + 6*x + 1)*sq
rt(1544809*sqrt(10) + 4885115)*log(-(243*sqrt(10)*x - (893*sqrt(10)*sqrt(3)
*x - 2824*sqrt(3)*x)*sqrt(1544809*sqrt(10) + 4885115) + 486*x - 486*sqrt(2*
x^2 + 3*x + 1) + 486)/x) + sqrt(3)*(4*x^4 + 12*x^3 + 13*x^2 + 6*x + 1)*sqrt
(-1544809*sqrt(10) + 4885115)*log((243*sqrt(10)*x + (893*sqrt(10)*sqrt(3)*x
 + 2824*sqrt(3)*x)*sqrt(-1544809*sqrt(10) + 4885115) - 486*x + 486*sqrt(2*x
^2 + 3*x + 1) - 486)/x) - sqrt(3)*(4*x^4 + 12*x^3 + 13*x^2 + 6*x + 1)*sqrt(
-1544809*sqrt(10) + 4885115)*log((243*sqrt(10)*x - (893*sqrt(10)*sqrt(3)*x
+ 2824*sqrt(3)*x)*sqrt(-1544809*sqrt(10) + 4885115) - 486*x + 486*sqrt(2*x^
2 + 3*x + 1) - 486)/x) + 76440*x^2 + 20*(460*x^3 + 1236*x^2 + 1071*x + 294)
*sqrt(2*x^2 + 3*x + 1) + 35280*x + 5880)/(4*x^4 + 12*x^3 + 13*x^2 + 6*x + 1
)
```

**Sympy** [F]
time $= 0.00$, size $= 0$, normalized size $= 0.00$

$$-\int \frac{x}{12 x^{6} \sqrt{2 x^{2}+3 x+1}+20 x^{5} \sqrt{2 x^{2}+3 x+1}-17 x^{4} \sqrt{2 x^{2}+3 x+1}-58 x^{3} \sqrt{2 x^{2}+3 x+1}-47 x^{2} \sqrt{2 x^{2}+3 x+1}-16 x \sqrt{2 x^{2}+3 x+1}-2 \sqrt{2 x^{2}+3 x+1}}\, dx-\int \frac{2}{12 x^{6} \sqrt{2 x^{2}+3 x+1}+20 x^{5} \sqrt{2 x^{2}+3 x+1}-17 x^{4} \sqrt{2 x^{2}+3 x+1}-58 x^{3} \sqrt{2 x^{2}+3 x+1}-47 x^{2} \sqrt{2 x^{2}+3 x+1}-16 x \sqrt{2 x^{2}+3 x+1}-2 \sqrt{2 x^{2}+3 x+1}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x**2+4*x+2)/(2*x**2+3*x+1)**(5/2),x)
```

```
[Out] -Integral(x/(12*x**6*sqrt(2*x**2 + 3*x + 1) + 20*x**5*sqrt(2*x**2 + 3*x + 1
) - 17*x**4*sqrt(2*x**2 + 3*x + 1) - 58*x**3*sqrt(2*x**2 + 3*x + 1) - 47*x*
*2*sqrt(2*x**2 + 3*x + 1) - 16*x*sqrt(2*x**2 + 3*x + 1) - 2*sqrt(2*x**2 + 3
*x + 1)), x) - Integral(2/(12*x**6*sqrt(2*x**2 + 3*x + 1) + 20*x**5*sqrt(2*
x**2 + 3*x + 1) - 17*x**4*sqrt(2*x**2 + 3*x + 1) - 58*x**3*sqrt(2*x**2 + 3*
x + 1) - 47*x**2*sqrt(2*x**2 + 3*x + 1) - 16*x*sqrt(2*x**2 + 3*x + 1) - 2*s
qrt(2*x**2 + 3*x + 1)), x)
```

**Giac** [A]
time $= 3.77$, size $= 121$, normalized size $= 0.61$

$$\frac{2\left(\left(4\left(115 x+309\right) x+1071\right) x+294\right)}{15\left(2 x^{2}+3 x+1\right)^{\frac{3}{2}}}+0.00115890443050800 \log \left(-\sqrt{2} x+\sqrt{2 x^{2}+3 x+1}+5.90976932712000\right)-36.0928986365333 \log \left(-\sqrt{2} x+\sqrt{2 x^{2}+3 x+1}-0.176527156327000\right)+36.0928986365333 \log \left(-\sqrt{2} x+\sqrt{2 x^{2}+3 x+1}-0.919278730509000\right)-0.00115890442528267 \log \left(-\sqrt{2} x+\sqrt{2 x^{2}+3 x+1}-1.04272727395000\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+x)/(-3*x^2+4*x+2)/(2*x^2+3*x+1)^(5/2),x, algorithm="giac")
```

```
[Out] 2/15*((4*(115*x + 309)*x + 1071)*x + 294)/(2*x^2 + 3*x + 1)^(3/2) + 0.00115
890443050800*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) + 5.90976932712000) - 3
6.0928986365333*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 0.176527156327000)
 + 36.0928986365333*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 0.919278730509
000) - 0.00115890442528267*log(-sqrt(2)*x + sqrt(2*x^2 + 3*x + 1) - 1.04272
727395000)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x+2}{(2\,x^2+3\,x+1)^{5/2}\,(-3\,x^2+4\,x+2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 2)/((3*x + 2*x^2 + 1)^(5/2)*(4*x - 3*x^2 + 2)),x)
```

```
[Out] int((x + 2)/((3*x + 2*x^2 + 1)^(5/2)*(4*x - 3*x^2 + 2)), x)
```

## 3.31 $\int \frac{1+x}{(4+2x+x^2)\sqrt{5+2x+x^2}} \, dx$

**Optimal**. Leaf size=15

$$-\tanh^{-1}\left(\sqrt{5+2x+x^2}\right)$$

```
[Out] -arctanh((x^2+2*x+5)^(1/2))
```

**Rubi** [A]
time = 0.01, antiderivative size = 15, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.077$, Rules used = {1038, 212}

$$-\tanh^{-1}\left(\sqrt{x^2+2x+5}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(1 + x)/((4 + 2*x + x^2)*Sqrt[5 + 2*x + x^2]),x]
```

```
[Out] -ArcTanh[Sqrt[5 + 2*x + x^2]]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 1038

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e
_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-2*g, Subst[Int[1/(b*d - a*e -
 b*x^2), x], x, Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0] &&
 EqQ[h*e - 2*g*f, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{1+x}{(4+2x+x^2)\sqrt{5+2x+x^2}} \, dx &= -\left(2\text{Subst}\left(\int \frac{1}{2-2x^2} \, dx, x, \sqrt{5+2x+x^2}\right)\right)\\ &= -\tanh^{-1}\left(\sqrt{5+2x+x^2}\right)\end{aligned}$$

**Mathematica** [A]
time = 0.15, size = 15, normalized size = 1.00

$$-\tanh^{-1}\left(\sqrt{5+2x+x^2}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(1 + x)/((4 + 2*x + x^2)*Sqrt[5 + 2*x + x^2]),x]
[Out] -ArcTanh[Sqrt[5 + 2*x + x^2]]
```

**Maple** [A]
time = 0.18, size = 14, normalized size = 0.93

| method | result | size |
|---|---|---|
| default | $-\operatorname{arctanh}\left(\sqrt{x^{2}+2x+5}\right)$ | 14 |
| trager | $\frac{\ln\left(\frac{-x^{2}+2\sqrt{x^{2}+2x+5}-2x-6}{x^{2}+2x+4}\right)}{2}$ | 37 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((1+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x,method=_RETURNVERBOSE)
[Out] -arctanh((x^2+2*x+5)^(1/2))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="maxima")
[Out] integrate((x + 1)/(sqrt(x^2 + 2*x + 5)*(x^2 + 2*x + 4)), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 49 vs. $2(13) = 26$.
time = 0.35, size = 49, normalized size = 3.27

$$\frac{1}{2}\log\left(x^{2}-\sqrt{x^{2}+2\,x+5}\,(x+2)+3\,x+6\right)-\frac{1}{2}\log\left(x^{2}-\sqrt{x^{2}+2\,x+5}\,x+x+4\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="fricas")
[Out] 1/2*log(x^2 - sqrt(x^2 + 2*x + 5)*(x + 2) + 3*x + 6) - 1/2*log(x^2 - sqrt(x
^2 + 2*x + 5)*x + x + 4)
```

**Sympy** [B] Leaf count of result is larger than twice the leaf count of optimal. 36 vs. $2(14) = 28$.
time = 2.39, size = 36, normalized size = 2.40

$$\frac{\log\left(-1+\frac{1}{\sqrt{x^{2}+2x+5}}\right)}{2}-\frac{\log\left(1+\frac{1}{\sqrt{x^{2}+2x+5}}\right)}{2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+x)/(x**2+2*x+4)/(x**2+2*x+5)**(1/2),x)
```

```
[Out] log(-1 + 1/sqrt(x**2 + 2*x + 5))/2 - log(1 + 1/sqrt(x**2 + 2*x + 5))/2
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 31 vs. $2(13) = 26$.
time = 3.77, size = 31, normalized size = 2.07

$$-\frac{1}{2}\log\left(\sqrt{x^2+2x+5}+1\right)+\frac{1}{2}\log\left(\sqrt{x^2+2x+5}-1\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="giac")
```

```
[Out] -1/2*log(sqrt(x^2 + 2*x + 5) + 1) + 1/2*log(sqrt(x^2 + 2*x + 5) - 1)
```

**Mupad** [B]
time = 3.76, size = 13, normalized size = 0.87

$$-\mathrm{atanh}\left(\sqrt{x^2+2x+5}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 1)/((2*x + x^2 + 4)*(2*x + x^2 + 5)^(1/2)),x)
```

```
[Out] -atanh((2*x + x^2 + 5)^(1/2))
```

## 3.32 $\int \frac{4+x}{(4+2x+x^2)\sqrt{5+2x+x^2}}\,dx$

**Optimal**. Leaf size=44

$$\sqrt{3}\tan^{-1}\left(\frac{1+x}{\sqrt{3}\sqrt{5+2x+x^2}}\right)-\tanh^{-1}\left(\sqrt{5+2x+x^2}\right)$$

```
[Out] -arctanh((x^2+2*x+5)^(1/2))+arctan(1/3*(1+x)*3^(1/2)/(x^2+2*x+5)^(1/2))*3^(
1/2)
```

**Rubi** [A]
time = 0.03, antiderivative size = 44, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.192$, Rules used = {1039, 996, 210, 1038, 212}

$$\sqrt{3}\,\text{ArcTan}\left(\frac{x+1}{\sqrt{3}\sqrt{x^2+2x+5}}\right)-\tanh^{-1}\left(\sqrt{x^2+2x+5}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(4 + x)/((4 + 2*x + x^2)*Sqrt[5 + 2*x + x^2]),x]
```

```
[Out] Sqrt[3]*ArcTan[(1 + x)/(Sqrt[3]*Sqrt[5 + 2*x + x^2])] - ArcTanh[Sqrt[5 + 2*
x + x^2]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 996

```
Int[1/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(
x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[1/(e*(b*e - 4*a*f) - (b*d - a*e)
*x^2), x], x, (e + 2*f*x)/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d,
e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0
]
```

Rule 1038

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e
_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-2*g, Subst[Int[1/(b*d - a*e -
 b*x^2), x], x, Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0] &&
 EqQ[h*e - 2*g*f, 0]
```

Rule 1039

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(h*e - 2*g*f)/(2*f), Int[1/(
(a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/(2*f), Int[(e + 2
*f*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c,
d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e
 - b*f, 0] && NeQ[h*e - 2*g*f, 0]
```

Rubi steps

$$\int \frac{4+x}{(4+2x+x^2)\sqrt{5+2x+x^2}}\,dx = \frac{1}{2}\int \frac{2+2x}{(4+2x+x^2)\sqrt{5+2x+x^2}}\,dx + 3\int \frac{1}{(4+2x+x^2)\sqrt{5+2x+}}$$

$$= -\left(2\text{Subst}\left(\int \frac{1}{2-2x^2}\,dx, x, \sqrt{5+2x+x^2}\right)\right) - 12\text{Subst}\left(\int \frac{1}{-24-}\right.$$

$$= \sqrt{3}\tan^{-1}\left(\frac{1+x}{\sqrt{3}\sqrt{5+2x+x^2}}\right) - \tanh^{-1}\left(\sqrt{5+2x+x^2}\right)$$

**Mathematica [A]**
time = 0.20, size = 55, normalized size = 1.25

$$-\sqrt{3}\tan^{-1}\left(\frac{4+2x+x^2-(1+x)\sqrt{5+2x+x^2}}{\sqrt{3}}\right) - \tanh^{-1}\left(\sqrt{5+2x+x^2}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(4 + x)/((4 + 2*x + x^2)*Sqrt[5 + 2*x + x^2]),x]
```

```
[Out] -(Sqrt[3]*ArcTan[(4 + 2*x + x^2 - (1 + x)*Sqrt[5 + 2*x + x^2])/Sqrt[3]]) -
ArcTanh[Sqrt[5 + 2*x + x^2]]
```

**Maple [A]**
time = 0.30, size = 40, normalized size = 0.91

| method | result |
|---|---|

| | |
|---|---|
| default | $-\operatorname{arctanh}\left(\sqrt{x^2+2x+5}\right)+\sqrt{3}\ \arctan\left(\frac{\sqrt{3}\ (2x+2)}{6\sqrt{x^2+2x+5}}\right)$ |
| trager | $\operatorname{RootOf}\left(\_Z^2+\_Z+1\right)\ln\left(\frac{40\operatorname{RootOf}\left(\_Z^2+\_Z+1\right)^2x+21\sqrt{x^2+2x+5}\ \operatorname{RootOf}\left(\_Z^2+\_Z+1\right)+51\operatorname{Root}}{\operatorname{RootOf}\left(\_Z^2+\_\right.}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x+4)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -arctanh((x^2+2*x+5)^(1/2))+3^(1/2)*arctan(1/6*3^(1/2)/(x^2+2*x+5)^(1/2)*(2
*x+2))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((4+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((x + 4)/(sqrt(x^2 + 2*x + 5)*(x^2 + 2*x + 4)), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 106 vs. $2(37) = 74$.
time = 0.36, size = 106, normalized size = 2.41

$$-\sqrt{3}\ \arctan\left(-\frac{1}{3}\sqrt{3}\ (x+2)+\frac{1}{3}\sqrt{3}\ \sqrt{x^2+2x+5}\right)+\sqrt{3}\ \arctan\left(-\frac{1}{3}\sqrt{3}\ x+\frac{1}{3}\sqrt{3}\ \sqrt{x^2+2x+5}\right)+\frac{1}{2}\log\left(x^2-\sqrt{x^2+2x+5}\ (x+2)+3x+6\right)-\frac{1}{2}\log\left(x^2-\sqrt{x^2+2x+5}\ x+x+4\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((4+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="fricas")
```

```
[Out] -sqrt(3)*arctan(-1/3*sqrt(3)*(x + 2) + 1/3*sqrt(3)*sqrt(x^2 + 2*x + 5)) + s
qrt(3)*arctan(-1/3*sqrt(3)*x + 1/3*sqrt(3)*sqrt(x^2 + 2*x + 5)) + 1/2*log(x
^2 - sqrt(x^2 + 2*x + 5)*(x + 2) + 3*x + 6) - 1/2*log(x^2 - sqrt(x^2 + 2*x
+ 5)*x + x + 4)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{x+4}{(x^2+2x+4)\sqrt{x^2+2x+5}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((4+x)/(x**2+2*x+4)/(x**2+2*x+5)**(1/2),x)
```

```
[Out] Integral((x + 4)/((x**2 + 2*x + 4)*sqrt(x**2 + 2*x + 5)), x)
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 108 vs. 2(37) = 74.
time = 3.79, size = 108, normalized size = 2.45

$$-\sqrt{3}\arctan\left(-\frac{1}{3}\sqrt{3}\left(x-\sqrt{x^2+2x+5}+2\right)\right)+\sqrt{3}\arctan\left(-\frac{1}{3}\sqrt{3}\left(x-\sqrt{x^2+2x+5}\right)\right)+\frac{1}{2}\log\left(\left(x-\sqrt{x^2+2x+5}\right)^2+4x-4\sqrt{x^2+2x+5}+7\right)-\frac{1}{2}\log\left(\left(x-\sqrt{x^2+2x+5}\right)^2+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((4+x)/(x^2+2*x+4)/(x^2+2*x+5)^(1/2),x, algorithm="giac")
```

```
[Out] -sqrt(3)*arctan(-1/3*sqrt(3)*(x - sqrt(x^2 + 2*x + 5) + 2)) + sqrt(3)*arcta
n(-1/3*sqrt(3)*(x - sqrt(x^2 + 2*x + 5))) + 1/2*log((x - sqrt(x^2 + 2*x + 5
))^2 + 4*x - 4*sqrt(x^2 + 2*x + 5) + 7) - 1/2*log((x - sqrt(x^2 + 2*x + 5))
^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.02

$$\int \frac{x+4}{\left(x^2+2\,x+4\right)\,\sqrt{x^2+2\,x+5}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + 4)/((2*x + x^2 + 4)*(2*x + x^2 + 5)^(1/2)),x)
```

```
[Out] int((x + 4)/((2*x + x^2 + 4)*(2*x + x^2 + 5)^(1/2)), x)
```

## 3.33 $\int \frac{1+2x}{(3+x+x^2)\sqrt{5+x+x^2}}\,dx$

**Optimal**. Leaf size=24

$$-\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{5+x+x^2}}{\sqrt{2}}\right)$$

```
[Out] -arctanh(1/2*(x^2+x+5)^(1/2)*2^(1/2))*2^(1/2)
```

**Rubi** [A]
time = 0.01, antiderivative size = 24, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.083$, Rules used = {1038, 212}

$$-\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{x^2+x+5}}{\sqrt{2}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(1 + 2*x)/((3 + x + x^2)*Sqrt[5 + x + x^2]),x]
```

```
[Out] -(Sqrt[2]*ArcTanh[Sqrt[5 + x + x^2]/Sqrt[2]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 1038

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e
_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-2*g, Subst[Int[1/(b*d - a*e -
 b*x^2), x], x, Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0] &&
 EqQ[h*e - 2*g*f, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{1+2x}{(3+x+x^2)\sqrt{5+x+x^2}}\,dx &= -\left(2\text{Subst}\left(\int \frac{1}{2-x^2}\,dx, x, \sqrt{5+x+x^2}\right)\right)\\ &= -\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{5+x+x^2}}{\sqrt{2}}\right)\end{aligned}$$

**Mathematica** [A]
time = 0.20, size = 24, normalized size = 1.00

$$-\sqrt{2}\tanh^{-1}\left(\frac{\sqrt{5+x+x^2}}{\sqrt{2}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(1 + 2*x)/((3 + x + x^2)*Sqrt[5 + x + x^2]),x]
[Out] -(Sqrt[2]*ArcTanh[Sqrt[5 + x + x^2]/Sqrt[2]])
```

**Maple** [A]
time = 0.32, size = 20, normalized size = 0.83

| method | result | size |
|---|---|---|
| default | $-\operatorname{arctanh}\left(\frac{\sqrt{x^2+x+5}\,\sqrt{2}}{2}\right)\sqrt{2}$ | 20 |
| trager | $\frac{\operatorname{RootOf}\left(\_Z^2-2\right)\ln\left(\frac{-\operatorname{RootOf}\left(\_Z^2-2\right)x^2-\operatorname{RootOf}\left(\_Z^2-2\right)x+4\sqrt{x^2+x+5}-7\operatorname{RootOf}\left(\_Z^2-2\right)}{x^2+x+3}\right)}{2}$ | 58 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x+1)/(x^2+x+3)/(x^2+x+5)^(1/2),x,method=_RETURNVERBOSE)
[Out] -arctanh(1/2*(x^2+x+5)^(1/2)*2^(1/2))*2^(1/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="maxima")
[Out] integrate((2*x + 1)/(sqrt(x^2 + x + 5)*(x^2 + x + 3)), x)
```

**Fricas** [A]
time = 0.34, size = 34, normalized size = 1.42

$$\frac{1}{2}\sqrt{2}\log\left(\frac{x^2-2\sqrt{2}\sqrt{x^2+x+5}+x+7}{x^2+x+3}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/2*sqrt(2)*log((x^2 - 2*sqrt(2)*sqrt(x^2 + x + 5) + x + 7)/(x^2 + x + 3))
```

**Sympy** [A]
time = 2.13, size = 68, normalized size = 2.83

$$2\left(\begin{cases} -\frac{\sqrt{2}\operatorname{acoth}\left(\frac{\sqrt{2}}{\sqrt{x^2+x+5}}\right)}{2} & \text{for } \frac{1}{x^2+x+5} > \frac{1}{2} \\ -\frac{\sqrt{2}\operatorname{atanh}\left(\frac{\sqrt{2}}{\sqrt{x^2+x+5}}\right)}{2} & \text{for } \frac{1}{x^2+x+5} < \frac{1}{2} \end{cases}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x**2+x+3)/(x**2+x+5)**(1/2),x)
```

```
[Out] 2*Piecewise((-sqrt(2)*acoth(sqrt(2)/sqrt(x**2 + x + 5))/2, 1/(x**2 + x + 5)
 > 1/2), (-sqrt(2)*atanh(sqrt(2)/sqrt(x**2 + x + 5))/2, 1/(x**2 + x + 5) <
1/2))
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 39 vs. 2(19) = 38.
time = 4.46, size = 39, normalized size = 1.62

$$-\frac{1}{2}\sqrt{2}\log\left(\sqrt{2}+\sqrt{x^2+x+5}\right)+\frac{1}{2}\sqrt{2}\log\left(-\sqrt{2}+\sqrt{x^2+x+5}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((1+2*x)/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="giac")
```

```
[Out] -1/2*sqrt(2)*log(sqrt(2) + sqrt(x^2 + x + 5)) + 1/2*sqrt(2)*log(-sqrt(2) +
sqrt(x^2 + x + 5))
```

**Mupad** [B]
time = 3.78, size = 19, normalized size = 0.79

$$-\sqrt{2}\operatorname{atanh}\left(\frac{\sqrt{2}\sqrt{x^2+x+5}}{2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x + 1)/((x + x^2 + 3)*(x + x^2 + 5)^(1/2)),x)
```

```
[Out] -2^(1/2)*atanh((2^(1/2)*(x + x^2 + 5)^(1/2))/2)
```

## 3.34 $\int \frac{x}{(3+x+x^2)\sqrt{5+x+x^2}}\,dx$

**Optimal**. Leaf size=56

$$-\frac{\tan^{-1}\left(\frac{\sqrt{\frac{2}{11}}(1+2x)}{\sqrt{5+x+x^2}}\right)}{\sqrt{22}}-\frac{\tanh^{-1}\left(\frac{\sqrt{5+x+x^2}}{\sqrt{2}}\right)}{\sqrt{2}}$$

```
[Out] -1/2*arctanh(1/2*(x^2+x+5)^(1/2)*2^(1/2))*2^(1/2)-1/22*arctan(1/11*(1+2*x)*
22^(1/2)/(x^2+x+5)^(1/2))*22^(1/2)
```

**Rubi** [A]

time = 0.03, antiderivative size = 56, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 20, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {1039, 996, 210, 1038, 212}

$$-\frac{\text{ArcTan}\left(\frac{\sqrt{\frac{2}{11}}(2x+1)}{\sqrt{x^2+x+5}}\right)}{\sqrt{22}}-\frac{\tanh^{-1}\left(\frac{\sqrt{x^2+x+5}}{\sqrt{2}}\right)}{\sqrt{2}}$$

Antiderivative was successfully verified.

```
[In] Int[x/((3 + x + x^2)*Sqrt[5 + x + x^2]),x]
```

```
[Out] -(ArcTan[(Sqrt[2/11]*(1 + 2*x))/Sqrt[5 + x + x^2]]/Sqrt[22]) - ArcTanh[Sqrt
[5 + x + x^2]/Sqrt[2]]/Sqrt[2]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 996

```
Int[1/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(
x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[1/(e*(b*e - 4*a*f) - (b*d - a*e)
```

```
*x^2), x], x, (e + 2*f*x)/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d,
e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0
]
```

Rule 1038

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e
_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-2*g, Subst[Int[1/(b*d - a*e -
 b*x^2), x], x, Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0] &&
 EqQ[h*e - 2*g*f, 0]
```

Rule 1039

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(h*e - 2*g*f)/(2*f), Int[1/(
(a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/(2*f), Int[(e + 2
*f*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c,
d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e
 - b*f, 0] && NeQ[h*e - 2*g*f, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{x}{(3+x+x^2)\sqrt{5+x+x^2}}\,dx &= -\left(\frac{1}{2}\int \frac{1}{(3+x+x^2)\sqrt{5+x+x^2}}\,dx\right)+\frac{1}{2}\int \frac{1+2x}{(3+x+x^2)\sqrt{5+x+}}\\ &= \text{Subst}\left(\int \frac{1}{-11-2x^2}\,dx,x,\frac{1+2x}{\sqrt{5+x+x^2}}\right)-\text{Subst}\left(\int \frac{1}{2-x^2}\,dx,x,\surd\right.\\ &= -\frac{\tan^{-1}\left(\frac{\sqrt{\frac{2}{11}}\,(1+2x)}{\sqrt{5+x+x^2}}\right)}{\sqrt{22}}-\frac{\tanh^{-1}\left(\frac{\sqrt{5+x+x^2}}{\sqrt{2}}\right)}{\sqrt{2}}\end{aligned}$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.13, size = 92, normalized size = 1.64

$$\text{RootSum}\left[23-2\#1+3\#1^2-2\#1^3+\#1^4\&,\frac{-5\log\left(-x+\sqrt{5+x+x^2}-\#1\right)+\log\left(-x+\sqrt{5+x+x^2}-\#1\right)\#1^2}{-1+3\#1-3\#1^2+2\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[x/((3 + x + x^2)*Sqrt[5 + x + x^2]),x]
```

```
[Out] RootSum[23 - 2*#1 + 3*#1^2 - 2*#1^3 + #1^4 & , (-5*Log[-x + Sqrt[5 + x + x^
2] - #1] + Log[-x + Sqrt[5 + x + x^2] - #1]*#1^2)/(-1 + 3*#1 - 3*#1^2 + 2*#
1^3) & ]
```

**Maple** [A]
time = 0.37, size = 45, normalized size = 0.80

| method | result |
|---|---|
| default | $-\frac{\operatorname{arctanh}\left(\frac{\sqrt{x^2+x+5}\,\sqrt{2}}{2}\right)\sqrt{2}}{2}-\frac{\arctan\left(\frac{(2x+1)\sqrt{22}}{11\sqrt{x^2+x+5}}\right)\sqrt{22}}{22}$ |
| trager | $\frac{22\ln\left(-\frac{-22990\,\mathrm{RootOf}\left(484\_Z^4-110\_Z^2+9\right)^5 x+2079\,\mathrm{RootOf}\left(484\_Z^4-110\_Z^2+9\right)^3 x+990\sqrt{x^2+x+5}\,\mathrm{RootOf}\left(484\_Z^4-110\right.}{22x\,\mathrm{RootOf}\left(\right.}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(x^2+x+3)/(x^2+x+5)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*arctanh(1/2*(x^2+x+5)^(1/2)*2^(1/2))*2^(1/2)-1/22*arctan(1/11*(2*x+1)*
22^(1/2)/(x^2+x+5)^(1/2))*22^(1/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(x/(sqrt(x^2 + x + 5)*(x^2 + x + 3)), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 307 vs. 2(44) = 88.
time = 0.35, size = 307, normalized size = 5.48

$$-\frac{1}{33}\sqrt{11}\sqrt{6}\sqrt{3}\arctan\left(\frac{2}{33}\sqrt{11}\sqrt{3}\sqrt{\sqrt{6}\sqrt{3}(2x+1)+6x^2-\sqrt{x^2+x+5}\left(2\sqrt{6}\sqrt{3}+6x+3\right)+6x+30}+\frac{1}{33}\sqrt{11}\left(2\sqrt{6}\sqrt{3}+6x+3\right)-\frac{2}{11}\sqrt{11}\sqrt{x^2+x+5}\right)+\frac{1}{33}\sqrt{11}\sqrt{6}\sqrt{3}\arctan\left(-\frac{1}{33}\sqrt{11}\left(2\sqrt{6}\sqrt{3}-6x-3\right)+\frac{1}{33}\sqrt{11}\sqrt{-12\sqrt{6}\sqrt{3}(2x+1)+72x^2+12\sqrt{x^2+x+5}\left(2\sqrt{6}\sqrt{3}-6x-3\right)+72x+360}-\frac{2}{11}\sqrt{11}\sqrt{x^2+x+5}\right)+\frac{1}{12}\sqrt{6}\sqrt{3}\log\left(12\sqrt{6}\sqrt{3}(2x+1)+72x^2-12\sqrt{x^2+x+5}\left(2\sqrt{6}\sqrt{3}+6x+3\right)+72x+360\right)-\frac{1}{12}\sqrt{6}\sqrt{3}\log\left(-12\sqrt{6}\sqrt{3}(2x+1)+72x^2+12\sqrt{x^2+x+5}\left(2\sqrt{6}\sqrt{3}-6x-3\right)+72x+360\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/33*sqrt(11)*sqrt(6)*sqrt(3)*arctan(2/33*sqrt(11)*sqrt(3)*sqrt(sqrt(6)*sq
rt(3)*(2*x + 1) + 6*x^2 - sqrt(x^2 + x + 5)*(2*sqrt(6)*sqrt(3) + 6*x + 3) +
 6*x + 30) + 1/33*sqrt(11)*(2*sqrt(6)*sqrt(3) + 6*x + 3) - 2/11*sqrt(11)*sq
```

```
rt(x^2 + x + 5)) + 1/33*sqrt(11)*sqrt(6)*sqrt(3)*arctan(-1/33*sqrt(11)*(2*s
qrt(6)*sqrt(3) - 6*x - 3) + 1/33*sqrt(11)*sqrt(-12*sqrt(6)*sqrt(3)*(2*x + 1
) + 72*x^2 + 12*sqrt(x^2 + x + 5)*(2*sqrt(6)*sqrt(3) - 6*x - 3) + 72*x + 36
0) - 2/11*sqrt(11)*sqrt(x^2 + x + 5)) + 1/12*sqrt(6)*sqrt(3)*log(12*sqrt(6)
*sqrt(3)*(2*x + 1) + 72*x^2 - 12*sqrt(x^2 + x + 5)*(2*sqrt(6)*sqrt(3) + 6*x
 + 3) + 72*x + 360) - 1/12*sqrt(6)*sqrt(3)*log(-12*sqrt(6)*sqrt(3)*(2*x + 1
) + 72*x^2 + 12*sqrt(x^2 + x + 5)*(2*sqrt(6)*sqrt(3) - 6*x - 3) + 72*x + 36
0)
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x}{(x^2 + x + 3)\sqrt{x^2 + x + 5}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(x**2+x+3)/(x**2+x+5)**(1/2),x)
```

```
[Out] Integral(x/((x**2 + x + 3)*sqrt(x**2 + x + 5)), x)
```

**Giac** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 133 vs. 2(44) = 88.
time = 4.05, size = 133, normalized size = 2.38

$$\frac{1}{22}\sqrt{11}\sqrt{2}\arctan\left(-\frac{1}{11}\sqrt{11}\left(2x + 2\sqrt{2} - 2\sqrt{x^2 + x + 5} + 1\right)\right) - \frac{1}{22}\sqrt{11}\sqrt{2}\arctan\left(-\frac{1}{11}\sqrt{11}\left(2x - 2\sqrt{2} - 2\sqrt{x^2 + x + 5} + 1\right)\right) + \frac{1}{4}\sqrt{2}\log\left(324\left(2x + 2\sqrt{2} - 2\sqrt{x^2 + x + 5} + 1\right)^2 + 3564\right) - \frac{1}{4}\sqrt{2}\log\left(324\left(2x - 2\sqrt{2} - 2\sqrt{x^2 + x + 5} + 1\right)^2 + 3564\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(x^2+x+3)/(x^2+x+5)^(1/2),x, algorithm="giac")
```

```
[Out] 1/22*sqrt(11)*sqrt(2)*arctan(-1/11*sqrt(11)*(2*x + 2*sqrt(2) - 2*sqrt(x^2 +
 x + 5) + 1)) - 1/22*sqrt(11)*sqrt(2)*arctan(-1/11*sqrt(11)*(2*x - 2*sqrt(2
) - 2*sqrt(x^2 + x + 5) + 1)) + 1/4*sqrt(2)*log(324*(2*x + 2*sqrt(2) - 2*sq
rt(x^2 + x + 5) + 1)^2 + 3564) - 1/4*sqrt(2)*log(324*(2*x - 2*sqrt(2) - 2*s
qrt(x^2 + x + 5) + 1)^2 + 3564)
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.02

$$\int \frac{x}{(x^2 + x + 3)\,\sqrt{x^2 + x + 5}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((x + x^2 + 3)*(x + x^2 + 5)^(1/2)),x)
```

```
[Out] int(x/((x + x^2 + 3)*(x + x^2 + 5)^(1/2)), x)
```

# 3.35 $\int \frac{A+Bx}{\sqrt{d+ex+fx^2}\,(ae+bex+bfx^2)^2}\,dx$

**Optimal**. Leaf size=249

$$-\frac{((Ab-2aB)e-b(Be-2Af)x)\sqrt{d+ex+fx^2}}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)}+\frac{(Be-2Af)\,(8aef-b(e^2+4df))\tanh^{-1}\left(\frac{\sqrt{bd}}{\sqrt{e}\,\sqrt{be-4}}\right.}{2e^{3/2}(bd-ae)^{3/2}f(be-4af)^{3/2}}$$

```
[Out] 1/2*(-2*A*f+B*e)*(8*a*e*f-b*(4*d*f+e^2))*arctanh((2*f*x+e)*(-a*e+b*d)^(1/2)
/e^(1/2)/(-4*a*f+b*e)^(1/2)/(f*x^2+e*x+d)^(1/2))/e^(3/2)/(-a*e+b*d)^(3/2)/f
/(-4*a*f+b*e)^(3/2)+1/2*B*arctanh(b^(1/2)*(f*x^2+e*x+d)^(1/2)/(-a*e+b*d)^(1
/2))/(-a*e+b*d)^(3/2)/f/b^(1/2)-((A*b-2*B*a)*e-b*(-2*A*f+B*e)*x)*(f*x^2+e*x
+d)^(1/2)/e/(-a*e+b*d)/(-4*a*f+b*e)/(b*f*x^2+b*e*x+a*e)
```

**Rubi** [A]
time = 0.54, antiderivative size = 249, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 36, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {1030, 1039, 996, 214, 1038}

$$\frac{(Be-2Af)\,(8aef-b(4df+e^2))\tanh^{-1}\left(\frac{(e+2fx)\sqrt{bd-ae}}{\sqrt{e}\,\sqrt{be-4af}\,\sqrt{d+ex+fx^2}}\right)}{2e^{3/2}f(bd-ae)^{3/2}(be-4af)^{3/2}}-\frac{\sqrt{d+ex+fx^2}\,(e(Ab-2aB)-bx(Be-2Af))}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)}+\frac{B\tanh^{-1}\left(\frac{\sqrt{b}\,\sqrt{d+ex+fx^2}}{\sqrt{bd-ae}}\right)}{2\sqrt{b}\,f(bd-ae)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(A + B*x)/(Sqrt[d + e*x + f*x^2]*(a*e + b*e*x + b*f*x^2)^2),x]
```

```
[Out] -((((A*b - 2*a*B)*e - b*(B*e - 2*A*f)*x)*Sqrt[d + e*x + f*x^2])/(e*(b*d - a
*e)*(b*e - 4*a*f)*(a*e + b*e*x + b*f*x^2))) + ((B*e - 2*A*f)*(8*a*e*f - b*(
e^2 + 4*d*f))*ArcTanh[(Sqrt[b*d - a*e]*(e + 2*f*x))/(Sqrt[e]*Sqrt[b*e - 4*a
*f]*Sqrt[d + e*x + f*x^2])]])/(2*e^(3/2)*(b*d - a*e)^(3/2)*f*(b*e - 4*a*f)^(
3/2)) + (B*ArcTanh[(Sqrt[b]*Sqrt[d + e*x + f*x^2])/Sqrt[b*d - a*e]])/(2*Sqr
t[b]*(b*d - a*e)^(3/2)*f)
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 996

```
Int[1/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(
x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[1/(e*(b*e - 4*a*f) - (b*d - a*e)
*x^2), x], x, (e + 2*f*x)/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d,
e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0
]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1038

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e
_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-2*g, Subst[Int[1/(b*d - a*e -
 b*x^2), x], x, Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e - b*f, 0] &&
 EqQ[h*e - 2*g*f, 0]
```

Rule 1039

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(h*e - 2*g*f)/(2*f), Int[1/(
(a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/(2*f), Int[(e + 2
*f*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c,
d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[c*e
 - b*f, 0] && NeQ[h*e - 2*g*f, 0]
```

Rubi steps

$$\int \frac{A+Bx}{\sqrt{d+ex+fx^2}\,(ae+bex+bfx^2)^2}\,dx = -\frac{((Ab-2aB)e-b(Be-2Af)x)\sqrt{d+ex+fx^2}}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)} - \frac{\int \frac{-\frac{1}{2}b(bd-}{}}{}$$

$$= -\frac{((Ab-2aB)e-b(Be-2Af)x)\sqrt{d+ex+fx^2}}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)} - \frac{B\int \frac{}{\sqrt{d}}}{}$$

$$= -\frac{((Ab-2aB)e-b(Be-2Af)x)\sqrt{d+ex+fx^2}}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)} + \frac{(Be)\text{Subs}}{}$$

$$= -\frac{((Ab-2aB)e-b(Be-2Af)x)\sqrt{d+ex+fx^2}}{e(bd-ae)(be-4af)\,(ae+bex+bfx^2)} + \frac{(Be-2A}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.89, size = 1014, normalized size = 4.07

Antiderivative was successfully verified.

```
[In] Integrate[(A + B*x)/(Sqrt[d + e*x + f*x^2]*(a*e + b*e*x + b*f*x^2)^2),x]
```

```
[Out] (Sqrt[d + x*(e + f*x)]*(-(B*e*(2*a + b*x)) + A*b*(e + 2*f*x)))/(e*(-(b*d) +
 a*e)*(b*e - 4*a*f)*(a*e + b*x*(e + f*x))) - (4*B*RootSum[-(b*d*e^2) + a*e^
3 + b*d^2*f + 2*b*d*e*Sqrt[f]*#1 - 4*a*e^2*Sqrt[f]*#1 + b*e^2*#1^2 - 2*b*d*
f*#1^2 + 4*a*e*f*#1^2 - 2*b*e*Sqrt[f]*#1^3 + b*f*#1^4 & , Log[-(Sqrt[f]*x)
+ Sqrt[d + e*x + f*x^2] - #1]/(-(b*d*e*Sqrt[f]) + 2*a*e^2*Sqrt[f] - b*e^2*#
1 + 2*b*d*f*#1 - 4*a*e*f*#1 + 3*b*e*Sqrt[f]*#1^2 - 2*b*f*#1^3) & ])/b - Roo
tSum[-(b*d*e^2) + a*e^3 + b*d^2*f + 2*b*d*e*Sqrt[f]*#1 - 4*a*e^2*Sqrt[f]*#1
 + b*e^2*#1^2 - 2*b*d*f*#1^2 + 4*a*e*f*#1^2 - 2*b*e*Sqrt[f]*#1^3 + b*f*#1^4
 & , (-5*b^2*B*d*e^2*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1] - A*b^2
*e^3*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1] + 6*a*b*B*e^3*Log[-(Sqr
t[f]*x) + Sqrt[d + e*x + f*x^2] - #1] - 4*A*b^2*d*e*f*Log[-(Sqrt[f]*x) + Sq
rt[d + e*x + f*x^2] - #1] + 28*a*b*B*d*e*f*Log[-(Sqrt[f]*x) + Sqrt[d + e*x
+ f*x^2] - #1] + 8*a*A*b*e^2*f*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #
1] - 32*a^2*B*e^2*f*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1] - 4*b^2*
B*d*e*Sqrt[f]*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1 + 2*A*b^2*e
^2*Sqrt[f]*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1 + 4*a*b*B*e^2*
Sqrt[f]*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1 + 8*A*b^2*d*f^(3/
2)*Log[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1 - 16*a*A*b*e*f^(3/2)*L
og[-(Sqrt[f]*x) + Sqrt[d + e*x + f*x^2] - #1]*#1 - b^2*B*e^2*Log[-(Sqrt[f]*
```

```
x) + Sqrt[d + e*x + f*x^2] - #1]*#1^2 + 4*a*b*B*e*f*Log[-(Sqrt[f]*x) + Sqrt
[d + e*x + f*x^2] - #1]*#1^2)/(-(b*d*e*Sqrt[f]) + 2*a*e^2*Sqrt[f] - b*e^2*#
1 + 2*b*d*f*#1 - 4*a*e*f*#1 + 3*b*e*Sqrt[f]*#1^2 - 2*b*f*#1^3) & ]/(2*b*e*(
b*d - a*e)*(b*e - 4*a*f))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1429 vs. $2(219) = 438$.

time = 0.22, size = 1430, normalized size = 5.74

| method | result |
|---|---|
| default | $-\frac{\left(2Abf-Bbe-B\sqrt{-eb\,(4fa-eb)}\right)\left(\frac{b\sqrt{\left(x+\frac{eb+\sqrt{-eb\,(4fa-eb)}}{2bf}\right)^{2}f-\frac{\sqrt{-eb\,(4fa-eb)}\,\left(x\cdots\right.}{\cdots}}}{(ae-bd)\left(x+\frac{eb+\sqrt{-eb\,(4fa-eb)}}{2bf}\cdots\right.}\cdots\right.}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((B*x+A)/(b*f*x^2+b*e*x+a*e)^2/(f*x^2+e*x+d)^(1/2),x,method=_RETURNVERBO
SE)
```

```
[Out] -1/2/f*(2*A*b*f-B*b*e-B*(-e*b*(4*a*f-b*e))^(1/2))/e/(4*a*f-b*e)/b^2*(b/(a*e
-b*d)/(x+1/2*(e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)*((x+1/2*(e*b+(-e*b*(4*a*f-
b*e))^(1/2))/b/f)^2*f-(-e*b*(4*a*f-b*e))^(1/2)/b*(x+1/2*(e*b+(-e*b*(4*a*f-b
*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(1/2)+1/2*(-e*b*(4*a*f-b*e))^(1/2)/(a*e-b*d
)/(-1/b*(a*e-b*d))^(1/2)*ln((-2/b*(a*e-b*d)-(-e*b*(4*a*f-b*e))^(1/2)/b*(x+1
/2*(e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)+2*(-1/b*(a*e-b*d))^(1/2)*((x+1/2*(e*
b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)^2*f-(-e*b*(4*a*f-b*e))^(1/2)/b*(x+1/2*(e*b
+(-e*b*(4*a*f-b*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(1/2))/(x+1/2*(e*b+(-e*b*(4*
a*f-b*e))^(1/2))/b/f)))+(2*A*f-B*e)/e/(4*a*f-b*e)/(-e*b*(4*a*f-b*e))^(1/2)/
(-1/b*(a*e-b*d))^(1/2)*ln((-2/b*(a*e-b*d)-(-e*b*(4*a*f-b*e))^(1/2)/b*(x+1/2
*(e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)+2*(-1/b*(a*e-b*d))^(1/2)*((x+1/2*(e*b+
(-e*b*(4*a*f-b*e))^(1/2))/b/f)^2*f-(-e*b*(4*a*f-b*e))^(1/2)/b*(x+1/2*(e*b+(
-e*b*(4*a*f-b*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(1/2))/(x+1/2*(e*b+(-e*b*(4*a*
f-b*e))^(1/2))/b/f))-(2*A*f-B*e)/e/(4*a*f-b*e)/(-e*b*(4*a*f-b*e))^(1/2)/(-1
```

```
/b*(a*e-b*d))^(1/2)*ln((-2/b*(a*e-b*d)+(-e*b*(4*a*f-b*e))^(1/2)/b*(x-1/2*(-
e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)+2*(-1/b*(a*e-b*d))^(1/2)*((x-1/2*(-e*b+(
-e*b*(4*a*f-b*e))^(1/2))/b/f)^2*f+(-e*b*(4*a*f-b*e))^(1/2)/b*(x-1/2*(-e*b+(
-e*b*(4*a*f-b*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(1/2))/(x-1/2*(-e*b+(-e*b*(4*a
*f-b*e))^(1/2))/b/f))-1/2/f*(2*A*b*f-B*b*e+B*(-e*b*(4*a*f-b*e))^(1/2))/e/(4
*a*f-b*e)/b^2*(b/(a*e-b*d)/(x-1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)*((x-
1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)^2*f+(-e*b*(4*a*f-b*e))^(1/2)/b*(x-
1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(1/2)-1/2*(-e*b*(4*
a*f-b*e))^(1/2)/(a*e-b*d)/(-1/b*(a*e-b*d))^(1/2)*ln((-2/b*(a*e-b*d)+(-e*b*(
4*a*f-b*e))^(1/2)/b*(x-1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)+2*(-1/b*(a*
e-b*d))^(1/2)*((x-1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)^2*f+(-e*b*(4*a*f
-b*e))^(1/2)/b*(x-1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)-1/b*(a*e-b*d))^(
1/2))/(x-1/2*(-e*b+(-e*b*(4*a*f-b*e))^(1/2))/b/f)))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(b*f*x^2+b*e*x+a*e)^2/(f*x^2+e*x+d)^(1/2),x, algorithm="m
axima")
```

```
[Out] integrate((B*x + A)/((b*f*x^2 + b*x*e + a*e)^2*sqrt(f*x^2 + x*e + d)), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(b*f*x^2+b*e*x+a*e)^2/(f*x^2+e*x+d)^(1/2),x, algorithm="f
ricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(b*f*x**2+b*e*x+a*e)**2/(f*x**2+e*x+d)**(1/2),x)
```

```
[Out] Timed out
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((B*x+A)/(b*f*x^2+b*e*x+a*e)^2/(f*x^2+e*x+d)^(1/2),x, algorithm="g
iac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Unable to divide, perhaps due to roun
ding error%%%{%%%{1,[2]%%%},[8,2,0,0,0]%%%}+%%%{%%{[%%%{-4,[1]%%%},0]:[1,0,
%%%{-1
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{A+B\,x}{{\left(b\,f\,x^2+b\,e\,x+a\,e\right)}^2\,\sqrt{f\,x^2+e\,x+d}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((A + B*x)/((a*e + b*e*x + b*f*x^2)^2*(d + e*x + f*x^2)^(1/2)),x)
```

```
[Out] int((A + B*x)/((a*e + b*e*x + b*f*x^2)^2*(d + e*x + f*x^2)^(1/2)), x)
```

# 3.36 $\int \frac{(g+hx)\sqrt{a+bx+cx^2}}{(ad+bdx+cdx^2)^2} \, dx$

**Optimal**. Leaf size=48

$$-\frac{2(bg-2ah+(2cg-bh)x)}{(b^2-4ac)\,d^2\sqrt{a+bx+cx^2}}$$

```
[Out] -2*(b*g-2*a*h+(-b*h+2*c*g)*x)/(-4*a*c+b^2)/d^2/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]

time = 0.03, antiderivative size = 48, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 36, $\frac{\text{number of rules}}{\text{integrand size}} = 0.056$, Rules used = {1012, 650}

$$-\frac{2(-2ah+x(2cg-bh)+bg)}{d^2\,(b^2-4ac)\sqrt{a+bx+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[((g + h*x)*Sqrt[a + b*x + c*x^2])/(a*d + b*d*x + c*d*x^2)^2,x]
```

```
[Out] (-2*(b*g - 2*a*h + (2*c*g - b*h)*x))/((b^2 - 4*a*c)*d^2*Sqrt[a + b*x + c*x^
2])
```

Rule 650

```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x] && NeQ[2*c*d - b*e, 0] && NeQ[b
^2 - 4*a*c, 0]
```

Rule 1012

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(c/f)^p, Int[(g + h*
x)^m*(d + e*x + f*x^2)^(p + q), x], x] /; FreeQ[{a, b, c, d, e, f, g, h, p,
 q}, x] && EqQ[c*d - a*f, 0] && EqQ[b*d - a*e, 0] && (IntegerQ[p] || GtQ[c/
f, 0]) && ( !IntegerQ[q] || LeafCount[d + e*x + f*x^2] <= LeafCount[a + b*x
 + c*x^2])
```

Rubi steps

$$\begin{aligned}
\int \frac{(g+hx)\sqrt{a+bx+cx^2}}{(ad+bdx+cdx^2)^2}\,dx &= \frac{\int \frac{g+hx}{(a+bx+cx^2)^{3/2}}\,dx}{d^2}\\
&= -\frac{2(bg-2ah+(2cg-bh)x)}{(b^2-4ac)\,d^2\sqrt{a+bx+cx^2}}
\end{aligned}$$

**Mathematica** [A]
time = 0.37, size = 46, normalized size = 0.96

$$\frac{-2bg+4ah-4cgx+2bhx}{(b^2-4ac)\,d^2\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[((g + h*x)*Sqrt[a + b*x + c*x^2])/(a*d + b*d*x + c*d*x^2)^2,x]
```

```
[Out] (-2*b*g + 4*a*h - 4*c*g*x + 2*b*h*x)/((b^2 - 4*a*c)*d^2*Sqrt[a + x*(b + c*x
)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 94 vs. $2(46) = 92$.
time = 0.17, size = 95, normalized size = 1.98

| method | result | size |
|---|---|---|
| gosper | $-\frac{2(bhx-2cgx+2ah-bg)}{\sqrt{c\,x^2+bx+a}\,d^2(4ac-b^2)}$ | 48 |
| trager | $-\frac{2(bhx-2cgx+2ah-bg)}{\sqrt{c\,x^2+bx+a}\,d^2(4ac-b^2)}$ | 48 |
| default | $\frac{h\left(-\frac{1}{c\sqrt{c\,x^2+bx+a}}-\frac{b(2cx+b)}{c(4ac-b^2)\sqrt{c\,x^2+bx+a}}\right)+\frac{2g(2cx+b)}{(4ac-b^2)\sqrt{c\,x^2+bx+a}}}{d^2}$ | 95 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^2,x,method=_RETURNVERBO
SE)
```

```
[Out] 1/d^2*(h*(-1/c/(c*x^2+b*x+a)^(1/2)-b/c*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^
(1/2))+2*g*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="m
axima")
```

```
[Out] integrate(sqrt(c*x^2 + b*x + a)*(h*x + g)/(c*d*x^2 + b*d*x + a*d)^2, x)
```

**Fricas** [A]
time = 0.50, size = 85, normalized size = 1.77

$$-\frac{2\sqrt{cx^2+bx+a}\,(bg-2\,ah+(2\,cg-bh)x)}{(b^2c-4\,ac^2)d^2x^2+(b^3-4\,abc)d^2x+(ab^2-4\,a^2c)d^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="f
ricas")
```

```
[Out] -2*sqrt(c*x^2 + b*x + a)*(b*g - 2*a*h + (2*c*g - b*h)*x)/((b^2*c - 4*a*c^2)
*d^2*x^2 + (b^3 - 4*a*b*c)*d^2*x + (a*b^2 - 4*a^2*c)*d^2)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\frac{\int \frac{g}{a\sqrt{a+bx+cx^2}+bx\sqrt{a+bx+cx^2}+cx^2\sqrt{a+bx+cx^2}}\,dx+\int \frac{hx}{a\sqrt{a+bx+cx^2}+bx\sqrt{a+bx+cx^2}+cx^2\sqrt{a+bx+cx^2}}\,dx}{d^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x**2+b*x+a)**(1/2)/(c*d*x**2+b*d*x+a*d)**2,x)
```

```
[Out] (Integral(g/(a*sqrt(a + b*x + c*x**2) + b*x*sqrt(a + b*x + c*x**2) + c*x**2
*sqrt(a + b*x + c*x**2)), x) + Integral(h*x/(a*sqrt(a + b*x + c*x**2) + b*x
*sqrt(a + b*x + c*x**2) + c*x**2*sqrt(a + b*x + c*x**2)), x))/d**2
```

**Giac** [A]
time = 3.21, size = 81, normalized size = 1.69

$$-\frac{2\left(\frac{(2\,cd^2g-bd^2h)x}{b^2d^4-4\,acd^4}+\frac{bd^2g-2\,ad^2h}{b^2d^4-4\,acd^4}\right)}{\sqrt{cx^2+bx+a}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^2,x, algorithm="g
iac")
```

```
[Out] -2*((2*c*d^2*g - b*d^2*h)*x/(b^2*d^4 - 4*a*c*d^4) + (b*d^2*g - 2*a*d^2*h)/(
b^2*d^4 - 4*a*c*d^4))/sqrt(c*x^2 + b*x + a)
```

**Mupad** [B]
time = 3.75, size = 49, normalized size = 1.02

$$\frac{4\,a\,h-2\,b\,g+2\,b\,h\,x-4\,c\,g\,x}{\left(b^2\,d^2-4\,a\,c\,d^2\right)\,\sqrt{c\,x^2+b\,x+a}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((g + h*x)*(a + b*x + c*x^2)^(1/2))/(a*d + b*d*x + c*d*x^2)^2,x)
```

```
[Out] (4*a*h - 2*b*g + 2*b*h*x - 4*c*g*x)/((b^2*d^2 - 4*a*c*d^2)*(a + b*x + c*x^2
)^(1/2))
```

# 3.37 $\int \frac{3+2x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

<u>**Optimal**</u>. Leaf size=17

$$\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

```
[Out] arctanh(x/(-x^2-4*x-3)^(1/2))
```

<u>**Rubi**</u> [A]
time = 0.01, antiderivative size = 17, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.062$, Rules used = {1041, 212}

$$\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(3 + 2*x)/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] ArcTanh[x/Sqrt[-3 - 4*x - x^2]]
```

<u>Rule 212</u>

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

<u>Rule 1041</u>

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

<u>Rubi steps</u>

$$\begin{aligned}\int \frac{3+2x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx &= 3\text{Subst}\left(\int \frac{1}{3-3x^2}\,dx, x, \frac{x}{\sqrt{-3-4x-x^2}}\right)\\ &= \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\end{aligned}$$

**Mathematica** [A]
time = 0.16, size = 17, normalized size = 1.00

$$\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(3 + 2*x)/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] ArcTanh[x/Sqrt[-3 - 4*x - x^2]]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 93 vs. $2(15) = 30$.
time = 0.14, size = 94, normalized size = 5.53

| method | result | size |
|---|---|---|
| trager | $-\frac{\ln\left(\frac{2x\sqrt{-x^2-4x-3}+4x+3}{2x^2+4x+3}\right)}{2}$ | 37 |
| default | $-\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\;\operatorname{arctanh}\left(\frac{3x}{\left(-\frac{3}{2}-x\right)\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}}\right)}{6\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ | 94 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((3+2*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/6*3^(1/2)*4^(1/2)/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2
-x))*(3*x^2/(-3/2-x)^2-12)^(1/2)*arctanh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)
^(1/2))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+2*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((2*x + 3)/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 56 vs. $2(15) = 30$.

time = 0.33, size = 56, normalized size = 3.29

$$-\frac{1}{4}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)+\frac{1}{4}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+2*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) + 1/4*log((2*sqrt(-x^2
- 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{2x+3}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+2*x)/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral((2*x + 3)/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 98 vs. 2(15) = 30.
time = 3.15, size = 98, normalized size = 5.76

$$\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)-\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+2*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] 1/2*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) - 1)
^2/(x + 2)^2 + 1) - 1/2*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt(-x
^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.06

$$\int\frac{2x+3}{\sqrt{-x^2-4x-3}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x + 3)/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int((2*x + 3)/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

## 3.38 $\int \frac{3+4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=86

$$\sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)-\sqrt{2}\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)+\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

```
[Out] arctanh(x/(-x^2-4*x-3)^(1/2))+arctan(1/2*(1+(-3-x)/(-x^2-4*x-3)^(1/2))*2^(1
/2))*2^(1/2)-arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(1/2))*2^(1/2))*2^(1/2)
```

**Rubi** [A]
time = 0.11, antiderivative size = 86, normalized size of antiderivative = 1.00, number of steps used = 13, number of rules used = 9, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.281$, Rules used = {1042, 1000, 12, 1040, 1175, 632, 210, 1041, 212}

$$\sqrt{2}\,\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)-\sqrt{2}\,\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)+\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(3 + 4*x)/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] Sqrt[2]*ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]] - Sqrt[2]*ArcTan
[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]] + ArcTanh[x/Sqrt[-3 - 4*x - x^
2]]
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
 && NeQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
```

```
0]))
```

Rubi steps

$$\int \frac{3+4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = -\left(3\int \frac{1}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx\right) - \int \frac{-6 -}{\sqrt{-3-4x-x^2}}$$

$$= \frac{1}{2}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx - \frac{1}{2}\int -\frac{4x}{\sqrt{-3-4x-x^2}}$$

$$= 2\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + 2\int \frac{x}{\sqrt{-3-4x-x^2}\,(3+4x+2x}$$

$$= \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + 16\text{Subst}\left(\int \frac{1+3x^2}{-4-8x^2-36x^4}\,dx, x,\right.$$

$$= \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) - \frac{2}{3}\text{Subst}\left(\int \frac{1}{\frac{1}{3}-\frac{2x}{3}+x^2}\,dx, x, \frac{}{\sqrt{-3}}\right.$$

$$= \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + \frac{4}{3}\text{Subst}\left(\int \frac{1}{-\frac{8}{9}-x^2}\,dx, x, \frac{2}{3}\left(-1+\right.\right.$$

$$= \sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right) - \sqrt{2}\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x}}}{\sqrt{2}}\right.$$

**Mathematica** [A]
time = 0.19, size = 51, normalized size = 0.59

$$-\sqrt{2}\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[(3 + 4*x)/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
[Out] -(Sqrt[2]*ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])]) + ArcTanh[x/Sqr
t[-3 - 4*x - x^2]]
```

**Maple** [A]
time = 0.35, size = 123, normalized size = 1.43

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}}{6}\right)-\operatorname{arctanh}\left(\frac{3x}{\left(-\frac{3}{2}-x\right)\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}}\right)\right)}{6\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ |
| trager | $\operatorname{RootOf}\left(4\_Z^2-4\_Z+3\right)\ln\left(-\frac{-4\operatorname{RootOf}\left(4\_Z^2-4\_Z+3\right)^2x+12\operatorname{RootOf}\left(4\_Z^2-4\_Z+3\right)x+6\sqrt{-x^2-4x}}{2\operatorname{RootOf}\left(4\_Z^2-4\_Z+3\right)x+x+3}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((3+4*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/6*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*arctan(1/6*(3*x^2/
(-3/2-x)^2-12)^(1/2)*2^(1/2))-arctanh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)^(1
/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2-x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+4*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate((4*x + 3)/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.34, size = 132, normalized size = 1.53

$$\frac{1}{2}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)+\frac{1}{2}\sqrt{2}\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)-\frac{1}{4}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)+\frac{1}{4}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+4*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/2*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x +
3)) + 1/2*sqrt(2)*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/
(2*x + 3)) - 1/4*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) + 1/4*log((
2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{4x+3}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+4*x)/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral((4*x + 3)/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac [B]** Leaf count of result is larger than twice the leaf count of optimal. 163 vs. 2(73) = 146.
time = 2.44, size = 163, normalized size = 1.90

$$\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)+\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)-\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((3+4*x)/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)) + sq
rt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)) + 1/2*lo
g(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) - 1)^2/(x
+ 2)^2 + 1) - 1/2*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt(-x^2 - 4
*x - 3) - 1)^2/(x + 2)^2 + 3)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{4\,x+3}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((4*x + 3)/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int((4*x + 3)/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

## 3.39 $\int \frac{(g+hx)\sqrt{a+bx+cx^2}}{(ad+bdx+cdx^2)^{3/2}}\,dx$

**Optimal**. Leaf size=136

$$-\frac{(2cg-bh)\sqrt{a+bx+cx^2}\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}\,d\sqrt{ad+bdx+cdx^2}}+\frac{h\sqrt{a+bx+cx^2}\log\left(a+bx+cx^2\right)}{2cd\sqrt{ad+bdx+cdx^2}}$$

```
[Out] 1/2*h*ln(c*x^2+b*x+a)*(c*x^2+b*x+a)^(1/2)/c/d/(c*d*x^2+b*d*x+a*d)^(1/2)-(-b
*h+2*c*g)*arctanh((2*c*x+b)/(-4*a*c+b^2)^(1/2))*(c*x^2+b*x+a)^(1/2)/c/d/(-4
*a*c+b^2)^(1/2)/(c*d*x^2+b*d*x+a*d)^(1/2)
```

**Rubi** [A]
time = 0.07, antiderivative size = 136, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 38, $\frac{\text{number of rules}}{\text{integrand size}} = 0.132$, Rules used = {1013, 648, 632, 212, 642}

$$\frac{h\sqrt{a+bx+cx^2}\log\left(a+bx+cx^2\right)}{2cd\sqrt{ad+bdx+cdx^2}}-\frac{\sqrt{a+bx+cx^2}\,(2cg-bh)\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{cd\sqrt{b^2-4ac}\,\sqrt{ad+bdx+cdx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[((g + h*x)*Sqrt[a + b*x + c*x^2])/(a*d + b*d*x + c*d*x^2)^(3/2),x]
```

```
[Out] -(((2*c*g - b*h)*Sqrt[a + b*x + c*x^2]*ArcTanh[(b + 2*c*x)/Sqrt[b^2 - 4*a*c
]])/(c*Sqrt[b^2 - 4*a*c]*d*Sqrt[a*d + b*d*x + c*d*x^2])) + (h*Sqrt[a + b*x
+ c*x^2]*Log[a + b*x + c*x^2])/(2*c*d*Sqrt[a*d + b*d*x + c*d*x^2])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 642

```
Int[((d_) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> S
imp[d*(Log[RemoveContent[a + b*x + c*x^2, x]]/b), x] /; FreeQ[{a, b, c, d,
```

```
e}, x] && EqQ[2*c*d - b*e, 0]
```

Rule 648

```
Int[((d_.) + (e_.)*(x_))/((a_) + (b_.)*(x_) + (c_.)*(x_)^2), x_Symbol] :> D
ist[(2*c*d - b*e)/(2*c), Int[1/(a + b*x + c*x^2), x], x] + Dist[e/(2*c), In
t[(b + 2*c*x)/(a + b*x + c*x^2), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ
[2*c*d - b*e, 0] && NeQ[b^2 - 4*a*c, 0] &&  !NiceSqrtQ[b^2 - 4*a*c]
```

Rule 1013

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[a^IntPart[p]*((a + b
*x + c*x^2)^FracPart[p]/(d^IntPart[p]*(d + e*x + f*x^2)^FracPart[p])), Int[
(g + h*x)^m*(d + e*x + f*x^2)^(p + q), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h, p, q}, x] && EqQ[c*d - a*f, 0] && EqQ[b*d - a*e, 0] &&  !IntegerQ[p] &
&  !IntegerQ[q] &&  !GtQ[c/f, 0]
```

Rubi steps

$$\int \frac{(g+hx)\sqrt{a+bx+cx^2}}{(ad+bdx+cdx^2)^{3/2}}\,dx = \frac{\sqrt{a+bx+cx^2}\int \frac{g+hx}{ad+bdx+cdx^2}\,dx}{\sqrt{ad+bdx+cdx^2}}$$
$$= \frac{\left(h\sqrt{a+bx+cx^2}\right)\int \frac{bd+2cdx}{ad+bdx+cdx^2}\,dx}{2cd\sqrt{ad+bdx+cdx^2}} + \frac{\left((2cdg-bdh)\sqrt{a+bx+cx^2}\right)\int \frac{}{a}}{2cd\sqrt{ad+bdx+cdx^2}}$$
$$= \frac{h\sqrt{a+bx+cx^2}\,\log\left(a+bx+cx^2\right)}{2cd\sqrt{ad+bdx+cdx^2}} - \frac{\left((2cdg-bdh)\sqrt{a+bx+cx^2}\right)\mathrm{Sub}}{cd\sqrt{ad+bd}}$$
$$= -\frac{(2cg-bh)\sqrt{a+bx+cx^2}\,\tanh^{-1}\left(\frac{b+2cx}{\sqrt{b^2-4ac}}\right)}{c\sqrt{b^2-4ac}\,d\sqrt{ad+bdx+cdx^2}} + \frac{h\sqrt{a+bx+cx^2}\,\log}{2cd\sqrt{ad+bdx}}$$

**Mathematica [A]**
time = 0.07, size = 108, normalized size = 0.79

$$\frac{(a+x(b+cx))^{3/2}\left((4cg-2bh)\tan^{-1}\left(\frac{b+2cx}{\sqrt{-b^2+4ac}}\right)+\sqrt{-b^2+4ac}\,h\log(a+x(b+cx))\right)}{2c\sqrt{-b^2+4ac}\,(d(a+x(b+cx)))^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[((g + h*x)*Sqrt[a + b*x + c*x^2])/(a*d + b*d*x + c*d*x^2)^(3/2),x
]
```

```
[Out] ((a + x*(b + c*x))^(3/2)*((4*c*g - 2*b*h)*ArcTan[(b + 2*c*x)/Sqrt[-b^2 + 4*
a*c]] + Sqrt[-b^2 + 4*a*c]*h*Log[a + x*(b + c*x)]))/(2*c*Sqrt[-b^2 + 4*a*c]
*(d*(a + x*(b + c*x)))^(3/2))
```

**Maple [A]**
time = 0.18, size = 121, normalized size = 0.89

| method | result |
|---|---|
| default | $\frac{\sqrt{d\left(c\,x^{2}+bx+a\right)}\,\left(h\ln\left(c\,x^{2}+bx+a\right)\sqrt{4ac-b^{2}}-2\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^{2}}}\right)bh+4\arctan\left(\frac{2cx+b}{\sqrt{4ac-b^{2}}}\right)cg\right)}{2\sqrt{c\,x^{2}+bx+a}\,d^{2}c\sqrt{4ac-b^{2}}}$ |
| risch | $\frac{\sqrt{c\,x^{2}+bx+a}\,\left(4ach-b^{2}h+\sqrt{-\left(bh-2cg\right)^{2}\left(4ac-b^{2}\right)}\right)\ln\left(-4abch+8a\,c^{2}g+b^{3}h-2b^{2}cg-2\sqrt{-\left(bh-2cg\cdots\right.}\right.}{2d\sqrt{d\left(c\,x^{2}+bx+a\right)}\,c\left(4ac-b^{2}\right)}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^(3/2),x,method=_RETURNV
ERBOSE)
```

```
[Out] 1/2/(c*x^2+b*x+a)^(1/2)*(d*(c*x^2+b*x+a))^(1/2)*(h*ln(c*x^2+b*x+a)*(4*a*c-b
^2)^(1/2)-2*arctan((2*c*x+b)/(4*a*c-b^2)^(1/2))*b*h+4*arctan((2*c*x+b)/(4*a
*c-b^2)^(1/2))*c*g)/d^2/c/(4*a*c-b^2)^(1/2)
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^(3/2),x, algorith
m="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas [F]**
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^(3/2),x, algorith
m="fricas")
```

```
[Out] integral(sqrt(c*d*x^2 + b*d*x + a*d)*sqrt(c*x^2 + b*x + a)*(h*x + g)/(c^2*d
^2*x^4 + 2*b*c*d^2*x^3 + 2*a*b*d^2*x + (b^2 + 2*a*c)*d^2*x^2 + a^2*d^2), x)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(g+hx)\sqrt{a+bx+cx^2}}{(d(a+bx+cx^2))^{\frac{3}{2}}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x**2+b*x+a)**(1/2)/(c*d*x**2+b*d*x+a*d)**(3/2),x)
```

```
[Out] Integral((g + h*x)*sqrt(a + b*x + c*x**2)/(d*(a + b*x + c*x**2))**(3/2), x)
```

**Giac** [F]
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)*(c*x^2+b*x+a)^(1/2)/(c*d*x^2+b*d*x+a*d)^(3/2),x, algorith
m="giac")
```

```
[Out] integrate(sqrt(c*x^2 + b*x + a)*(h*x + g)/(c*d*x^2 + b*d*x + a*d)^(3/2), x)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{(g+h\,x)\,\sqrt{c\,x^2+b\,x+a}}{{\left(c\,d\,x^2+b\,d\,x+a\,d\right)}^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((g + h*x)*(a + b*x + c*x^2)^(1/2))/(a*d + b*d*x + c*d*x^2)^(3/2),x)
```

```
[Out] int(((g + h*x)*(a + b*x + c*x^2)^(1/2))/(a*d + b*d*x + c*d*x^2)^(3/2), x)
```

# 3.40 $\int x^2 \sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx$

**Optimal**. Leaf size=212

$$-\frac{acx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{5d(a+bx)}-\frac{(8bc-15adx)\sqrt{a^2+2abx+b^2}}{60d^2(a+bx)}$$

```
[Out] 1/5*b*x^2*(d*x^2+c)^(3/2)*((b*x+a)^2)^(1/2)/d/(b*x+a)-1/60*(-15*a*d*x+8*b*c
)*(d*x^2+c)^(3/2)*((b*x+a)^2)^(1/2)/d^2/(b*x+a)-1/8*a*c^2*arctanh(x*d^(1/2)
/(d*x^2+c)^(1/2))*((b*x+a)^2)^(1/2)/d^(3/2)/(b*x+a)-1/8*a*c*x*((b*x+a)^2)^(
1/2)*(d*x^2+c)^(1/2)/d/(b*x+a)
```

**Rubi** [A]
time = 0.07, antiderivative size = 212, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 35, $\frac{\text{number of rules}}{\text{integrand size}} = 0.171$, Rules used = {1015, 847, 794, 201, 223, 212}

$$-\frac{ac^2\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{8d^{3/2}(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}(8bc-15adx)}{60d^2(a+bx)}-\frac{acx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{5d(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[x^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] -1/8*(a*c*x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(d*(a + b*x)) +
(b*x^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*(c + d*x^2)^(3/2))/(5*d*(a + b*x)) - (
(8*b*c - 15*a*d*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*(c + d*x^2)^(3/2))/(60*d^2
*(a + b*x)) - (a*c^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt
[c + d*x^2]])/(8*d^(3/2)*(a + b*x))
```

Rule 201

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^p/(n*p
+ 1)), x] + Dist[a*n*(p/(n*p + 1)), Int[(a + b*x^n)^(p - 1), x], x] /; Free
Q[{a, b}, x] && IGtQ[n, 0] && GtQ[p, 0] && (IntegerQ[2*p] || (EqQ[n, 2] &&
IntegerQ[4*p]) || (EqQ[n, 2] && IntegerQ[3*p]) || LtQ[Denominator[p + 1/n],
 Denominator[p]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] && !GtQ[a, 0]
```

Rule 794

```
Int[((d_.) + (e_.)*(x_))*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_), x
_Symbol] :> Simp[((e*f + d*g)*(2*p + 3) + 2*e*g*(p + 1)*x)*((a + c*x^2)^(p
+ 1)/(2*c*(p + 1)*(2*p + 3))), x] - Dist[(a*e*g - c*d*f*(2*p + 3))/(c*(2*p
+ 3)), Int[(a + c*x^2)^p, x], x] /; FreeQ[{a, c, d, e, f, g, p}, x] && !Le
Q[p, -1]
```

Rule 847

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Simp[g*(d + e*x)^m*((a + c*x^2)^(p + 1)/(c*(m + 2*p + 2))
), x] + Dist[1/(c*(m + 2*p + 2)), Int[(d + e*x)^(m - 1)*(a + c*x^2)^p*Simp[
c*d*f*(m + 2*p + 2) - a*e*g*m + c*(e*f*(m + 2*p + 2) + d*g*m)*x, x], x], x]
 /; FreeQ[{a, c, d, e, f, g, p}, x] && NeQ[c*d^2 + a*e^2, 0] && GtQ[m, 0] &
& NeQ[m + 2*p + 2, 0] && (IntegerQ[m] || IntegerQ[p] || IntegersQ[2*m, 2*p]
) && !(IGtQ[m, 0] && EqQ[f, 0])
```

Rule 1015

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4
*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(b + 2*c*x)^(2
*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, f, g, h, m, p, q}, x] && Eq
Q[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int x^2\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int x^2(2ab+2b^2x)\sqrt{c+dx^2}\,dx}{2ab+2b^2x}$$

$$= \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{5d(a+bx)} + \frac{\sqrt{a^2+2abx+b^2x^2}\,\int x(-\ldots}{5d\,(2ab\ldots}$$

$$= \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{5d(a+bx)} - \frac{(8bc-15adx)\sqrt{a^2+2abx\ldots}}{60d^2(a+b\ldots}$$

$$= -\frac{acx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)} + \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c\ldots}{5d(a+bx)}$$

$$= -\frac{acx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)} + \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c\ldots}{5d(a+bx)}$$

$$= -\frac{acx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)} + \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c\ldots}{5d(a+bx)}$$

**Mathematica** [A]
time = 0.16, size = 107, normalized size = 0.50

$$\frac{\sqrt{(a+bx)^2}\left(\sqrt{c+dx^2}\,(15adx(c+2dx^2)+8b(-2c^2+cdx^2+3d^2x^4))+15ac^2\sqrt{d}\,\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)}{120d^2(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(Sqrt[c + d*x^2]*(15*a*d*x*(c + 2*d*x^2) + 8*b*(-2*c^2 +
 c*d*x^2 + 3*d^2*x^4)) + 15*a*c^2*Sqrt[d]*Log[-(Sqrt[d]*x) + Sqrt[c + d*x^2
]]))/(120*d^2*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.09, size = 103, normalized size = 0.49

| method | result |
|---|---|
| default | $\frac{\mathrm{csgn}(bx+a)\left(24(dx^2+c)^{\frac{3}{2}}d^{\frac{3}{2}}bx^2+30(dx^2+c)^{\frac{3}{2}}d^{\frac{3}{2}}ax-16(dx^2+c)^{\frac{3}{2}}\sqrt{d}\,bc-15\sqrt{dx^2+c}\,d^{\frac{3}{2}}acx-15\ln\left(\sqrt{d}\,x+\sqrt{dx^2+\ldots}\right.\right.}{120d^{\frac{5}{2}}}$ |
| risch | $\frac{(24bx^4d^2+30ax^3d^2+8bcx^2d+15acxd-16c^2b)\sqrt{dx^2+c}\,\sqrt{(bx+a)^2}}{120d^2(bx+a)} - \frac{c^2a\ln\left(\sqrt{d}\,x+\sqrt{dx^2+c}\right)\sqrt{(bx+a\ldots}}{8d^{\frac{3}{2}}(bx+a)}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/120*csgn(b*x+a)*(24*(d*x^2+c)^(3/2)*d^(3/2)*b*x^2+30*(d*x^2+c)^(3/2)*d^(3
/2)*a*x-16*(d*x^2+c)^(3/2)*d^(1/2)*b*c-15*(d*x^2+c)^(1/2)*d^(3/2)*a*c*x-15*
ln(d^(1/2)*x+(d*x^2+c)^(1/2))*a*c^2*d)/d^(5/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2)*x^2, x)
```

**Fricas [A]**
time = 0.37, size = 175, normalized size = 0.83

$$\left[\frac{15\,ac^{2}\sqrt{d}\,\log\left(-2\,dx^{2}+2\,\sqrt{dx^{2}+c}\,\sqrt{d}\,x-c\right)+2\,(24\,bd^{2}x^{4}+30\,ad^{2}x^{3}+8\,bcdx^{2}+15\,acdx-16\,bc^{2})\sqrt{dx^{2}+c}}{240\,d^{2}}, \frac{15\,ac^{2}\sqrt{-d}\,\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^{2}+c}}\right)+(24\,bd^{2}x^{4}+30\,ad^{2}x^{3}+8\,bcdx^{2}+15\,acdx-16\,bc^{2})\sqrt{dx^{2}+c}}{120\,d^{2}}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="fricas")
```

```
[Out] [1/240*(15*a*c^2*sqrt(d)*log(-2*d*x^2 + 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) +
2*(24*b*d^2*x^4 + 30*a*d^2*x^3 + 8*b*c*d*x^2 + 15*a*c*d*x - 16*b*c^2)*sqrt(
d*x^2 + c))/d^2, 1/120*(15*a*c^2*sqrt(-d)*arctan(sqrt(-d)*x/sqrt(d*x^2 + c)
) + (24*b*d^2*x^4 + 30*a*d^2*x^3 + 8*b*c*d*x^2 + 15*a*c*d*x - 16*b*c^2)*sqr
t(d*x^2 + c))/d^2]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int x^{2}\sqrt{c+dx^{2}}\,\sqrt{(a+bx)^{2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2),x)
```

```
[Out] Integral(x**2*sqrt(c + d*x**2)*sqrt((a + b*x)**2), x)
```

**Giac [A]**
time = 2.68, size = 117, normalized size = 0.55

$$\frac{ac^{2}\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^{2}+c}\right|\right)\mathrm{sgn}(bx+a)}{8\,d^{\frac{3}{2}}}+\frac{1}{120}\,\sqrt{dx^{2}+c}\left(\left(2\left(3\,(4\,bx\mathrm{sgn}(bx+a)+5\,a\mathrm{sgn}(bx+a))x+\frac{4\,bc\mathrm{sgn}(bx+a)}{d}\right)x+\frac{15\,ac\mathrm{sgn}(bx+a)}{d}\right)x-\frac{16\,bc^{2}\mathrm{sgn}(bx+a)}{d^{2}}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="giac")
```

```
[Out] 1/8*a*c^2*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a)/d^(3/2) + 1/1
20*sqrt(d*x^2 + c)*((2*(3*(4*b*x*sgn(b*x + a) + 5*a*sgn(b*x + a))*x + 4*b*c
*sgn(b*x + a)/d)*x + 15*a*c*sgn(b*x + a)/d)*x - 16*b*c^2*sgn(b*x + a)/d^2)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int x^2\sqrt{(a+b\,x)^2}\,\sqrt{d\,x^2+c}\;dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2),x)
```

```
[Out] int(x^2*((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2), x)
```

# 3.41 $\int x\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx$

**Optimal**. Leaf size=161

$$-\frac{bcx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{(4a+3bx)\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{12d(a+bx)}-\frac{bc^2\sqrt{a^2+2abx+b^2x^2}\ \mathrm{ta}}{8d^{3/2}(a+}$$

```
[Out] 1/12*(3*b*x+4*a)*(d*x^2+c)^(3/2)*((b*x+a)^2)^(1/2)/d/(b*x+a)-1/8*b*c^2*arct
anh(x*d^(1/2)/(d*x^2+c)^(1/2))*((b*x+a)^2)^(1/2)/d^(3/2)/(b*x+a)-1/8*b*c*x*
((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/d/(b*x+a)
```

**Rubi** [A]
time = 0.04, antiderivative size = 161, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 33, $\frac{\text{number of rules}}{\text{integrand size}} = 0.152$, Rules used = {1015, 794, 201, 223, 212}

$$-\frac{bc^2\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{8d^{3/2}(a+bx)}-\frac{bcx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{(4a+3bx)\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{12d(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] -1/8*(b*c*x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(d*(a + b*x)) +
((4*a + 3*b*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*(c + d*x^2)^(3/2))/(12*d*(a +
b*x)) - (b*c^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt[c + d
*x^2]])/(8*d^(3/2)*(a + b*x))
```

Rule 201

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^p/(n*p
+ 1)), x] + Dist[a*n*(p/(n*p + 1)), Int[(a + b*x^n)^(p - 1), x], x] /; Free
Q[{a, b}, x] && IGtQ[n, 0] && GtQ[p, 0] && (IntegerQ[2*p] || (EqQ[n, 2] &&
IntegerQ[4*p]) || (EqQ[n, 2] && IntegerQ[3*p]) || LtQ[Denominator[p + 1/n],
 Denominator[p]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 794

```
Int[((d_.) + (e_.)*(x_))*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_), x
_Symbol] :> Simp[((e*f + d*g)*(2*p + 3) + 2*e*g*(p + 1)*x)*((a + c*x^2)^(p
+ 1)/(2*c*(p + 1)*(2*p + 3))), x] - Dist[(a*e*g - c*d*f*(2*p + 3))/(c*(2*p
+ 3)), Int[(a + c*x^2)^p, x], x] /; FreeQ[{a, c, d, e, f, g, p}, x] &&  !Le
Q[p, -1]
```

Rule 1015

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4
*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(b + 2*c*x)^(2
*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, f, g, h, m, p, q}, x] && Eq
Q[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int x\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx=\frac{\sqrt{a^2+2abx+b^2x^2}\int x(2ab+2b^2x)\sqrt{c+dx^2}\,dx}{2ab+2b^2x}$$

$$=\frac{(4a+3bx)\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{12d(a+bx)}-\frac{\left(b^2c\sqrt{a^2+2abx+b\right.}{2d\,(2ab}$$

$$=-\frac{bcx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{(4a+3bx)\sqrt{a^2+2abx+b^2}}{12d(a+bx)}$$

$$=-\frac{bcx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{(4a+3bx)\sqrt{a^2+2abx+b^2}}{12d(a+bx)}$$

$$=-\frac{bcx\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{8d(a+bx)}+\frac{(4a+3bx)\sqrt{a^2+2abx+b^2}}{12d(a+bx)}$$

**Mathematica** [A]
time = 0.13, size = 95, normalized size = 0.59

$$\frac{\sqrt{(a+bx)^2}\left(\sqrt{d}\,\sqrt{c+dx^2}\,(8a(c+dx^2)+3bx(c+2dx^2))+3bc^2\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)}{24d^{3/2}(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(Sqrt[d]*Sqrt[c + d*x^2]*(8*a*(c + d*x^2) + 3*b*x*(c + 2
*d*x^2)) + 3*b*c^2*Log[-(Sqrt[d]*x) + Sqrt[c + d*x^2]]))/(24*d^(3/2)*(a + b
*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.08, size = 83, normalized size = 0.52

| method | result | size |
|---|---|---|
| default | $\frac{\mathrm{csgn}(bx+a)\left(6(d\,x^2+c)^{\frac{3}{2}}\sqrt{d}\,bx+8a(d\,x^2+c)^{\frac{3}{2}}\sqrt{d}-3\sqrt{d\,x^2+c}\,\sqrt{d}\,bcx-3\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)b\,c^2\right)}{24d^{\frac{3}{2}}}$ | 83 |
| risch | $\frac{(6bd\,x^3+8ad\,x^2+3bcx+8ac)\sqrt{d\,x^2+c}\,\sqrt{(bx+a)^2}}{24d(bx+a)}-\frac{c^2b\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)\sqrt{(bx+a)^2}}{8d^{\frac{3}{2}}(bx+a)}$ | 97 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/24*csgn(b*x+a)*(6*(d*x^2+c)^(3/2)*d^(1/2)*b*x+8*a*(d*x^2+c)^(3/2)*d^(1/2)
-3*(d*x^2+c)^(1/2)*d^(1/2)*b*c*x-3*ln(d^(1/2)*x+(d*x^2+c)^(1/2))*b*c^2)/d^(
3/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2)*x, x)
```

**Fricas** [A]
time = 0.40, size = 157, normalized size = 0.98

$$\left[\frac{3\,bc^2\sqrt{d}\,\log\left(-2\,dx^2+2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)+2\,(6\,bd^2x^3+8\,ad^2x^2+3\,bcdx+8\,acd)\sqrt{dx^2+c}}{48\,d^2},\ \frac{3\,bc^2\sqrt{-d}\,\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)+(6\,bd^2x^3+8\,ad^2x^2+3\,bcdx+8\,acd)\sqrt{dx^2+c}}{24\,d^2}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="fricas")
```

```
[Out] [1/48*(3*b*c^2*sqrt(d)*log(-2*d*x^2 + 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + 2*
(6*b*d^2*x^3 + 8*a*d^2*x^2 + 3*b*c*d*x + 8*a*c*d)*sqrt(d*x^2 + c))/d^2, 1/2
4*(3*b*c^2*sqrt(-d)*arctan(sqrt(-d)*x/sqrt(d*x^2 + c)) + (6*b*d^2*x^3 + 8*a
*d^2*x^2 + 3*b*c*d*x + 8*a*c*d)*sqrt(d*x^2 + c))/d^2]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int x\sqrt{c+dx^2}\,\sqrt{(a+bx)^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2),x)
```

```
[Out] Integral(x*sqrt(c + d*x**2)*sqrt((a + b*x)**2), x)
```

**Giac [A]**
time = 3.30, size = 98, normalized size = 0.61

$$\frac{bc^2\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^2+c}\right|\right)\operatorname{sgn}(bx+a)}{8\,d^{\frac{3}{2}}}+\frac{1}{24}\sqrt{dx^2+c}\left(\left(2\,(3\,bx\operatorname{sgn}(bx+a)+4\,a\operatorname{sgn}(bx+a))x+\frac{3\,bc\operatorname{sgn}(bx+a)}{d}\right)x+\frac{8\,ac\operatorname{sgn}(bx+a)}{d}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="giac")
```

```
[Out] 1/8*b*c^2*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a)/d^(3/2) + 1/2
4*sqrt(d*x^2 + c)*((2*(3*b*x*sgn(b*x + a) + 4*a*sgn(b*x + a))*x + 3*b*c*sgn
(b*x + a)/d)*x + 8*a*c*sgn(b*x + a)/d)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int x\,\sqrt{(a+b\,x)^2}\,\sqrt{d\,x^2+c}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2),x)
```

```
[Out] int(x*((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2), x)
```

# 3.42 $\int \sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx$

**Optimal**. Leaf size=148

$$\frac{ax\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)}+\frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{3d(a+bx)}+\frac{ac\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}}{\sqrt{c+}}\right.}{2\sqrt{d}\,(a+bx)}$$

```
[Out] 1/3*b*(d*x^2+c)^(3/2)*((b*x+a)^2)^(1/2)/d/(b*x+a)+1/2*a*c*arctanh(x*d^(1/2)
/(d*x^2+c)^(1/2))*((b*x+a)^2)^(1/2)/(b*x+a)/d^(1/2)+1/2*a*x*((b*x+a)^2)^(1/
2)*(d*x^2+c)^(1/2)/(b*x+a)
```

**Rubi** [A]
time = 0.03, antiderivative size = 148, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 32, $\frac{\text{number of rules}}{\text{integrand size}} = 0.156$, Rules used = {984, 655, 201, 223, 212}

$$\frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{3d(a+bx)}+\frac{ax\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)}+\frac{ac\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{2\sqrt{d}\,(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] (a*x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(2*(a + b*x)) + (b*Sqrt
[a^2 + 2*a*b*x + b^2*x^2]*(c + d*x^2)^(3/2))/(3*d*(a + b*x)) + (a*c*Sqrt[a^
2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt[c + d*x^2]])/(2*Sqrt[d]*(a
+ b*x))
```

Rule 201

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^p/(n*p
+ 1)), x] + Dist[a*n*(p/(n*p + 1)), Int[(a + b*x^n)^(p - 1), x], x] /; Free
Q[{a, b}, x] && IGtQ[n, 0] && GtQ[p, 0] && (IntegerQ[2*p] || (EqQ[n, 2] &&
IntegerQ[4*p]) || (EqQ[n, 2] && IntegerQ[3*p]) || LtQ[Denominator[p + 1/n],
 Denominator[p]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 655

```
Int[((d_) + (e_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[e*((
a + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[d, Int[(a + c*x^2)^p, x], x] /
; FreeQ[{a, c, d, e, p}, x] && NeQ[p, -1]
```

Rule 984

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f_.)*(x_)^2)^(q_.), x_
Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)
^(2*FracPart[p])), Int[(b + 2*c*x)^(2*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a,
 b, c, d, f, p, q}, x] && EqQ[b^2 - 4*a*c, 0] &&  !IntegerQ[p]
```

Rubi steps

$$\int \sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\int (2ab+2b^2x)\sqrt{c+dx^2}\,dx}{2ab+2b^2x}$$

$$= \frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3/2}}{3d(a+bx)} + \frac{\left(2ab\sqrt{a^2+2abx+b^2x^2}\right)\int \sqrt{c}}{2ab+2b^2x}$$

$$= \frac{ax\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)} + \frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3}}{3d(a+bx)}$$

$$= \frac{ax\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)} + \frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3}}{3d(a+bx)}$$

$$= \frac{ax\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)} + \frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2)^{3}}{3d(a+bx)}$$

**Mathematica [A]**
time = 0.15, size = 84, normalized size = 0.57

$$\frac{\sqrt{(a+bx)^2}\left(\sqrt{c+dx^2}\,(3adx+2b(c+dx^2))-3ac\sqrt{d}\,\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)}{6d(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2],x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(Sqrt[c + d*x^2]*(3*a*d*x + 2*b*(c + d*x^2)) - 3*a*c*Sqr
t[d]*Log[-(Sqrt[d]*x) + Sqrt[c + d*x^2]]))/(6*d*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.08, size = 65, normalized size = 0.44

| method | result | size |
|---|---|---|
| default | $\frac{\operatorname{csgn}(bx+a)\left(2b(d\,x^2+c)^{\frac{3}{2}}\sqrt{d}+3\sqrt{d\,x^2+c}\,d^{\frac{3}{2}}ax+3\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)acd\right)}{6d^{\frac{3}{2}}}$ | 65 |
| risch | $\frac{(2bd\,x^2+3adx+2bc)\sqrt{d\,x^2+c}\,\sqrt{(bx+a)^2}}{6d(bx+a)}+\frac{ac\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)\sqrt{(bx+a)^2}}{2\sqrt{d}\,(bx+a)}$ | 88 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/6*csgn(b*x+a)*(2*b*(d*x^2+c)^(3/2)*d^(1/2)+3*(d*x^2+c)^(1/2)*d^(3/2)*a*x+
3*ln(d^(1/2)*x+(d*x^2+c)^(1/2))*a*c*d)/d^(3/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2), x)
```

**Fricas** [A]
time = 0.51, size = 128, normalized size = 0.86

$$\left[\frac{3\,ac\sqrt{d}\,\log\left(-2\,dx^2-2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)+2\,(2\,bdx^2+3\,adx+2\,bc)\sqrt{dx^2+c}}{12\,d},-\frac{3\,ac\sqrt{-d}\,\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)-(2\,bdx^2+3\,adx+2\,bc)\sqrt{dx^2+c}}{6\,d}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="fricas")
```

```
[Out] [1/12*(3*a*c*sqrt(d)*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + 2*(2
*b*d*x^2 + 3*a*d*x + 2*b*c)*sqrt(d*x^2 + c))/d, -1/6*(3*a*c*sqrt(-d)*arctan
(sqrt(-d)*x/sqrt(d*x^2 + c)) - (2*b*d*x^2 + 3*a*d*x + 2*b*c)*sqrt(d*x^2 + c
))/d]
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int\sqrt{c+dx^2}\,\sqrt{(a+bx)^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2),x)
```

```
[Out] Integral(sqrt(c + d*x**2)*sqrt((a + b*x)**2), x)
```

**Giac [A]**
time = 2.93, size = 79, normalized size = 0.53

$$-\frac{ac\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^2+c}\right|\right)\operatorname{sgn}(bx+a)}{2\sqrt{d}}+\frac{1}{6}\sqrt{dx^2+c}\left((2\,bx\operatorname{sgn}(bx+a)+3\,a\operatorname{sgn}(bx+a))x+\frac{2\,bc\operatorname{sgn}(bx+a)}{d}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2),x, algorithm="giac")
```

```
[Out] -1/2*a*c*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a)/sqrt(d) + 1/6*
sqrt(d*x^2 + c)*((2*b*x*sgn(b*x + a) + 3*a*sgn(b*x + a))*x + 2*b*c*sgn(b*x
+ a)/d)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \sqrt{{(a+b\,x)}^2}\,\sqrt{d\,x^2+c}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2),x)
```

```
[Out] int(((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2), x)
```

# 3.43 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{x}\,dx$

**Optimal**. Leaf size=160

$$\frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2(a+bx)}+\frac{bc\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{2\sqrt{d}\,(a+bx)}-\frac{a\sqrt{c}\,\sqrt{a^2+2abx+}}{}$$

```
[Out] -a*arctanh((d*x^2+c)^(1/2)/c^(1/2))*c^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)+1/2*b
*c*arctanh(x*d^(1/2)/(d*x^2+c)^(1/2))*((b*x+a)^2)^(1/2)/(b*x+a)/d^(1/2)+1/2
*(b*x+2*a)*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/(b*x+a)
```

**Rubi** [A]
time = 0.08, antiderivative size = 160, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 8, integrand size = 35, $\frac{\text{number of rules}}{\text{integrand size}} = 0.229$, Rules used = {1015, 829, 858, 223, 212, 272, 65, 214}

$$\frac{\sqrt{a^2+2abx+b^2x^2}\,(2a+bx)\sqrt{c+dx^2}}{2(a+bx)}+\frac{bc\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{2\sqrt{d}\,(a+bx)}-\frac{a\sqrt{c}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{c+dx^2}}{\sqrt{c}}\right)}{a+bx}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x,x]
```

```
[Out] ((2*a + b*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(2*(a + b*x)) +
 (b*c*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt[c + d*x^2]])/(
2*Sqrt[d]*(a + b*x)) - (a*Sqrt[c]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[Sqr
t[c + d*x^2]/Sqrt[c]])/(a + b*x)
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 829

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(c*e*f*(m + 2*p + 2) - g*c*d*(2*p
+ 1) + g*c*e*(m + 2*p + 1)*x)*((a + c*x^2)^p/(c*e^2*(m + 2*p + 1)*(m + 2*p
+ 2))), x] + Dist[2*(p/(c*e^2*(m + 2*p + 1)*(m + 2*p + 2))), Int[(d + e*x)^
m*(a + c*x^2)^(p - 1)*Simp[f*a*c*e^2*(m + 2*p + 2) + a*c*d*e*g*m - (c^2*f*d
*e*(m + 2*p + 2) - g*(c^2*d^2*(2*p + 1) + a*c*e^2*(m + 2*p + 1)))*x, x], x]
, x] /; FreeQ[{a, c, d, e, f, g, m}, x] && NeQ[c*d^2 + a*e^2, 0] && GtQ[p,
0] && (IntegerQ[p] ||  !RationalQ[m] || (GeQ[m, -1] && LtQ[m, 0])) &&  !ILt
Q[m + 2*p, 0] && (IntegerQ[m] || IntegerQ[p] || IntegersQ[2*m, 2*p])
```

Rule 858

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + c*x^2)^p, x], x] + D
ist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + c*x^2)^p, x], x] /; FreeQ[{a, c, d,
 e, f, g, m, p}, x] && NeQ[c*d^2 + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1015

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4
*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(b + 2*c*x)^(2
*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, f, g, h, m, p, q}, x] && Eq
Q[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{(2ab+2b^2x)\sqrt{c+dx^2}}{x}\,dx}{2ab+2b^2x}$$

$$= \frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2(a+bx)} + \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{4ab}{x\sqrt{}}}{2d\,(2ab+2b^2x}$$

$$= \frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2(a+bx)} + \frac{\left(2abc\sqrt{a^2+2abx+b^2x^2}\right.}{2ab+2b}$$

$$= \frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2(a+bx)} + \frac{\left(abc\sqrt{a^2+2abx+b^2x^2}\right.}{2a}$$

$$= \frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2(a+bx)} + \frac{bc\sqrt{a^2+2abx+b^2x^2}\,\tan}{2\sqrt{d}\,(a+}$$

$$= \frac{(2a+bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2(a+bx)} + \frac{bc\sqrt{a^2+2abx+b^2x^2}\,\tan}{2\sqrt{d}\,(a+}$$

**Mathematica** [A]
time = 0.18, size = 118, normalized size = 0.74

$$\frac{\sqrt{(a+bx)^2}\left(\sqrt{d}\,(2a+bx)\sqrt{c+dx^2}+4a\sqrt{c}\sqrt{d}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x-\sqrt{c+dx^2}}{\sqrt{c}}\right)-bc\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)}{2\sqrt{d}\,(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x,x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(Sqrt[d]*(2*a + b*x)*Sqrt[c + d*x^2] + 4*a*Sqrt[c]*Sqrt[
d]*ArcTanh[(Sqrt[d]*x - Sqrt[c + d*x^2])/Sqrt[c]] - b*c*Log[-(Sqrt[d]*x) +
Sqrt[c + d*x^2]]))/(2*Sqrt[d]*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.10, size = 92, normalized size = 0.58

| method | result |
|---|---|
| default | $\frac{\mathrm{csgn}(bx+a)\left(\sqrt{d\,x^2+c}\,\sqrt{d}\,bx-2\sqrt{d}\,\ln\left(\frac{2c+2\sqrt{c}\,\sqrt{d\,x^2+c}}{x}\right)\sqrt{c}\,a+2\sqrt{d\,x^2+c}\,\sqrt{d}\,a+\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2}\right.\right.}{2\sqrt{d}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x,x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*csgn(b*x+a)*((d*x^2+c)^(1/2)*d^(1/2)*b*x-2*d^(1/2)*ln(2*(c^(1/2)*(d*x^2
+c)^(1/2)+c)/x)*c^(1/2)*a+2*(d*x^2+c)^(1/2)*d^(1/2)*a+ln(d^(1/2)*x+(d*x^2+c
)^(1/2))*b*c)/d^(1/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2)/x, x)
```

**Fricas [A]**
time = 0.43, size = 341, normalized size = 2.13

$$\left[\frac{bc\sqrt{d}\log\left(-2\,dx^2-2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)+2\,a\sqrt{c}\,d\log\left(-\frac{dx^2-2\,\sqrt{dx^2+c}\,\sqrt{c}+2\,c}{x^2}\right)+2\,(bdx+2\,ad)\sqrt{dx^2+c}}{4\,d},\ -\frac{bc\sqrt{-d}\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)-a\sqrt{c}\,d\log\left(-\frac{dx^2-2\,\sqrt{dx^2+c}\,\sqrt{c}+2\,c}{x^2}\right)-(bdx+2\,ad)\sqrt{dx^2+c}}{2\,d},\ \frac{4\,a\sqrt{-c}\,d\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^2+c}}\right)+bc\sqrt{d}\log\left(-2\,dx^2-2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)+2\,(bdx+2\,ad)\sqrt{dx^2+c}}{4\,d},\ -\frac{bc\sqrt{-d}\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)-2\,a\sqrt{-c}\,d\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^2+c}}\right)-(bdx+2\,ad)\sqrt{dx^2+c}}{2\,d}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x,x, algorithm="fricas")
```

```
[Out] [1/4*(b*c*sqrt(d)*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + 2*a*sqr
t(c)*d*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(c) + 2*c)/x^2) + 2*(b*d*x + 2*a
*d)*sqrt(d*x^2 + c))/d, -1/2*(b*c*sqrt(-d)*arctan(sqrt(-d)*x/sqrt(d*x^2 + c
)) - a*sqrt(c)*d*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(c) + 2*c)/x^2) - (b*d
*x + 2*a*d)*sqrt(d*x^2 + c))/d, 1/4*(4*a*sqrt(-c)*d*arctan(sqrt(-c)/sqrt(d*
x^2 + c)) + b*c*sqrt(d)*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + 2
*(b*d*x + 2*a*d)*sqrt(d*x^2 + c))/d, -1/2*(b*c*sqrt(-d)*arctan(sqrt(-d)*x/s
qrt(d*x^2 + c)) - 2*a*sqrt(-c)*d*arctan(sqrt(-c)/sqrt(d*x^2 + c)) - (b*d*x
+ 2*a*d)*sqrt(d*x^2 + c))/d]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c + dx^{2}}\,\sqrt{\left(a + bx\right)^{2}}}{x}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2)/x,x)
```

[Out] Integral(sqrt(c + d*x**2)*sqrt((a + b*x)**2)/x, x)

**Giac** [A]
time = 3.22, size = 102, normalized size = 0.64

$$\frac{2\,ac\arctan\left(-\frac{\sqrt{d}\,x-\sqrt{dx^2+c}}{\sqrt{-c}}\right)\mathrm{sgn}(bx+a)}{\sqrt{-c}}-\frac{bc\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^2+c}\right|\right)\mathrm{sgn}(bx+a)}{2\,\sqrt{d}}+\frac{1}{2}\,\sqrt{dx^2+c}\,(bx\mathrm{sgn}(bx+a)+2\,a\mathrm{sgn}(bx+a))$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x,x, algorithm="giac")
```

```
[Out] 2*a*c*arctan(-(sqrt(d)*x - sqrt(d*x^2 + c))/sqrt(-c))*sgn(b*x + a)/sqrt(-c)
 - 1/2*b*c*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a)/sqrt(d) + 1/
2*sqrt(d*x^2 + c)*(b*x*sgn(b*x + a) + 2*a*sgn(b*x + a))
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+c}}{x} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x,x)
```

```
[Out] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x, x)
```

# 3.44 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{x^2}\,dx$

**Optimal**. Leaf size=156

$$-\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{x(a+bx)}+\frac{a\sqrt{d}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{a+bx}-\frac{b\sqrt{c}\,\sqrt{a^2+2a}}{}$$

```
[Out] -b*arctanh((d*x^2+c)^(1/2)/c^(1/2))*c^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)+a*arc
tanh(x*d^(1/2)/(d*x^2+c)^(1/2))*d^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)-(-b*x+a)*
((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x/(b*x+a)
```

**Rubi** [A]

time = 0.07, antiderivative size = 156, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 8, integrand size = 35, $\frac{\text{number of rules}}{\text{integrand size}} = 0.229$, Rules used = {1015, 827, 858, 223, 212, 272, 65, 214}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(a-bx)\sqrt{c+dx^2}}{x(a+bx)}+\frac{a\sqrt{d}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{a+bx}-\frac{b\sqrt{c}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{\sqrt{c+dx^2}}{\sqrt{c}}\right)}{a+bx}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x^2,x]
```

```
[Out] -(((a - b*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(x*(a + b*x)))
+ (a*Sqrt[d]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt[c + d*x
^2]])/(a + b*x) - (b*Sqrt[c]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[Sqrt[c +
 d*x^2]/Sqrt[c]])/(a + b*x)
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 827

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(e*f*(m + 2*p + 2) - d*g*(2*p + 1)
 + e*g*(m + 1)*x)*((a + c*x^2)^p/(e^2*(m + 1)*(m + 2*p + 2))), x] + Dist[p/
(e^2*(m + 1)*(m + 2*p + 2)), Int[(d + e*x)^(m + 1)*(a + c*x^2)^(p - 1)*Simp
[g*(2*a*e + 2*a*e*m) + (g*(2*c*d + 4*c*d*p) - 2*c*e*f*(m + 2*p + 2))*x, x],
 x], x] /; FreeQ[{a, c, d, e, f, g, m}, x] && NeQ[c*d^2 + a*e^2, 0] && Rati
onalQ[p] && p > 0 && (LtQ[m, -1] || EqQ[p, 1] || (IntegerQ[p] &&  !Rational
Q[m])) && NeQ[m, -1] &&  !ILtQ[m + 2*p + 1, 0] && (IntegerQ[m] || IntegerQ[
p] || IntegersQ[2*m, 2*p])
```

Rule 858

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + c*x^2)^p, x], x] + D
ist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + c*x^2)^p, x], x] /; FreeQ[{a, c, d,
 e, f, g, m, p}, x] && NeQ[c*d^2 + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1015

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4
*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(b + 2*c*x)^(2
*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, f, g, h, m, p, q}, x] && Eq
Q[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x^2}dx = \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{(2ab+2b^2x)\sqrt{c+dx^2}}{x^2}dx}{2ab+2b^2x}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x(a+bx)} - \frac{\sqrt{a^2+2abx+b^2x^2}\int}{2(2ab+2b^2}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x(a+bx)} + \frac{\left(2b^2c\sqrt{a^2+2abx+b^2x\right.}{2ab+}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x(a+bx)} + \frac{\left(b^2c\sqrt{a^2+2abx+b^2x^2\right.}{}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x(a+bx)} + \frac{a\sqrt{d}\sqrt{a^2+2abx+b^2x}}{a-}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x(a+bx)} + \frac{a\sqrt{d}\sqrt{a^2+2abx+b^2x}}{a-}$$

**Mathematica** [A]
time = 0.18, size = 109, normalized size = 0.70

$$\frac{\sqrt{(a+bx)^2}\left((-a+bx)\sqrt{c+dx^2}+2b\sqrt{c}\,x\tanh^{-1}\left(\frac{\sqrt{d}\,x-\sqrt{c+dx^2}}{\sqrt{c}}\right)-a\sqrt{d}\,x\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)}{x(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x^2,x]
```

```
[Out] (Sqrt[(a + b*x)^2]*((-a + b*x)*Sqrt[c + d*x^2] + 2*b*Sqrt[c]*x*ArcTanh[(Sqr
t[d]*x - Sqrt[c + d*x^2])/Sqrt[c]] - a*Sqrt[d]*x*Log[-(Sqrt[d]*x) + Sqrt[c
+ d*x^2]]))/(x*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.09, size = 120, normalized size = 0.77

| method | result |
|---|---|
| risch | $-\frac{a\sqrt{d\,x^2+c}\,\sqrt{(bx+a)^2}}{x(bx+a)}+\frac{\left(b\sqrt{d\,x^2+c}+a\sqrt{d}\,\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)-b\sqrt{c}\,\ln\left(\frac{2c+2\sqrt{c}\,\sqrt{d\,x^2+}}{x}\right.\right.}{bx+a}$ |

| | |
|---|---|
| default | $-\frac{\operatorname{csgn}(bx+a)\left(-\sqrt{d\,x^{2}+c}\,d^{\frac{3}{2}}a\,x^{2}+\sqrt{d}\,c^{\frac{3}{2}}\ln\left(\frac{2c+2\sqrt{c}\,\sqrt{d\,x^{2}+c}}{x}\right)bx+a\left(d\,x^{2}+c\right)^{\frac{3}{2}}\sqrt{d}-\sqrt{d\,x^{2}+c}\,\sqrt{d}\,bcx-}{cx\sqrt{d}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^2,x,method=_RETURNVERBOSE)
```

```
[Out] -csgn(b*x+a)*(-(d*x^2+c)^(1/2)*d^(3/2)*a*x^2+d^(1/2)*c^(3/2)*ln(2*(c^(1/2)*
(d*x^2+c)^(1/2)+c)/x)*b*x+a*(d*x^2+c)^(3/2)*d^(1/2)-(d*x^2+c)^(1/2)*d^(1/2)
*b*c*x-ln(d^(1/2)*x+(d*x^2+c)^(1/2))*a*c*d*x)/c/x/d^(1/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^2,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2)/x^2, x)
```

**Fricas [A]**
time = 0.36, size = 333, normalized size = 2.13

$$\left[\frac{a\sqrt{d}\,x\log\left(-2\,dx^{2}-2\,\sqrt{dx^{2}+c}\,\sqrt{d}\,x-c\right)+b\sqrt{c}\,x\log\left(-\frac{dx^{2}-2\,\sqrt{dx^{2}+c}\,\sqrt{c}+2\,c}{x^{2}}\right)+2\,\sqrt{dx^{2}+c}\,(bx-a)}{2\,x},\ -\frac{2\,a\sqrt{-d}\,x\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^{2}+c}}\right)-b\sqrt{c}\,x\log\left(-\frac{dx^{2}-2\,\sqrt{dx^{2}+c}\,\sqrt{c}+2\,c}{x^{2}}\right)-2\,\sqrt{dx^{2}+c}\,(bx-a)}{2\,x},\ \frac{2\,b\sqrt{-c}\,x\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^{2}+c}}\right)+a\sqrt{d}\,x\log\left(-2\,dx^{2}-2\,\sqrt{dx^{2}+c}\,\sqrt{d}\,x-c\right)+2\,\sqrt{dx^{2}+c}\,(bx-a)}{2\,x},\ -\frac{a\sqrt{-d}\,x\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^{2}+c}}\right)-b\sqrt{-c}\,x\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^{2}+c}}\right)-\sqrt{dx^{2}+c}\,(bx-a)}{x}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^2,x, algorithm="fricas")
```

```
[Out] [1/2*(a*sqrt(d)*x*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + b*sqrt(
c)*x*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(c) + 2*c)/x^2) + 2*sqrt(d*x^2 + c
)*(b*x - a))/x, -1/2*(2*a*sqrt(-d)*x*arctan(sqrt(-d)*x/sqrt(d*x^2 + c)) - b
*sqrt(c)*x*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(c) + 2*c)/x^2) - 2*sqrt(d*x
^2 + c)*(b*x - a))/x, 1/2*(2*b*sqrt(-c)*x*arctan(sqrt(-c)/sqrt(d*x^2 + c))
+ a*sqrt(d)*x*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + 2*sqrt(d*x^
2 + c)*(b*x - a))/x, -(a*sqrt(-d)*x*arctan(sqrt(-d)*x/sqrt(d*x^2 + c)) - b*
sqrt(-c)*x*arctan(sqrt(-c)/sqrt(d*x^2 + c)) - sqrt(d*x^2 + c)*(b*x - a))/x]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c+dx^{2}}\,\sqrt{\left(a+bx\right)^{2}}}{x^{2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2)/x**2,x)
[Out] Integral(sqrt(c + d*x**2)*sqrt((a + b*x)**2)/x**2, x)
```

**Giac [A]**
time = 5.59, size = 126, normalized size = 0.81

$$\frac{2\,bc\arctan\left(-\frac{\sqrt{d}\,x-\sqrt{dx^2+c}}{\sqrt{-c}}\right)\mathrm{sgn}(bx+a)}{\sqrt{-c}}-a\sqrt{d}\,\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^2+c}\right|\right)\mathrm{sgn}(bx+a)+\sqrt{dx^2+c}\,b\mathrm{sgn}(bx+a)+\frac{2\,ac\sqrt{d}\,\mathrm{sgn}(bx+a)}{\left(\sqrt{d}\,x-\sqrt{dx^2+c}\right)^2-c}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^2,x, algorithm="giac")
[Out] 2*b*c*arctan(-(sqrt(d)*x - sqrt(d*x^2 + c))/sqrt(-c))*sgn(b*x + a)/sqrt(-c)
 - a*sqrt(d)*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a) + sqrt(d*x
^2 + c)*b*sgn(b*x + a) + 2*a*c*sqrt(d)*sgn(b*x + a)/((sqrt(d)*x - sqrt(d*x^
2 + c))^2 - c)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+c}}{x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x^2,x)
[Out] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x^2, x)
```

# 3.45 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{x^3}\,dx$

**Optimal**. Leaf size=161

$$-\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\sqrt{c+dx^2}}{2x^2(a+bx)}+\frac{b\sqrt{d}\sqrt{a^2+2abx+b^2x^2}\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{a+bx}-\frac{ad\sqrt{a^2+2abx}}{}$$

```
[Out] -1/2*a*d*arctanh((d*x^2+c)^(1/2)/c^(1/2))*((b*x+a)^2)^(1/2)/(b*x+a)/c^(1/2)
+b*arctanh(x*d^(1/2)/(d*x^2+c)^(1/2))*d^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)-1/2
*(2*b*x+a)*((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^2/(b*x+a)
```

**Rubi** [A]

time = 0.07, antiderivative size = 161, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 8, integrand size = 35, $\frac{\text{number of rules}}{\text{integrand size}} = 0.229$, Rules used = {1015, 825, 858, 223, 212, 272, 65, 214}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(a+2bx)\sqrt{c+dx^2}}{2x^2(a+bx)}+\frac{b\sqrt{d}\sqrt{a^2+2abx+b^2x^2}\tanh^{-1}\left(\frac{\sqrt{d}\,x}{\sqrt{c+dx^2}}\right)}{a+bx}-\frac{ad\sqrt{a^2+2abx+b^2x^2}\tanh^{-1}\left(\frac{\sqrt{c+dx^2}}{\sqrt{c}}\right)}{2\sqrt{c}\,(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x^3,x]
```

```
[Out] -1/2*((a + 2*b*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/(x^2*(a +
b*x)) + (b*Sqrt[d]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(Sqrt[d]*x)/Sqrt[c
 + d*x^2]])/(a + b*x) - (a*d*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[Sqrt[c +
 d*x^2]/Sqrt[c]])/(2*Sqrt[c]*(a + b*x))
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 825

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Simp[(-(d + e*x)^(m + 1))*((a + c*x^2)^p/(e^2*(m + 1)*(m
+ 2)*(c*d^2 + a*e^2)))*((d*g - e*f*(m + 2))*(c*d^2 + a*e^2) - 2*c*d^2*p*(e*
f - d*g) - e*(g*(m + 1)*(c*d^2 + a*e^2) + 2*c*d*p*(e*f - d*g))*x), x] - Dis
t[p/(e^2*(m + 1)*(m + 2)*(c*d^2 + a*e^2)), Int[(d + e*x)^(m + 2)*(a + c*x^2
)^(p - 1)*Simp[2*a*c*e*(e*f - d*g)*(m + 2) - c*(2*c*d*(d*g*(2*p + 1) - e*f*
(m + 2*p + 2)) - 2*a*e^2*g*(m + 1))*x, x], x], x] /; FreeQ[{a, c, d, e, f,
g}, x] && NeQ[c*d^2 + a*e^2, 0] && GtQ[p, 0] && LtQ[m, -2] && LtQ[m + 2*p,
0] &&  !ILtQ[m + 2*p + 3, 0]
```

Rule 858

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p
_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + c*x^2)^p, x], x] + D
ist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + c*x^2)^p, x], x] /; FreeQ[{a, c, d,
 e, f, g, m, p}, x] && NeQ[c*d^2 + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1015

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4
*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(b + 2*c*x)^(2
*p)*(d + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, f, g, h, m, p, q}, x] && Eq
Q[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{x^3}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{(2ab+2b^2x)\sqrt{c+dx^2}}{x^3}\,dx}{2ab+2b^2x}$$

$$= -\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2x^2(a+bx)} - \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{\cdots}{x\cdots}}{4c\,(2ab+2b\cdots}$$

$$= -\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2x^2(a+bx)} + \frac{\left(abd\sqrt{a^2+2abx+b^2x^2}\cdots\right.}{2ab+2\cdots}$$

$$= -\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2x^2(a+bx)} + \frac{\left(abd\sqrt{a^2+2abx+b^2x^2}\cdots\right.}{2\cdots}$$

$$= -\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2x^2(a+bx)} + \frac{b\sqrt{d}\,\sqrt{a^2+2abx+b^2x\cdots}}{a+\cdots}$$

$$= -\frac{(a+2bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2}}{2x^2(a+bx)} + \frac{b\sqrt{d}\,\sqrt{a^2+2abx+b^2x\cdots}}{a+\cdots}$$

**Mathematica** [A]
time = 0.25, size = 124, normalized size = 0.77

$$\frac{\sqrt{(a+bx)^2}\left(2adx^2\tanh^{-1}\left(\frac{\sqrt{d}\,x-\sqrt{c+dx^2}}{\sqrt{c}}\right)-\sqrt{c}\left((a+2bx)\sqrt{c+dx^2}+2b\sqrt{d}\,x^2\log\left(-\sqrt{d}\,x+\sqrt{c+dx^2}\right)\right)\right)}{2\sqrt{c}\,x^2(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + d*x^2])/x^3,x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(2*a*d*x^2*ArcTanh[(Sqrt[d]*x - Sqrt[c + d*x^2])/Sqrt[c]
] - Sqrt[c]*((a + 2*b*x)*Sqrt[c + d*x^2] + 2*b*Sqrt[d]*x^2*Log[-(Sqrt[d]*x)
 + Sqrt[c + d*x^2]])))/(2*Sqrt[c]*x^2*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.11, size = 141, normalized size = 0.88

| method | result |
|---|---|
| risch | $-\frac{(2bx+a)\sqrt{(bx+a)^2}\,\sqrt{d\,x^2+c}}{2x^2(bx+a)}+\frac{\left(\sqrt{d}\,b\ln\left(\sqrt{d}\,x+\sqrt{d\,x^2+c}\right)-\frac{da\ln\left(\frac{2c+2\sqrt{c}\,\sqrt{d\,x^2+c}}{x}\right)}{2\sqrt{c}}\right)\sqrt{(bx\cdots}}{bx+a}$ |

| | |
|---|---|
| default | $-\frac{\operatorname{csgn}(bx+a)\left(-2\sqrt{d\,x^2+c}\,d^{\frac{3}{2}}b\,x^3+\sqrt{c}\,\ln\left(\frac{2c+2\sqrt{c}\,\sqrt{d\,x^2+c}}{x}\right)d^{\frac{3}{2}}a\,x^2+2\left(d\,x^2+c\right)^{\frac{3}{2}}\sqrt{d}\,bx-\sqrt{d\,x^2+c}\,d^{\frac{3}{2}}a}{2c\,x^2\sqrt{d}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^3,x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*csgn(b*x+a)*(-2*(d*x^2+c)^(1/2)*d^(3/2)*b*x^3+c^(1/2)*ln(2*(c^(1/2)*(d
*x^2+c)^(1/2)+c)/x)*d^(3/2)*a*x^2+2*(d*x^2+c)^(3/2)*d^(1/2)*b*x-(d*x^2+c)^(
1/2)*d^(3/2)*a*x^2-2*ln(d^(1/2)*x+(d*x^2+c)^(1/2))*b*c*d*x^2+a*(d*x^2+c)^(3
/2)*d^(1/2))/c/x^2/d^(1/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^3,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + c)*sqrt((b*x + a)^2)/x^3, x)
```

**Fricas [A]**
time = 0.41, size = 377, normalized size = 2.34

$$\left[\frac{2\,bc\sqrt{d}\,x^2\log\left(-2\,dx^2-2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)+a\sqrt{c}\,dx^2\log\left(-\frac{dx^2-2\,\sqrt{dx^2+c}\,\sqrt{c}+2\,c}{x^2}\right)-2\,(2\,bcx+ac)\sqrt{dx^2+c}}{4\,cx^2},\ -\frac{4\,bc\sqrt{-d}\,x^2\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)-a\sqrt{c}\,dx^2\log\left(-\frac{dx^2-2\,\sqrt{dx^2+c}\,\sqrt{c}+2\,c}{x^2}\right)+2\,(2\,bcx+ac)\sqrt{dx^2+c}}{4\,cx^2},\ \frac{a\sqrt{-c}\,dx^2\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^2+c}}\right)+bc\sqrt{d}\,x^2\log\left(-2\,dx^2-2\,\sqrt{dx^2+c}\,\sqrt{d}\,x-c\right)-(2\,bcx+ac)\sqrt{dx^2+c}}{2\,cx^2},\ -\frac{2\,bc\sqrt{-d}\,x^2\arctan\left(\frac{\sqrt{-d}\,x}{\sqrt{dx^2+c}}\right)-a\sqrt{-c}\,dx^2\arctan\left(\frac{\sqrt{-c}}{\sqrt{dx^2+c}}\right)+(2\,bcx+ac)\sqrt{dx^2+c}}{2\,cx^2}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^3,x, algorithm="fricas")
```

```
[Out] [1/4*(2*b*c*sqrt(d)*x^2*log(-2*d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) + a
*sqrt(c)*d*x^2*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sqrt(c) + 2*c)/x^2) - 2*(2*b
*c*x + a*c)*sqrt(d*x^2 + c))/(c*x^2), -1/4*(4*b*c*sqrt(-d)*x^2*arctan(sqrt(
-d)*x/sqrt(d*x^2 + c)) - a*sqrt(c)*d*x^2*log(-(d*x^2 - 2*sqrt(d*x^2 + c)*sq
rt(c) + 2*c)/x^2) + 2*(2*b*c*x + a*c)*sqrt(d*x^2 + c))/(c*x^2), 1/2*(a*sqrt
(-c)*d*x^2*arctan(sqrt(-c)/sqrt(d*x^2 + c)) + b*c*sqrt(d)*x^2*log(-2*d*x^2
- 2*sqrt(d*x^2 + c)*sqrt(d)*x - c) - (2*b*c*x + a*c)*sqrt(d*x^2 + c))/(c*x^
2), -1/2*(2*b*c*sqrt(-d)*x^2*arctan(sqrt(-d)*x/sqrt(d*x^2 + c)) - a*sqrt(-c
)*d*x^2*arctan(sqrt(-c)/sqrt(d*x^2 + c)) + (2*b*c*x + a*c)*sqrt(d*x^2 + c))
/(c*x^2)]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c+dx^2}\,\sqrt{(a+bx)^2}}{x^3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+c)**(1/2)/x**3,x)
```

```
[Out] Integral(sqrt(c + d*x**2)*sqrt((a + b*x)**2)/x**3, x)
```

**Giac [A]**
time = 4.63, size = 199, normalized size = 1.24

$$\frac{ad\arctan\left(-\frac{\sqrt{d}\,x-\sqrt{dx^2+c}}{\sqrt{-c}}\right)\operatorname{sgn}(bx+a)}{\sqrt{-c}} - b\sqrt{d}\,\log\left(\left|-\sqrt{d}\,x+\sqrt{dx^2+c}\right|\right)\operatorname{sgn}(bx+a) + \frac{\left(\sqrt{d}\,x-\sqrt{dx^2+c}\right)^3 ad\operatorname{sgn}(bx+a) + 2\left(\sqrt{d}\,x-\sqrt{dx^2+c}\right)^2 bc\sqrt{d}\operatorname{sgn}(bx+a) + \left(\sqrt{d}\,x-\sqrt{dx^2+c}\right)acd\operatorname{sgn}(bx+a) - 2bc^2\sqrt{d}\operatorname{sgn}(bx+a)}{\left(\left(\sqrt{d}\,x-\sqrt{dx^2+c}\right)^2-c\right)^2}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+c)^(1/2)/x^3,x, algorithm="giac")
```

```
[Out] a*d*arctan(-(sqrt(d)*x - sqrt(d*x^2 + c))/sqrt(-c))*sgn(b*x + a)/sqrt(-c) -
 b*sqrt(d)*log(abs(-sqrt(d)*x + sqrt(d*x^2 + c)))*sgn(b*x + a) + ((sqrt(d)*
x - sqrt(d*x^2 + c))^3*a*d*sgn(b*x + a) + 2*(sqrt(d)*x - sqrt(d*x^2 + c))^2
*b*c*sqrt(d)*sgn(b*x + a) + (sqrt(d)*x - sqrt(d*x^2 + c))*a*c*d*sgn(b*x + a
) - 2*b*c^2*sqrt(d)*sgn(b*x + a))/((sqrt(d)*x - sqrt(d*x^2 + c))^2 - c)^2
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+c}}{x^3} \,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x^3,x)
```

```
[Out] int((((a + b*x)^2)^(1/2)*(c + d*x^2)^(1/2))/x^3, x)
```

# 3.46 $\int x^2 \sqrt{a^2+2abx+b^2x^2} \sqrt{c+ex+dx^2}\, dx$

**Optimal**. Leaf size=317

$$-\frac{(2ad(4cd-5e^2)-b(12cde-7e^3))\,(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{128d^4(a+bx)}+\frac{bx^2\sqrt{a^2+2abx+b^2x^2}}{5d(a+}$$

```
[Out] 1/5*b*x^2*(d*x^2+e*x+c)^(3/2)*((b*x+a)^2)^(1/2)/d/(b*x+a)-1/240*(32*b*c*d+5
0*a*d*e-35*b*e^2-6*d*(10*a*d-7*b*e)*x)*(d*x^2+e*x+c)^(3/2)*((b*x+a)^2)^(1/2
)/d^3/(b*x+a)-1/256*(4*c*d-e^2)*(8*a*c*d^2-10*a*d*e^2-12*b*c*d*e+7*b*e^3)*a
rctanh(1/2*(2*d*x+e)/d^(1/2)/(d*x^2+e*x+c)^(1/2))*((b*x+a)^2)^(1/2)/d^(9/2)
/(b*x+a)-1/128*(2*a*d*(4*c*d-5*e^2)-b*(12*c*d*e-7*e^3))*(2*d*x+e)*((b*x+a)^
2)^(1/2)*(d*x^2+e*x+c)^(1/2)/d^4/(b*x+a)
```

**Rubi** [A]
time = 0.19, antiderivative size = 317, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 6, integrand size = 38, $\frac{\text{number of rules}}{\text{integrand size}} = 0.158$, Rules used = {1014, 846, 793, 626, 635, 212}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(2dx+e)\sqrt{c+dx^2+ex}\,(2ad(4cd-5e^2)-b(12cde-7e^3))}{128d^4(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2+ex)^{3/2}\,(-6dx(10ad-7be)+50ade+32bcd-35be^2)}{240d^3(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,(4cd-e^2)\,(8acd^2-10ade^2-12bcde+7be^3)\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{256d^{9/2}(a+bx)}+\frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2+ex)^{3/2}}{5d(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[x^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

```
[Out] -1/128*((2*a*d*(4*c*d - 5*e^2) - b*(12*c*d*e - 7*e^3))*(e + 2*d*x)*Sqrt[a^2
 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/(d^4*(a + b*x)) + (b*x^2*Sqrt[
a^2 + 2*a*b*x + b^2*x^2]*(c + e*x + d*x^2)^(3/2))/(5*d*(a + b*x)) - ((32*b*
c*d + 50*a*d*e - 35*b*e^2 - 6*d*(10*a*d - 7*b*e)*x)*Sqrt[a^2 + 2*a*b*x + b^
2*x^2]*(c + e*x + d*x^2)^(3/2))/(240*d^3*(a + b*x)) - ((4*c*d - e^2)*(8*a*c
*d^2 - 12*b*c*d*e - 10*a*d*e^2 + 7*b*e^3)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Arc
Tanh[(e + 2*d*x)/(2*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(256*d^(9/2)*(a + b*x)
)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 793

```
Int[((d_.) + (e_.)*(x_))*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(
x_)^2)^(p_), x_Symbol] :> Simp[(-(b*e*g*(p + 2) - c*(e*f + d*g)*(2*p + 3) -
 2*c*e*g*(p + 1)*x))*((a + b*x + c*x^2)^(p + 1)/(2*c^2*(p + 1)*(2*p + 3))),
 x] + Dist[(b^2*e*g*(p + 2) - 2*a*c*e*g + c*(2*c*d*f - b*(e*f + d*g))*(2*p
+ 3))/(2*c^2*(2*p + 3)), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, g, p}, x] && NeQ[b^2 - 4*a*c, 0] &&  !LeQ[p, -1]
```

Rule 846

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[g*(d + e*x)^m*((a + b*x + c*x^2)^(p +
1)/(c*(m + 2*p + 2))), x] + Dist[1/(c*(m + 2*p + 2)), Int[(d + e*x)^(m - 1)
*(a + b*x + c*x^2)^p*Simp[m*(c*d*f - a*e*g) + d*(2*c*f - b*g)*(p + 1) + (m*
(c*e*f + c*d*g - b*e*g) + e*(p + 1)*(2*c*f - b*g))*x, x], x], x] /; FreeQ[{
a, b, c, d, e, f, g, p}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a
*e^2, 0] && GtQ[m, 0] && NeQ[m + 2*p + 2, 0] && (IntegerQ[m] || IntegerQ[p]
 || IntegersQ[2*m, 2*p]) &&  !(IGtQ[m, 0] && EqQ[f, 0])
```

Rule 1014

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^Fr
acPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(
b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, e, f, g,
 h, m, p, q}, x] && EqQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int x^2\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int x^2(2ab+2b^2x)\,\sqrt{c+ex+dx^2}\,dx}{2ab+2b^2x}$$

$$= \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+ex+dx^2)^{3/2}}{5d(a+bx)} + \frac{\sqrt{a^2+2abx+b}}{}$$

$$= \frac{bx^2\sqrt{a^2+2abx+b^2x^2}\,(c+ex+dx^2)^{3/2}}{5d(a+bx)} - \frac{(32bcd+50ade}{}$$

$$= -\frac{(8acd^2-12bcde-10ade^2+7be^3)\,(e+2dx)\sqrt{a^2+2abx+}}{128d^4(a+bx)}$$

$$= -\frac{(8acd^2-12bcde-10ade^2+7be^3)\,(e+2dx)\sqrt{a^2+2abx+}}{128d^4(a+bx)}$$

$$= -\frac{(8acd^2-12bcde-10ade^2+7be^3)\,(e+2dx)\sqrt{a^2+2abx+}}{128d^4(a+bx)}$$

**Mathematica [A]**
time = 0.63, size = 236, normalized size = 0.74

$$\frac{\sqrt{(a+bx)^2}\left(2\sqrt{d}\,\sqrt{c+x(e+dx)}\,(10ad(15e^3-10de^2x+8d^2ex^2+48d^3x^3+4cd(-13e+6dx))+b(-256c^2d^2-105e^4+70de^3x-56d^2e^2x^2+48d^3ex^3+384d^4x^4+4cd(115e^2-58dex+32d^2x^2)))+15(4cd-e^2)(2ad(4cd-5e^2)+b(-12cde+7e^3))\log\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)}{3840d^{9/2}(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(2*Sqrt[d]*Sqrt[c + x*(e + d*x)]*(10*a*d*(15*e^3 - 10*d*
e^2*x + 8*d^2*e*x^2 + 48*d^3*x^3 + 4*c*d*(-13*e + 6*d*x)) + b*(-256*c^2*d^2
 - 105*e^4 + 70*d*e^3*x - 56*d^2*e^2*x^2 + 48*d^3*e*x^3 + 384*d^4*x^4 + 4*c
*d*(115*e^2 - 58*d*e*x + 32*d^2*x^2))) + 15*(4*c*d - e^2)*(2*a*d*(4*c*d - 5
*e^2) + b*(-12*c*d*e + 7*e^3))*Log[e + 2*d*x - 2*Sqrt[d]*Sqrt[c + x*(e + d*
x)]]))/(3840*d^(9/2)*(a + b*x))
```

**Maple [C]** Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.16, size = 530, normalized size = 1.67

| method | result |
|---|---|
| risch | $\frac{(384b\,x^4d^4+480a\,d^4x^3+48b\,d^3e\,x^3+80a\,d^3e\,x^2+128bc\,d^3x^2-56b\,d^2e^2x^2+240ac\,d^3x-100a\,d^2e^2x-232bc\,d^2ex+70bd\,e^3x-520ac\,d^2e-}{1920d^4(bx+a)}$ |

| | |
|---|---|
| default | $-\frac{\mathrm{csgn}(bx+a)\left(-768d^{\frac{9}{2}}\left(d\,x^{2}+ex+c\right)^{\frac{3}{2}}b\,x^{2}-960d^{\frac{9}{2}}\left(d\,x^{2}+ex+c\right)^{\frac{3}{2}}ax+672d^{\frac{7}{2}}\left(d\,x^{2}+ex+c\right)^{\frac{3}{2}}bex+800d^{\frac{7}{2}}\left(d\,x^{2}+ex+c\right)^{\frac{3}{2}}ae+512d^{\frac{7}{2}}(d\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/3840*csgn(b*x+a)*(-768*d^(9/2)*(d*x^2+e*x+c)^(3/2)*b*x^2-960*d^(9/2)*(d*
x^2+e*x+c)^(3/2)*a*x+672*d^(7/2)*(d*x^2+e*x+c)^(3/2)*b*e*x+800*d^(7/2)*(d*x
^2+e*x+c)^(3/2)*a*e+512*d^(7/2)*(d*x^2+e*x+c)^(3/2)*b*c-560*d^(5/2)*(d*x^2+
e*x+c)^(3/2)*b*e^2+480*d^(9/2)*(d*x^2+e*x+c)^(1/2)*a*c*x-600*d^(7/2)*(d*x^2
+e*x+c)^(1/2)*a*e^2*x-720*d^(7/2)*(d*x^2+e*x+c)^(1/2)*b*c*e*x+420*d^(5/2)*(
d*x^2+e*x+c)^(1/2)*b*e^3*x+240*d^(7/2)*(d*x^2+e*x+c)^(1/2)*a*c*e-300*d^(5/2
)*(d*x^2+e*x+c)^(1/2)*a*e^3-360*d^(5/2)*(d*x^2+e*x+c)^(1/2)*b*c*e^2+210*d^(
3/2)*(d*x^2+e*x+c)^(1/2)*b*e^4+480*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*
d*x+e)/d^(1/2))*a*c^2*d^4-720*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e
)/d^(1/2))*a*c*d^3*e^2+150*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d
^(1/2))*a*d^2*e^4-720*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2
))*b*c^2*d^3*e+600*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*
b*c*d^2*e^3-105*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*b*d
*e^5)/d^(11/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2)*x^2, x)
```

**Fricas [A]**
time = 0.39, size = 516, normalized size = 1.63

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="fricas")
```

```
[Out] [-1/7680*(15*(32*a*c^2*d^3 - 48*b*c^2*d^2*e - 48*a*c*d^2*e^2 + 40*b*c*d*e^3
 + 10*a*d*e^4 - 7*b*e^5)*sqrt(d)*log(8*d^2*x^2 + 8*d*x*e + 4*sqrt(d*x^2 + x
*e + c)*(2*d*x + e)*sqrt(d) + 4*c*d + e^2) - 4*(384*b*d^5*x^4 + 480*a*d^5*x
^3 + 128*b*c*d^4*x^2 + 240*a*c*d^4*x - 256*b*c^2*d^3 - 105*b*d*e^4 + 10*(7*
b*d^2*x + 15*a*d^2)*e^3 - 4*(14*b*d^3*x^2 + 25*a*d^3*x - 115*b*c*d^2)*e^2 +
```

```
8*(6*b*d^4*x^3 + 10*a*d^4*x^2 - 29*b*c*d^3*x - 65*a*c*d^3)*e)*sqrt(d*x^2 +
 x*e + c))/d^5, 1/3840*(15*(32*a*c^2*d^3 - 48*b*c^2*d^2*e - 48*a*c*d^2*e^2
+ 40*b*c*d*e^3 + 10*a*d*e^4 - 7*b*e^5)*sqrt(-d)*arctan(1/2*sqrt(d*x^2 + x*e
 + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) + 2*(384*b*d^5*x^4 + 48
0*a*d^5*x^3 + 128*b*c*d^4*x^2 + 240*a*c*d^4*x - 256*b*c^2*d^3 - 105*b*d*e^4
 + 10*(7*b*d^2*x + 15*a*d^2)*e^3 - 4*(14*b*d^3*x^2 + 25*a*d^3*x - 115*b*c*d
^2)*e^2 + 8*(6*b*d^4*x^3 + 10*a*d^4*x^2 - 29*b*c*d^3*x - 65*a*c*d^3)*e)*sqr
t(d*x^2 + x*e + c))/d^5]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int x^2 \sqrt{c + dx^2 + ex}\, \sqrt{(a + bx)^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2),x)
[Out] Integral(x**2*sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2), x)
```

**Giac [A]**
time = 4.87, size = 368, normalized size = 1.16

$$\frac{1}{1920}\sqrt{dx^2+xe+c}\left(2\left(4\left(6\left(8\,bx\mathrm{sgn}(bx+a)+\frac{10\,ad^4\mathrm{sgn}(bx+a)+bd^3e\mathrm{sgn}(bx+a)}{d^4}\right)x+\frac{16\,bcd^3\mathrm{sgn}(bx+a)+10\,ad^3e\mathrm{sgn}(bx+a)-7\,bd^2e^2\mathrm{sgn}(bx+a)}{d^4}\right)x+\frac{120\,acd^3\mathrm{sgn}(bx+a)-116\,bcd^2e\mathrm{sgn}(bx+a)-50\,ad^2e^2\mathrm{sgn}(bx+a)+35\,bde^3\mathrm{sgn}(bx+a)}{d^4}\right)x-\frac{256\,bc^2d^2\mathrm{sgn}(bx+a)+520\,acd^2e\mathrm{sgn}(bx+a)-460\,bcde^2\mathrm{sgn}(bx+a)-150\,ade^3\mathrm{sgn}(bx+a)+105\,be^4\mathrm{sgn}(bx+a)}{d^4}\right)+\frac{\left(32\,ac^2d^3\mathrm{sgn}(bx+a)-48\,bc^2d^2e\mathrm{sgn}(bx+a)-48\,acd^2e^2\mathrm{sgn}(bx+a)+40\,bcde^3\mathrm{sgn}(bx+a)+10\,ade^4\mathrm{sgn}(bx+a)-7\,be^5\mathrm{sgn}(bx+a)\right)\log\left(\left|-2\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)\sqrt{d}-e\right|\right)}{256\,d^{\frac{9}{2}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="giac")
[Out] 1/1920*sqrt(d*x^2 + x*e + c)*(2*(4*(6*(8*b*x*sgn(b*x + a) + (10*a*d^4*sgn(b
*x + a) + b*d^3*e*sgn(b*x + a))/d^4)*x + (16*b*c*d^3*sgn(b*x + a) + 10*a*d^
3*e*sgn(b*x + a) - 7*b*d^2*e^2*sgn(b*x + a))/d^4)*x + (120*a*c*d^3*sgn(b*x
+ a) - 116*b*c*d^2*e*sgn(b*x + a) - 50*a*d^2*e^2*sgn(b*x + a) + 35*b*d*e^3*
sgn(b*x + a))/d^4)*x - (256*b*c^2*d^2*sgn(b*x + a) + 520*a*c*d^2*e*sgn(b*x
+ a) - 460*b*c*d*e^2*sgn(b*x + a) - 150*a*d*e^3*sgn(b*x + a) + 105*b*e^4*sg
n(b*x + a))/d^4) + 1/256*(32*a*c^2*d^3*sgn(b*x + a) - 48*b*c^2*d^2*e*sgn(b*
x + a) - 48*a*c*d^2*e^2*sgn(b*x + a) + 40*b*c*d*e^3*sgn(b*x + a) + 10*a*d*e
^4*sgn(b*x + a) - 7*b*e^5*sgn(b*x + a))*log(abs(-2*(sqrt(d)*x - sqrt(d*x^2
+ x*e + c))*sqrt(d) - e))/d^(9/2)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int x^2\,\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+e\,x+c}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2),x)
[Out] int(x^2*((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2), x)
```

# 3.47 $\int x\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}\,dx$

**Optimal**. Leaf size=227

$$-\frac{(4bcd+8ade-5be^2)\,(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{64d^3(a+bx)}+\frac{(8ad-5be+6bdx)\sqrt{a^2+2abx+b^2x}}{24d^2(a+bx)}$$

```
[Out] 1/24*(6*b*d*x+8*a*d-5*b*e)*(d*x^2+e*x+c)^(3/2)*((b*x+a)^2)^(1/2)/d^2/(b*x+a
)-1/128*(4*c*d-e^2)*(8*a*d*e+4*b*c*d-5*b*e^2)*arctanh(1/2*(2*d*x+e)/d^(1/2)
/(d*x^2+e*x+c)^(1/2))*((b*x+a)^2)^(1/2)/d^(7/2)/(b*x+a)-1/64*(8*a*d*e+4*b*c
*d-5*b*e^2)*(2*d*x+e)*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/d^3/(b*x+a)
```

**Rubi** [A]
time = 0.08, antiderivative size = 227, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 36, $\frac{\text{number of rules}}{\text{integrand size}} = 0.139$, Rules used = {1014, 793, 626, 635, 212}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(4cd-e^2)\,(8ade+4bcd-5be^2)\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{128d^{7/2}(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,(2dx+e)\sqrt{c+dx^2+ex}\,(8ade+4bcd-5be^2)}{64d^3(a+bx)}+\frac{\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2+ex)^{3/2}\,(8ad+6bdx-5be)}{24d^2(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

```
[Out] -1/64*((4*b*c*d + 8*a*d*e - 5*b*e^2)*(e + 2*d*x)*Sqrt[a^2 + 2*a*b*x + b^2*x
^2]*Sqrt[c + e*x + d*x^2])/(d^3*(a + b*x)) + ((8*a*d - 5*b*e + 6*b*d*x)*Sqr
t[a^2 + 2*a*b*x + b^2*x^2]*(c + e*x + d*x^2)^(3/2))/(24*d^2*(a + b*x)) - ((
4*c*d - e^2)*(4*b*c*d + 8*a*d*e - 5*b*e^2)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Ar
cTanh[(e + 2*d*x)/(2*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(128*d^(7/2)*(a + b*x
))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 793

```
Int[((d_.) + (e_.)*(x_))*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(
x_)^2)^(p_), x_Symbol] :> Simp[(-(b*e*g*(p + 2) - c*(e*f + d*g)*(2*p + 3) -
 2*c*e*g*(p + 1)*x))*((a + b*x + c*x^2)^(p + 1)/(2*c^2*(p + 1)*(2*p + 3))),
 x] + Dist[(b^2*e*g*(p + 2) - 2*a*c*e*g + c*(2*c*d*f - b*(e*f + d*g))*(2*p
+ 3))/(2*c^2*(2*p + 3)), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c,
 d, e, f, g, p}, x] && NeQ[b^2 - 4*a*c, 0] &&  !LeQ[p, -1]
```

Rule 1014

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^Fr
acPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(
b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, e, f, g,
 h, m, p, q}, x] && EqQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int x\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int x(2ab+2b^2x)\,\sqrt{c+ex+dx^2}\,dx}{2ab+2b^2x}$$

$$= \frac{(8ad-5be+6bdx)\sqrt{a^2+2abx+b^2x^2}\,(c+ex+dx^2)^{3/2}}{24d^2(a+bx)} - \ldots$$

$$= -\frac{(4bcd+8ade-5be^2)\,(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex\ldots}}{64d^3(a+bx)}$$

$$= -\frac{(4bcd+8ade-5be^2)\,(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex\ldots}}{64d^3(a+bx)}$$

$$= -\frac{(4bcd+8ade-5be^2)\,(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex\ldots}}{64d^3(a+bx)}$$

**Mathematica [A]**
time = 0.46, size = 176, normalized size = 0.78

$$\frac{\sqrt{(a+bx)^2}\,\left(2\sqrt{d}\,\sqrt{c+x(e+dx)}\,(8ad(8cd-3e^2+2dex+8d^2x^2)+b(15e^3-10de^2x+8d^2ex^2+48d^3x^3+4cd(-13e+6dx)))+3(4cd-e^2)\,(4bcd+8ade-5be^2)\log\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)}{384d^{7/2}(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[x*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(2*Sqrt[d]*Sqrt[c + x*(e + d*x)]*(8*a*d*(8*c*d - 3*e^2 +
 2*d*e*x + 8*d^2*x^2) + b*(15*e^3 - 10*d*e^2*x + 8*d^2*e*x^2 + 48*d^3*x^3 +
 4*c*d*(-13*e + 6*d*x))) + 3*(4*c*d - e^2)*(4*b*c*d + 8*a*d*e - 5*b*e^2)*Lo
g[e + 2*d*x - 2*Sqrt[d]*Sqrt[c + x*(e + d*x)]]))/(384*d^(7/2)*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.10, size = 381, normalized size = 1.68

| method | result |
|---|---|
| risch | $\frac{\left(48 b\,x^{3} d^{3}+64 a\,d^{3} x^{2}+8 b\,d^{2} e\,x^{2}+16 a\,d^{2} e x+24 b c\,d^{2} x-10 b d\,e^{2} x+64 a c\,d^{2}-24 d\,e^{2} a-52 b c d e+15 e^{3} b\right) \sqrt{d\,x^{2}+e x+c}\,\sqrt{(b x+a}}{192 d^{3} \left(b x+a\right)}$ |
| default | $\mathrm{csgn}\left(b x+a\right) \left(96 \left(d\,x^{2}+e x+c\right)^{\frac{3}{2}} d^{\frac{7}{2}} b x+128 \left(d\,x^{2}+e x+c\right)^{\frac{3}{2}} d^{\frac{7}{2}} a-80 \left(d\,x^{2}+e x+c\right)^{\frac{3}{2}} d^{\frac{5}{2}} b e-96 \sqrt{d\,x^{2}+e x+c}\, d^{\frac{7}{2}} a e x-48 \sqrt{d\,x^{2}+}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/384*csgn(b*x+a)*(96*(d*x^2+e*x+c)^(3/2)*d^(7/2)*b*x+128*(d*x^2+e*x+c)^(3/
2)*d^(7/2)*a-80*(d*x^2+e*x+c)^(3/2)*d^(5/2)*b*e-96*(d*x^2+e*x+c)^(1/2)*d^(7
/2)*a*e*x-48*(d*x^2+e*x+c)^(1/2)*d^(7/2)*b*c*x+60*(d*x^2+e*x+c)^(1/2)*d^(5/
2)*b*e^2*x-48*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a*e^2-24*(d*x^2+e*x+c)^(1/2)*d^(5
/2)*b*c*e+30*(d*x^2+e*x+c)^(1/2)*d^(3/2)*b*e^3-96*ln(1/2*(2*(d*x^2+e*x+c)^(
1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*a*c*d^3*e+24*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d
^(1/2)+2*d*x+e)/d^(1/2))*a*d^2*e^3-48*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)
+2*d*x+e)/d^(1/2))*b*c^2*d^3+72*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x
+e)/d^(1/2))*b*c*d^2*e^2-15*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/
d^(1/2))*b*d*e^4)/d^(9/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2)*x, x)
```

**Fricas** [A]
time = 0.52, size = 394, normalized size = 1.74

$$\left[\frac{3 \, {\left(16 \, b c^{2} d^{2} + 32 \, a c d^{2} e - 24 \, b c d e^{2} - 8 \, a d e^{3} + 5 \, b e^{4}\right)} \sqrt{d} \log\left(8 \, d^{2} x^{2} + 8 \, d x e - 4 \, \sqrt{d x^{2} + x e + c} {\left(2 \, d x + e\right)} \sqrt{d} + 4 \, c d + e^{2}\right) + 4 \, {\left(48 \, b d^{4} x^{3} + 64 \, a d^{4} x^{2} + 24 \, b c d^{3} x + 64 \, a c d^{3} + 15 \, b d e^{3} - 2 \, {\left(5 \, b d^{2} x + 12 \, a d^{2}\right)} e^{2} + 4 \, {\left(2 \, b d^{3} x^{2} + 4 \, a d^{3} x - 13 \, b c d^{2}\right)} e\right)} \sqrt{d x^{2} + x e + c}}{768 \, d^{4}}, \frac{3 \, {\left(16 \, b c^{2} d^{2} + 32 \, a c d^{2} e - 24 \, b c d e^{2} - 8 \, a d e^{3} + 5 \, b e^{4}\right)} \sqrt{-d} \arctan\left(\frac{\sqrt{d x^{2} + x e + c} {\left(2 \, d x + e\right)} \sqrt{-d}}{2 \, {\left(d^{2} x^{2} + d x e + c d\right)}}\right) + 2 \, {\left(48 \, b d^{4} x^{3} + 64 \, a d^{4} x^{2} + 24 \, b c d^{3} x + 64 \, a c d^{3} + 15 \, b d e^{3} - 2 \, {\left(5 \, b d^{2} x + 12 \, a d^{2}\right)} e^{2} + 4 \, {\left(2 \, b d^{3} x^{2} + 4 \, a d^{3} x - 13 \, b c d^{2}\right)} e\right)} \sqrt{d x^{2} + x e + c}}{384 \, d^{4}}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="fricas")
```

```
[Out] [1/768*(3*(16*b*c^2*d^2 + 32*a*c*d^2*e - 24*b*c*d*e^2 - 8*a*d*e^3 + 5*b*e^4
)*sqrt(d)*log(8*d^2*x^2 + 8*d*x*e - 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqr
t(d) + 4*c*d + e^2) + 4*(48*b*d^4*x^3 + 64*a*d^4*x^2 + 24*b*c*d^3*x + 64*a*
c*d^3 + 15*b*d*e^3 - 2*(5*b*d^2*x + 12*a*d^2)*e^2 + 4*(2*b*d^3*x^2 + 4*a*d^
3*x - 13*b*c*d^2)*e)*sqrt(d*x^2 + x*e + c))/d^4, 1/384*(3*(16*b*c^2*d^2 + 3
2*a*c*d^2*e - 24*b*c*d*e^2 - 8*a*d*e^3 + 5*b*e^4)*sqrt(-d)*arctan(1/2*sqrt(
d*x^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) + 2*(48*b*d^
4*x^3 + 64*a*d^4*x^2 + 24*b*c*d^3*x + 64*a*c*d^3 + 15*b*d*e^3 - 2*(5*b*d^2*
x + 12*a*d^2)*e^2 + 4*(2*b*d^3*x^2 + 4*a*d^3*x - 13*b*c*d^2)*e)*sqrt(d*x^2
+ x*e + c))/d^4]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int x\sqrt{c + dx^2 + ex}\,\sqrt{(a + bx)^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2),x)
```

```
[Out] Integral(x*sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2), x)
```

**Giac [A]**
time = 4.21, size = 268, normalized size = 1.18

$$\frac{1}{192}\sqrt{dx^2 + xe + c}\left(2\left(4\left(6\,bx\mathrm{sgn}(bx+a) + \frac{8\,ad^3\mathrm{sgn}(bx+a) + bd^2e\mathrm{sgn}(bx+a)}{d^3}\right)x + \frac{12\,bcd^2\mathrm{sgn}(bx+a) + 8\,ad^2e\mathrm{sgn}(bx+a) - 5\,bde^2\mathrm{sgn}(bx+a)}{d^3}\right)x + \frac{64\,acd^2\mathrm{sgn}(bx+a) - 52\,bcde\mathrm{sgn}(bx+a) - 24\,ade^2\mathrm{sgn}(bx+a) + 15\,be^3\mathrm{sgn}(bx+a)}{d^3}\right) + \frac{(16\,bc^2d^2\mathrm{sgn}(bx+a) + 32\,acd^2e\mathrm{sgn}(bx+a) - 24\,bcde^2\mathrm{sgn}(bx+a) - 8\,ade^3\mathrm{sgn}(bx+a) + 5\,be^4\mathrm{sgn}(bx+a))\log\left(\left|-2\left(\sqrt{d}x - \sqrt{dx^2 + xe + c}\right)\sqrt{d} - e\right|\right)}{128\,d^{\frac{7}{2}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="giac")
```

```
[Out] 1/192*sqrt(d*x^2 + x*e + c)*(2*(4*(6*b*x*sgn(b*x + a) + (8*a*d^3*sgn(b*x +
a) + b*d^2*e*sgn(b*x + a))/d^3)*x + (12*b*c*d^2*sgn(b*x + a) + 8*a*d^2*e*sg
n(b*x + a) - 5*b*d*e^2*sgn(b*x + a))/d^3)*x + (64*a*c*d^2*sgn(b*x + a) - 52
*b*c*d*e*sgn(b*x + a) - 24*a*d*e^2*sgn(b*x + a) + 15*b*e^3*sgn(b*x + a))/d^
3) + 1/128*(16*b*c^2*d^2*sgn(b*x + a) + 32*a*c*d^2*e*sgn(b*x + a) - 24*b*c*
d*e^2*sgn(b*x + a) - 8*a*d*e^3*sgn(b*x + a) + 5*b*e^4*sgn(b*x + a))*log(abs
(-2*(sqrt(d)*x - sqrt(d*x^2 + x*e + c))*sqrt(d) - e))/d^(7/2)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int x\,\sqrt{{(a+b\,x)}^2}\,\sqrt{d\,x^2+e\,x+c}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2),x)
[Out] int(x*((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2), x)
```

# 3.48 $\int \sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}\,dx$

**Optimal**. Leaf size=198

$$\frac{(2ad-be)(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{8d^2(a+bx)}+\frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+ex+dx^2)^{3/2}}{3d(a+bx)}+\frac{(2ad-b}{}$$

```
[Out] 1/3*b*(d*x^2+e*x+c)^(3/2)*((b*x+a)^2)^(1/2)/d/(b*x+a)+1/16*(2*a*d-b*e)*(4*c
*d-e^2)*arctanh(1/2*(2*d*x+e)/d^(1/2)/(d*x^2+e*x+c)^(1/2))*((b*x+a)^2)^(1/2
)/d^(5/2)/(b*x+a)+1/8*(2*a*d-b*e)*(2*d*x+e)*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)
^(1/2)/d^2/(b*x+a)
```

**Rubi** [A]
time = 0.06, antiderivative size = 198, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 35, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {983, 654, 626, 635, 212}

$$\frac{\sqrt{a^2+2abx+b^2x^2}\,(4cd-e^2)\,(2ad-be)\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{16d^{5/2}(a+bx)}+\frac{\sqrt{a^2+2abx+b^2x^2}\,(2dx+e)(2ad-be)\sqrt{c+dx^2+ex}}{8d^2(a+bx)}+\frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+dx^2+ex)^{3/2}}{3d(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

```
[Out] ((2*a*d - b*e)*(e + 2*d*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x
^2])/(8*d^2*(a + b*x)) + (b*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*(c + e*x + d*x^2)
^(3/2))/(3*d*(a + b*x)) + ((2*a*d - b*e)*(4*c*d - e^2)*Sqrt[a^2 + 2*a*b*x +
 b^2*x^2]*ArcTanh[(e + 2*d*x)/(2*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(16*d^(5/
2)*(a + b*x))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 983

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f_.)*(x_)
^2)^(q_.), x_Symbol] :> Dist[(a + b*x + c*x^2)^FracPart[p]/((4*c)^IntPart[p
]*(b + 2*c*x)^(2*FracPart[p])), Int[(b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q,
x], x] /; FreeQ[{a, b, c, d, e, f, p, q}, x] && EqQ[b^2 - 4*a*c, 0] &&  !In
tegerQ[p]
```

Rubi steps

$$\int \sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int (2ab+2b^2x)\sqrt{c+ex+dx^2}\,dx}{2ab+2b^2x}$$

$$= \frac{b\sqrt{a^2+2abx+b^2x^2}\,(c+ex+dx^2)^{3/2}}{3d(a+bx)} + \frac{\left(b(2ad-be)\sqrt{a^2+2a\right.}{d}$$

$$= \frac{(2ad-be)(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{8d^2(a+bx)} + \frac{b\sqrt{a}}{}$$

$$= \frac{(2ad-be)(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{8d^2(a+bx)} + \frac{b\sqrt{a}}{}$$

$$= \frac{(2ad-be)(e+2dx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{8d^2(a+bx)} + \frac{b\sqrt{a}}{}$$

**Mathematica** [A]
time = 0.40, size = 136, normalized size = 0.69

$$\frac{\sqrt{(a+bx)^2}\left(2\sqrt{d}\,\sqrt{c+x(e+dx)}\,(6ad(e+2dx)+b(8cd-3e^2+2dex+8d^2x^2))-3(2ad-be)\,(4cd-e^2)\log\left(d^2\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)\right)}{48d^{5/2}(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2],x]
```

[Out] (Sqrt[(a + b*x)^2]*(2*Sqrt[d]*Sqrt[c + x*(e + d*x)]*(6*a*d*(e + 2*d*x) + b*(8*c*d - 3*e^2 + 2*d*e*x + 8*d^2*x^2)) - 3*(2*a*d - b*e)*(4*c*d - e^2)*Log[d^2*(e + 2*d*x - 2*Sqrt[d]*Sqrt[c + x*(e + d*x)])]))/(48*d^(5/2)*(a + b*x))

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.12, size = 257, normalized size = 1.30

| method | result |
|---|---|
| risch | $\frac{(8b\,x^2d^2+12a\,d^2x+2bdex+6ade+8bcd-3e^2b)\sqrt{d\,x^2+ex+c}\;\sqrt{(bx+a)^2}}{24d^2(bx+a)}+\frac{\left(\frac{\ln\left(\frac{\frac{e}{2}+dx}{\sqrt{d}}+\sqrt{d\,x^2+ex+c}\right)ac}{2\sqrt{d}}-\right.}{}$ |
| default | $\frac{\mathrm{csgn}(bx+a)\left(16(d\,x^2+ex+c)^{\frac{3}{2}}d^{\frac{5}{2}}b+24\sqrt{d\,x^2+ex+c}\;d^{\frac{7}{2}}ax-12\sqrt{d\,x^2+ex+c}\;d^{\frac{5}{2}}bex+12\sqrt{d\,x^2+ex+c}\;d^{\frac{5}{2}}a\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

[In] int(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x,method=_RETURNVERBOSE)

[Out] 1/48*csgn(b*x+a)*(16*(d*x^2+e*x+c)^(3/2)*d^(5/2)*b+24*(d*x^2+e*x+c)^(1/2)*d^(7/2)*a*x-12*(d*x^2+e*x+c)^(1/2)*d^(5/2)*b*e*x+12*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a*e-6*(d*x^2+e*x+c)^(1/2)*d^(3/2)*b*e^2+24*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*a*c*d^3-6*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*a*d^2*e^2-12*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*b*c*d^2*e+3*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*b*d*e^3)/d^(7/2)

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="maxima")

[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2), x)

**Fricas** [A]
time = 0.41, size = 292, normalized size = 1.47

$$\left[\frac{3\,(8\,acd^2-4\,bcde-2\,ade^2+be^3)\sqrt{d}\,\log\left(8\,d^2x^2+8\,dxe+4\,\sqrt{dx^2+xe+c}\,(2\,dx+e)\sqrt{d}+4\,cd+e^2\right)+4\,(8\,bd^3x^2+12\,ad^3x+8\,bcd^2-3\,bde^2+2\,(bd^2x+3\,ad^2)e)\sqrt{dx^2+xe+c}}{96\,d^3},-\frac{3\,(8\,acd^2-4\,bcde-2\,ade^2+be^3)\sqrt{-d}\,\arctan\left(\frac{\sqrt{dx^2+xe+c}\,(2\,dx+e)\sqrt{-d}}{2\,(d^2x^2+dxe+cd)}\right)-2\,(8\,bd^3x^2+12\,ad^3x+8\,bcd^2-3\,bde^2+2\,(bd^2x+3\,ad^2)e)\sqrt{dx^2+xe+c}}{48\,d^3}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="fricas")

```
[Out] [1/96*(3*(8*a*c*d^2 - 4*b*c*d*e - 2*a*d*e^2 + b*e^3)*sqrt(d)*log(8*d^2*x^2
+ 8*d*x*e + 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(d) + 4*c*d + e^2) + 4*
(8*b*d^3*x^2 + 12*a*d^3*x + 8*b*c*d^2 - 3*b*d*e^2 + 2*(b*d^2*x + 3*a*d^2)*e
)*sqrt(d*x^2 + x*e + c))/d^3, -1/48*(3*(8*a*c*d^2 - 4*b*c*d*e - 2*a*d*e^2 +
 b*e^3)*sqrt(-d)*arctan(1/2*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2
*x^2 + d*x*e + c*d)) - 2*(8*b*d^3*x^2 + 12*a*d^3*x + 8*b*c*d^2 - 3*b*d*e^2
+ 2*(b*d^2*x + 3*a*d^2)*e)*sqrt(d*x^2 + x*e + c))/d^3]
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{c + dx^2 + ex}\,\sqrt{(a + bx)^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2),x)
```

```
[Out] Integral(sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2), x)
```

**Giac** [A]
time = 2.06, size = 185, normalized size = 0.93

$$\frac{1}{24}\sqrt{dx^2 + xe + c}\left(2\left(4\,bx\mathrm{sgn}(bx + a) + \frac{6\,ad^2\mathrm{sgn}(bx + a) + bde\mathrm{sgn}(bx + a)}{d^2}\right)x + \frac{8\,bcd\mathrm{sgn}(bx + a) + 6\,ade\mathrm{sgn}(bx + a) - 3\,be^2\mathrm{sgn}(bx + a)}{d^2}\right) - \frac{(8\,acd^2\mathrm{sgn}(bx + a) - 4\,bcde\mathrm{sgn}(bx + a) - 2\,ade^2\mathrm{sgn}(bx + a) + be^3\mathrm{sgn}(bx + a))\log\left(\left|-2\left(\sqrt{d}\,x - \sqrt{dx^2 + xe + c}\right)\sqrt{d} - e\right|\right)}{16\,d^{\frac{5}{2}}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2),x, algorithm="giac")
```

```
[Out] 1/24*sqrt(d*x^2 + x*e + c)*(2*(4*b*x*sgn(b*x + a) + (6*a*d^2*sgn(b*x + a) +
 b*d*e*sgn(b*x + a))/d^2)*x + (8*b*c*d*sgn(b*x + a) + 6*a*d*e*sgn(b*x + a)
- 3*b*e^2*sgn(b*x + a))/d^2) - 1/16*(8*a*c*d^2*sgn(b*x + a) - 4*b*c*d*e*sgn
(b*x + a) - 2*a*d*e^2*sgn(b*x + a) + b*e^3*sgn(b*x + a))*log(abs(-2*(sqrt(d
)*x - sqrt(d*x^2 + x*e + c))*sqrt(d) - e))/d^(5/2)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+e\,x+c}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2),x)
```

```
[Out] int(((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2), x)
```

# 3.49 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x}\,dx$

**Optimal**. Leaf size=211

$$\frac{(4ad+be+2bdx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4d(a+bx)}+\frac{(4bcd+4ade-be^2)\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\Big(}{8d^{3/2}(a+bx)}$$

```
[Out] 1/8*(4*a*d*e+4*b*c*d-b*e^2)*arctanh(1/2*(2*d*x+e)/d^(1/2)/(d*x^2+e*x+c)^(1/
2))*((b*x+a)^2)^(1/2)/d^(3/2)/(b*x+a)-a*arctanh(1/2*(e*x+2*c)/c^(1/2)/(d*x^
2+e*x+c)^(1/2))*c^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)+1/4*(2*b*d*x+4*a*d+b*e)*(
(b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/d/(b*x+a)
```

**Rubi** [A]
time = 0.11, antiderivative size = 211, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 38, $\frac{\text{number of rules}}{\text{integrand size}} = 0.158$, Rules used = {1014, 828, 857, 635, 212, 738}

$$\frac{\sqrt{a^2+2abx+b^2x^2}\,(4ade+4bcd-be^2)\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{8d^{3/2}(a+bx)}+\frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2+ex}\,(4ad+2bdx+be)}{4d(a+bx)}-\frac{a\sqrt{c}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{2c+ex}{2\sqrt{c}\,\sqrt{c+dx^2+ex}}\right)}{a+bx}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x,x]
```

```
[Out] ((4*a*d + b*e + 2*b*d*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2
])/(4*d*(a + b*x)) + ((4*b*c*d + 4*a*d*e - b*e^2)*Sqrt[a^2 + 2*a*b*x + b^2*
x^2]*ArcTanh[(e + 2*d*x)/(2*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(8*d^(3/2)*(a
+ b*x)) - (a*Sqrt[c]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(2*c + e*x)/(2*S
qrt[c]*Sqrt[c + e*x + d*x^2])])/(a + b*x)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 828

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(c*e*f*(m + 2*p + 2)
 - g*(c*d + 2*c*d*p - b*e*p) + g*c*e*(m + 2*p + 1)*x)*((a + b*x + c*x^2)^p/
(c*e^2*(m + 2*p + 1)*(m + 2*p + 2))), x] - Dist[p/(c*e^2*(m + 2*p + 1)*(m +
 2*p + 2)), Int[(d + e*x)^m*(a + b*x + c*x^2)^(p - 1)*Simp[c*e*f*(b*d - 2*a
*e)*(m + 2*p + 2) + g*(a*e*(b*e - 2*c*d*m + b*e*m) + b*d*(b*e*p - c*d - 2*c
*d*p)) + (c*e*f*(2*c*d - b*e)*(m + 2*p + 2) + g*(b^2*e^2*(p + m + 1) - 2*c^
2*d^2*(1 + 2*p) - c*e*(b*d*(m - 2*p) + 2*a*e*(m + 2*p + 1))))*x, x], x], x]
 /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2
- b*d*e + a*e^2, 0] && GtQ[p, 0] && (IntegerQ[p] ||  !RationalQ[m] || (GeQ[
m, -1] && LtQ[m, 0])) &&  !ILtQ[m + 2*p, 0] && (IntegerQ[m] || IntegerQ[p]
|| IntegersQ[2*m, 2*p])
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1014

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^Fr
acPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(
b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, e, f, g,
 h, m, p, q}, x] && EqQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int \frac{(2ab+2b^2x)\sqrt{c+ex+dx^2}}{x}\,dx}{2ab+2b^2x}$$

$$= \frac{(4ad+be+2bdx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4d(a+bx)} - \frac{\sqrt{a^2}}{}$$

$$= \frac{(4ad+be+2bdx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4d(a+bx)} + \frac{\Big(2ab}{}$$

$$= \frac{(4ad+be+2bdx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4d(a+bx)} - \frac{\Big(4ab}{}$$

$$= \frac{(4ad+be+2bdx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4d(a+bx)} + \frac{(4bc}{}$$

**Mathematica** [A]
time = 0.48, size = 150, normalized size = 0.71

$$\frac{\sqrt{(a+bx)^2}\left(2\sqrt{d}\,\sqrt{c+x(e+dx)}\,(4ad+b(e+2dx))+16a\sqrt{c}\,d^{3/2}\tanh^{-1}\left(\frac{\sqrt{d}\,x-\sqrt{c+x(e+dx)}}{\sqrt{c}}\right)+(-4ade+b(-4cd+e^2))\log\left(d\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)\right)}{8d^{3/2}(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x,x]
```

```
[Out] (Sqrt[(a + b*x)^2]*(2*Sqrt[d]*Sqrt[c + x*(e + d*x)]*(4*a*d + b*(e + 2*d*x))
 + 16*a*Sqrt[c]*d^(3/2)*ArcTanh[(Sqrt[d]*x - Sqrt[c + x*(e + d*x)])/Sqrt[c]
] + (-4*a*d*e + b*(-4*c*d + e^2))*Log[d*(e + 2*d*x - 2*Sqrt[d]*Sqrt[c + x*(
e + d*x)])]))/(8*d^(3/2)*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.13, size = 215, normalized size = 1.02

| method | result |
|---|---|
| default | $\frac{\mathrm{csgn}(bx+a)\left(4\sqrt{d\,x^2+ex+c}\,d^{\frac{5}{2}}bx-8\sqrt{c}\,d^{\frac{5}{2}}\ln\left(\frac{2c+ex+2\sqrt{c}\,\sqrt{d\,x^2+ex+c}}{x}\right)a+8\sqrt{d\,x^2+ex+c}\,d^{\frac{5}{2}}a+2}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x,x,method=_RETURNVERBOSE)
```

```
[Out] 1/8*csgn(b*x+a)*(4*(d*x^2+e*x+c)^(1/2)*d^(5/2)*b*x-8*c^(1/2)*d^(5/2)*ln((2*
c+e*x+2*c^(1/2)*(d*x^2+e*x+c)^(1/2))/x)*a+8*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a+2
```

```
*(d*x^2+e*x+c)^(1/2)*d^(3/2)*b*e+4*d^2*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2
)+2*d*x+e)/d^(1/2))*a*e+4*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^
(1/2))*b*c*d^2-ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*b*d*
e^2)/d^(5/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2)/x, x)
```

**Fricas [A]**
time = 0.81, size = 683, normalized size = 3.24

$$\left[\frac{8a\sqrt{c}d^2\log\left(\frac{4cdx^2+x^2e^2+8cxe-4\sqrt{dx^2+xe+c}(xe+2c)\sqrt{c}+8c^2}{x^2}\right)-(4bcd+4ade-be^2)\sqrt{d}\log\left(8d^2x^2+8dxe-4\sqrt{dx^2+xe+c}(2dx+e)\sqrt{d}+4cd+e^2\right)+4(2bd^2x+4ad^2+bde)\sqrt{dx^2+xe+c}}{16d^2}, \frac{4a\sqrt{c}d^2\log\left(\frac{4cdx^2+x^2e^2+8cxe-4\sqrt{dx^2+xe+c}(xe+2c)\sqrt{c}+8c^2}{x^2}\right)-(4bcd+4ade-be^2)\sqrt{-d}\arctan\left(\frac{\sqrt{dx^2+xe+c}(2dx+e)\sqrt{-d}}{2(d^2x^2+dxe+cd)}\right)+2(2bd^2x+4ad^2+bde)\sqrt{dx^2+xe+c}}{8d^2}, \frac{16a\sqrt{-c}d^2\arctan\left(\frac{\sqrt{dx^2+xe+c}(xe+2c)\sqrt{-c}}{2(cdx^2+cxe+c^2)}\right)-(4bcd+4ade-be^2)\sqrt{d}\log\left(8d^2x^2+8dxe-4\sqrt{dx^2+xe+c}(2dx+e)\sqrt{d}+4cd+e^2\right)+4(2bd^2x+4ad^2+bde)\sqrt{dx^2+xe+c}}{16d^2}, \frac{8a\sqrt{-c}d^2\arctan\left(\frac{\sqrt{dx^2+xe+c}(xe+2c)\sqrt{-c}}{2(cdx^2+cxe+c^2)}\right)-(4bcd+4ade-be^2)\sqrt{-d}\arctan\left(\frac{\sqrt{dx^2+xe+c}(2dx+e)\sqrt{-d}}{2(d^2x^2+dxe+cd)}\right)+2(2bd^2x+4ad^2+bde)\sqrt{dx^2+xe+c}}{8d^2}\right]$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x,x, algorithm="fricas")
```

```
[Out] [1/16*(8*a*sqrt(c)*d^2*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e - 4*sqrt(d*x^2 +
x*e + c)*(x*e + 2*c)*sqrt(c) + 8*c^2)/x^2) - (4*b*c*d + 4*a*d*e - b*e^2)*sq
rt(d)*log(8*d^2*x^2 + 8*d*x*e - 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(d)
 + 4*c*d + e^2) + 4*(2*b*d^2*x + 4*a*d^2 + b*d*e)*sqrt(d*x^2 + x*e + c))/d^
2, 1/8*(4*a*sqrt(c)*d^2*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e - 4*sqrt(d*x^2 +
 x*e + c)*(x*e + 2*c)*sqrt(c) + 8*c^2)/x^2) - (4*b*c*d + 4*a*d*e - b*e^2)*s
qrt(-d)*arctan(1/2*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*
x*e + c*d)) + 2*(2*b*d^2*x + 4*a*d^2 + b*d*e)*sqrt(d*x^2 + x*e + c))/d^2, 1
/16*(16*a*sqrt(-c)*d^2*arctan(1/2*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(-c
)/(c*d*x^2 + c*x*e + c^2)) - (4*b*c*d + 4*a*d*e - b*e^2)*sqrt(d)*log(8*d^2*
x^2 + 8*d*x*e - 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(d) + 4*c*d + e^2)
+ 4*(2*b*d^2*x + 4*a*d^2 + b*d*e)*sqrt(d*x^2 + x*e + c))/d^2, 1/8*(8*a*sqrt
(-c)*d^2*arctan(1/2*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(-c)/(c*d*x^2 + c
*x*e + c^2)) - (4*b*c*d + 4*a*d*e - b*e^2)*sqrt(-d)*arctan(1/2*sqrt(d*x^2 +
 x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) + 2*(2*b*d^2*x + 4*
a*d^2 + b*d*e)*sqrt(d*x^2 + x*e + c))/d^2]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c + dx^2 + ex}\,\sqrt{(a + bx)^2}}{x}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2)/x,x)
```

```
[Out] Integral(sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2)/x, x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x,x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Warning, integration of abs or sign a
ssumes constant sign by intervals (correct if the argument is real):Check [
abs(sa
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+e\,x+c}}{x} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x,x)
```

```
[Out] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x, x)
```

# 3.50 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x^2}\,dx$

**Optimal**. Leaf size=202

$$-\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x(a+bx)}+\frac{(2ad+be)\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{e+2dx}{2\sqrt{d}\,\sqrt{c+ex+dx^2}}\right.}{2\sqrt{d}\,(a+bx)}$$

```
[Out] -1/2*(a*e+2*b*c)*arctanh(1/2*(e*x+2*c)/c^(1/2)/(d*x^2+e*x+c)^(1/2))*((b*x+a
)^2)^(1/2)/(b*x+a)/c^(1/2)+1/2*(2*a*d+b*e)*arctanh(1/2*(2*d*x+e)/d^(1/2)/(d
*x^2+e*x+c)^(1/2))*((b*x+a)^2)^(1/2)/(b*x+a)/d^(1/2)-(-b*x+a)*((b*x+a)^2)^(
1/2)*(d*x^2+e*x+c)^(1/2)/x/(b*x+a)
```

**Rubi** [A]
time = 0.10, antiderivative size = 202, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 38, $\frac{\text{number of rules}}{\text{integrand size}} = 0.158$, Rules used = {1014, 826, 857, 635, 212, 738}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(a-bx)\sqrt{c+dx^2+ex}}{x(a+bx)}+\frac{\sqrt{a^2+2abx+b^2x^2}\,(2ad+be)\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{2\sqrt{d}\,(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,(ae+2bc)\tanh^{-1}\left(\frac{2c+ex}{2\sqrt{c}\,\sqrt{c+dx^2+ex}}\right)}{2\sqrt{c}\,(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x^2,x]
```

```
[Out] -(((a - b*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/(x*(a + b
*x))) + ((2*a*d + b*e)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(e + 2*d*x)/(2
*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(2*Sqrt[d]*(a + b*x)) - ((2*b*c + a*e)*Sq
rt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(2*c + e*x)/(2*Sqrt[c]*Sqrt[c + e*x + d
*x^2])])/(2*Sqrt[c]*(a + b*x))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 826

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(e*f*(m + 2*p + 2) -
 d*g*(2*p + 1) + e*g*(m + 1)*x)*((a + b*x + c*x^2)^p/(e^2*(m + 1)*(m + 2*p
+ 2))), x] + Dist[p/(e^2*(m + 1)*(m + 2*p + 2)), Int[(d + e*x)^(m + 1)*(a +
 b*x + c*x^2)^(p - 1)*Simp[g*(b*d + 2*a*e + 2*a*e*m + 2*b*d*p) - f*b*e*(m +
 2*p + 2) + (g*(2*c*d + b*e + b*e*m + 4*c*d*p) - 2*c*e*f*(m + 2*p + 2))*x,
x], x], x] /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] && RationalQ[p] && p > 0 && (LtQ[m, -1] || Eq
Q[p, 1] || (IntegerQ[p] &&  !RationalQ[m])) && NeQ[m, -1] &&  !ILtQ[m + 2*p
 + 1, 0] && (IntegerQ[m] || IntegerQ[p] || IntegersQ[2*m, 2*p])
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1014

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^Fr
acPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(
b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, e, f, g,
 h, m, p, q}, x] && EqQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x^2}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\int \frac{(2ab+2b^2x)\sqrt{c+ex+dx^2}}{x^2}\,dx}{2ab+2b^2x}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x(a+bx)} - \frac{\sqrt{a^2+2abx+ \cdots}}{\cdots}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x(a+bx)} + \frac{\Big(b(2bc+ae)\sqrt{\cdots}}{\cdots}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x(a+bx)} - \frac{\Big(2b(2bc+ae)\cdots}{\cdots}$$

$$= -\frac{(a-bx)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x(a+bx)} + \frac{(2ad+be)\sqrt{a\cdots}}{\cdots}$$

**Mathematica** [A]
time = 0.49, size = 151, normalized size = 0.75

$$-\frac{\sqrt{(a+bx)^2}\left(2\sqrt{d}\,(2bc+ae)x\tanh^{-1}\left(\frac{-\sqrt{d}\,x+\sqrt{c+x(e+dx)}}{\sqrt{c}}\right)+\sqrt{c}\left(2\sqrt{d}\,(a-bx)\sqrt{c+x(e+dx)}+(2ad+be)x\log\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)\right)}{2\sqrt{c}\,\sqrt{d}\,x(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x^2,x]
```

```
[Out] -1/2*(Sqrt[(a + b*x)^2]*(2*Sqrt[d]*(2*b*c + a*e)*x*ArcTanh[(-(Sqrt[d]*x) +
Sqrt[c + x*(e + d*x)])/Sqrt[c]] + Sqrt[c]*(2*Sqrt[d]*(a - b*x)*Sqrt[c + x*(
e + d*x)] + (2*a*d + b*e)*x*Log[e + 2*d*x - 2*Sqrt[d]*Sqrt[c + x*(e + d*x)]
])))/(Sqrt[c]*Sqrt[d]*x*(a + b*x))
```

**Maple** [C] Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.13, size = 249, normalized size = 1.23

| method | result |
|---|---|
| risch | $-\frac{a\sqrt{d\,x^2+ex+c}\,\sqrt{(bx+a)^2}}{x(bx+a)} + \frac{\left(b\sqrt{d\,x^2+ex+c}+\frac{eb\ln\left(\frac{\frac{e}{2}+dx}{\sqrt{d}}+\sqrt{d\,x^2+ex+c}\right)}{2\sqrt{d}}+a\sqrt{d}\,\ln\left(\frac{\frac{e}{2}+dx}{\sqrt{d}}\cdots\right.\right.}{\cdots}$ |

| | |
|---|---|
| default | $-\frac{\mathrm{csgn}(bx+a)\left(-2\sqrt{dx^2+ex+c}\,d^{\frac{5}{2}}ax^2+2d^{\frac{3}{2}}c^{\frac{3}{2}}\ln\left(\frac{2c+ex+2\sqrt{c}\,\sqrt{dx^2+ex+c}}{x}\right)bx+d^{\frac{3}{2}}\sqrt{c}\,\ln\left(\frac{2c+ex+2\sqrt{c}}{}\right.\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^2,x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*csgn(b*x+a)*(-2*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a*x^2+2*d^(3/2)*c^(3/2)*ln
((2*c+e*x+2*c^(1/2)*(d*x^2+e*x+c)^(1/2))/x)*b*x+d^(3/2)*c^(1/2)*ln((2*c+e*x
+2*c^(1/2)*(d*x^2+e*x+c)^(1/2))/x)*a*e*x+2*(d*x^2+e*x+c)^(3/2)*d^(3/2)*a-2*
(d*x^2+e*x+c)^(1/2)*d^(3/2)*a*e*x-2*(d*x^2+e*x+c)^(1/2)*d^(3/2)*b*c*x-2*ln(
1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*a*c*d^2*x-ln(1/2*(2*(d
*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*d*b*c*e*x)/c/x/d^(3/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^2,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2)/x^2, x)
```

**Fricas [A]**
time = 0.58, size = 691, normalized size = 3.42

[illegible]

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^2,x, algorithm="fricas")
```

```
[Out] [1/4*((2*a*c*d*x + b*c*x*e)*sqrt(d)*log(8*d^2*x^2 + 8*d*x*e + 4*sqrt(d*x^2
+ x*e + c)*(2*d*x + e)*sqrt(d) + 4*c*d + e^2) + (2*b*c*d*x + a*d*x*e)*sqrt(
c)*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e - 4*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)
*sqrt(c) + 8*c^2)/x^2) + 4*(b*c*d*x - a*c*d)*sqrt(d*x^2 + x*e + c))/(c*d*x)
, -1/4*(2*(2*a*c*d*x + b*c*x*e)*sqrt(-d)*arctan(1/2*sqrt(d*x^2 + x*e + c)*(
2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) - (2*b*c*d*x + a*d*x*e)*sqrt(c
)*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e - 4*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)*
sqrt(c) + 8*c^2)/x^2) - 4*(b*c*d*x - a*c*d)*sqrt(d*x^2 + x*e + c))/(c*d*x),
 1/4*(2*(2*b*c*d*x + a*d*x*e)*sqrt(-c)*arctan(1/2*sqrt(d*x^2 + x*e + c)*(x*
e + 2*c)*sqrt(-c)/(c*d*x^2 + c*x*e + c^2)) + (2*a*c*d*x + b*c*x*e)*sqrt(d)*
log(8*d^2*x^2 + 8*d*x*e + 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(d) + 4*c
*d + e^2) + 4*(b*c*d*x - a*c*d)*sqrt(d*x^2 + x*e + c))/(c*d*x), 1/2*((2*b*c
*d*x + a*d*x*e)*sqrt(-c)*arctan(1/2*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(
```

```
-c)/(c*d*x^2 + c*x*e + c^2)) - (2*a*c*d*x + b*c*x*e)*sqrt(-d)*arctan(1/2*sq
rt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) + 2*(b*c*
d*x - a*c*d)*sqrt(d*x^2 + x*e + c))/(c*d*x)]
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c+dx^2+ex}\,\sqrt{(a+bx)^2}}{x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2)/x**2,x)
```

```
[Out] Integral(sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2)/x**2, x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^2,x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Warning, integration of abs or sign a
ssumes constant sign by intervals (correct if the argument is real):Check [
abs(sa
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{(a+b\,x)^2}\,\sqrt{d\,x^2+e\,x+c}}{x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x^2,x)
```

```
[Out] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x^2, x)
```

# 3.51 $\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x^3}\,dx$

**Optimal**. Leaf size=215

$$-\frac{(2ac+(4bc+ae)x)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4cx^2(a+bx)}+\frac{b\sqrt{d}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{e+2dx}{2\sqrt{d}\,\sqrt{c+e}}\right.}{a+bx}$$

```
[Out] -1/8*(4*a*c*d-a*e^2+4*b*c*e)*arctanh(1/2*(e*x+2*c)/c^(1/2)/(d*x^2+e*x+c)^(1
/2))*((b*x+a)^2)^(1/2)/c^(3/2)/(b*x+a)+b*arctanh(1/2*(2*d*x+e)/d^(1/2)/(d*x
^2+e*x+c)^(1/2))*d^(1/2)*((b*x+a)^2)^(1/2)/(b*x+a)-1/4*(2*a*c+(a*e+4*b*c)*x
)*((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/c/x^2/(b*x+a)
```

**Rubi** [A]
time = 0.11, antiderivative size = 215, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 6, integrand size = 38, $\frac{\text{number of rules}}{\text{integrand size}} = 0.158$, Rules used = {1014, 824, 857, 635, 212, 738}

$$-\frac{\sqrt{a^2+2abx+b^2x^2}\,(4acd-ae^2+4bce)\tanh^{-1}\left(\frac{2c+ex}{2\sqrt{c}\,\sqrt{c+dx^2+ex}}\right)}{8c^{3/2}(a+bx)}-\frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+dx^2+ex}\,(x(ae+4bc)+2ac)}{4cx^2(a+bx)}+\frac{b\sqrt{d}\,\sqrt{a^2+2abx+b^2x^2}\,\tanh^{-1}\left(\frac{2dx+e}{2\sqrt{d}\,\sqrt{c+dx^2+ex}}\right)}{a+bx}$$

Antiderivative was successfully verified.

```
[In] Int[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x^3,x]
```

```
[Out] -1/4*((2*a*c + (4*b*c + a*e)*x)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x
+ d*x^2])/(c*x^2*(a + b*x)) + (b*Sqrt[d]*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcT
anh[(e + 2*d*x)/(2*Sqrt[d]*Sqrt[c + e*x + d*x^2])])/(a + b*x) - ((4*a*c*d +
 4*b*c*e - a*e^2)*Sqrt[a^2 + 2*a*b*x + b^2*x^2]*ArcTanh[(2*c + e*x)/(2*Sqrt
[c]*Sqrt[c + e*x + d*x^2])])/(8*c^(3/2)*(a + b*x))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 824

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(d + e*x)^(m + 1))*((a + b*x + c*x^2
)^p/(e^2*(m + 1)*(m + 2)*(c*d^2 - b*d*e + a*e^2)))*((d*g - e*f*(m + 2))*(c*
d^2 - b*d*e + a*e^2) - d*p*(2*c*d - b*e)*(e*f - d*g) - e*(g*(m + 1)*(c*d^2
- b*d*e + a*e^2) + p*(2*c*d - b*e)*(e*f - d*g))*x), x] - Dist[p/(e^2*(m + 1
)*(m + 2)*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(m + 2)*(a + b*x + c*x^2)
^(p - 1)*Simp[2*a*c*e*(e*f - d*g)*(m + 2) + b^2*e*(d*g*(p + 1) - e*f*(m + p
 + 2)) + b*(a*e^2*g*(m + 1) - c*d*(d*g*(2*p + 1) - e*f*(m + 2*p + 2))) - c*
(2*c*d*(d*g*(2*p + 1) - e*f*(m + 2*p + 2)) - e*(2*a*e*g*(m + 1) - b*(d*g*(m
 - 2*p) + e*f*(m + 2*p + 2))))*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, g}
, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && GtQ[p, 0] &
& LtQ[m, -2] && LtQ[m + 2*p, 0] &&  !ILtQ[m + 2*p + 3, 0]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1014

```
Int[((g_.) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_
) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Dist[(a + b*x + c*x^2)^Fr
acPart[p]/((4*c)^IntPart[p]*(b + 2*c*x)^(2*FracPart[p])), Int[(g + h*x)^m*(
b + 2*c*x)^(2*p)*(d + e*x + f*x^2)^q, x], x] /; FreeQ[{a, b, c, d, e, f, g,
 h, m, p, q}, x] && EqQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{x^3}\,dx = \frac{\sqrt{a^2+2abx+b^2x^2}\,\int \frac{(2ab+2b^2x)\sqrt{c+ex+dx^2}}{x^3}\,dx}{2ab+2b^2x}$$

$$= -\frac{(2ac+(4bc+ae)x)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4cx^2(a+bx)} - \ldots$$

$$= -\frac{(2ac+(4bc+ae)x)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4cx^2(a+bx)} + \ldots$$

$$= -\frac{(2ac+(4bc+ae)x)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4cx^2(a+bx)} + \ldots$$

$$= -\frac{(2ac+(4bc+ae)x)\sqrt{a^2+2abx+b^2x^2}\,\sqrt{c+ex+dx^2}}{4cx^2(a+bx)} + \ldots$$

**Mathematica [A]**
time = 0.60, size = 155, normalized size = 0.72

$$-\frac{\sqrt{(a+bx)^2}\left((4acd+4bce-ae^2)\,x^2\tanh^{-1}\left(\frac{-\sqrt{d}\,x+\sqrt{c+x(e+dx)}}{\sqrt{c}}\right)+\sqrt{c}\left((2ac+4bcx+aex)\sqrt{c+x(e+dx)}+4bc\sqrt{d}\,x^2\log\left(e+2dx-2\sqrt{d}\,\sqrt{c+x(e+dx)}\right)\right)\right)}{4c^{3/2}x^2(a+bx)}$$

Antiderivative was successfully verified.

```
[In] Integrate[(Sqrt[a^2 + 2*a*b*x + b^2*x^2]*Sqrt[c + e*x + d*x^2])/x^3,x]
```

```
[Out] -1/4*(Sqrt[(a + b*x)^2]*((4*a*c*d + 4*b*c*e - a*e^2)*x^2*ArcTanh[(-(Sqrt[d]
*x) + Sqrt[c + x*(e + d*x)])/Sqrt[c]] + Sqrt[c]*((2*a*c + 4*b*c*x + a*e*x)*
Sqrt[c + x*(e + d*x)] + 4*b*c*Sqrt[d]*x^2*Log[e + 2*d*x - 2*Sqrt[d]*Sqrt[c
+ x*(e + d*x)]])))/(c^(3/2)*x^2*(a + b*x))
```

**Maple [C]** Result contains higher order function than in optimal. Order 9 vs. order 3.
time = 0.12, size = 359, normalized size = 1.67

| method | result |
|---|---|
| risch | $-\frac{\sqrt{d\,x^2+ex+c}\,(aex+4bcx+2ac)\sqrt{(bx+a)^2}}{4x^2c(bx+a)}+\frac{\left(b\sqrt{d}\,\ln\left(\frac{\frac{e}{2}+dx}{\sqrt{d}}+\sqrt{d\,x^2+ex+c}\right)-\frac{\ln\left(\frac{2c+ex+2\sqrt{c}\,\ldots}{\ldots}\right)}{2\ldots}\ldots\right.}{\ldots}$ |

| | |
|---|---|
| default | $-\frac{\mathrm{csgn}(bx+a)\left(4d^{\frac{5}{2}}c^{\frac{3}{2}}\ln\left(\frac{2c+ex+2\sqrt{c}\,\sqrt{d\,x^2+ex+c}}{x}\right)a\,x^2+2\sqrt{d\,x^2+ex+c}\,d^{\frac{5}{2}}ae\,x^3-8\sqrt{d\,x^2+ex+c}\,d^{\frac{5}{2}}b\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^3,x,method=_RETURNVERBOSE)
```

```
[Out] -1/8*csgn(b*x+a)*(4*d^(5/2)*c^(3/2)*ln((2*c+e*x+2*c^(1/2)*(d*x^2+e*x+c)^(1/
2))/x)*a*x^2+2*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a*e*x^3-8*(d*x^2+e*x+c)^(1/2)*d^
(5/2)*b*c*x^3+4*d^(3/2)*c^(3/2)*ln((2*c+e*x+2*c^(1/2)*(d*x^2+e*x+c)^(1/2))/
x)*b*e*x^2-4*(d*x^2+e*x+c)^(1/2)*d^(5/2)*a*c*x^2-d^(3/2)*c^(1/2)*ln((2*c+e*
x+2*c^(1/2)*(d*x^2+e*x+c)^(1/2))/x)*a*e^2*x^2-2*(d*x^2+e*x+c)^(3/2)*d^(3/2)
*a*e*x+8*(d*x^2+e*x+c)^(3/2)*d^(3/2)*b*c*x+2*(d*x^2+e*x+c)^(1/2)*d^(3/2)*a*
e^2*x^2-8*(d*x^2+e*x+c)^(1/2)*d^(3/2)*b*c*e*x^2+4*(d*x^2+e*x+c)^(3/2)*d^(3/
2)*a*c-8*ln(1/2*(2*(d*x^2+e*x+c)^(1/2)*d^(1/2)+2*d*x+e)/d^(1/2))*b*c^2*d^2*
x^2)/c^2/x^2/d^(3/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^3,x, algorithm="maxima")
```

```
[Out] integrate(sqrt(d*x^2 + x*e + c)*sqrt((b*x + a)^2)/x^3, x)
```

**Fricas [A]**
time = 0.61, size = 745, normalized size = 3.47

[illegible]

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^3,x, algorithm="fricas")
```

```
[Out] [1/16*(8*b*c^2*sqrt(d)*x^2*log(8*d^2*x^2 + 8*d*x*e + 4*sqrt(d*x^2 + x*e + c
)*(2*d*x + e)*sqrt(d) + 4*c*d + e^2) - (4*a*c*d*x^2 + 4*b*c*x^2*e - a*x^2*e
^2)*sqrt(c)*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e + 4*sqrt(d*x^2 + x*e + c)*(x
*e + 2*c)*sqrt(c) + 8*c^2)/x^2) - 4*(4*b*c^2*x + a*c*x*e + 2*a*c^2)*sqrt(d*
x^2 + x*e + c))/(c^2*x^2), -1/16*(16*b*c^2*sqrt(-d)*x^2*arctan(1/2*sqrt(d*x
^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2 + d*x*e + c*d)) + (4*a*c*d*x^2
+ 4*b*c*x^2*e - a*x^2*e^2)*sqrt(c)*log((4*c*d*x^2 + x^2*e^2 + 8*c*x*e + 4*s
qrt(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(c) + 8*c^2)/x^2) + 4*(4*b*c^2*x + a*c
*x*e + 2*a*c^2)*sqrt(d*x^2 + x*e + c))/(c^2*x^2), 1/8*(4*b*c^2*sqrt(d)*x^2*
log(8*d^2*x^2 + 8*d*x*e + 4*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(d) + 4*c
```

```
*d + e^2) + (4*a*c*d*x^2 + 4*b*c*x^2*e - a*x^2*e^2)*sqrt(-c)*arctan(1/2*sqr
t(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(-c)/(c*d*x^2 + c*x*e + c^2)) - 2*(4*b*c
^2*x + a*c*x*e + 2*a*c^2)*sqrt(d*x^2 + x*e + c))/(c^2*x^2), -1/8*(8*b*c^2*s
qrt(-d)*x^2*arctan(1/2*sqrt(d*x^2 + x*e + c)*(2*d*x + e)*sqrt(-d)/(d^2*x^2
+ d*x*e + c*d)) - (4*a*c*d*x^2 + 4*b*c*x^2*e - a*x^2*e^2)*sqrt(-c)*arctan(1
/2*sqrt(d*x^2 + x*e + c)*(x*e + 2*c)*sqrt(-c)/(c*d*x^2 + c*x*e + c^2)) + 2*
(4*b*c^2*x + a*c*x*e + 2*a*c^2)*sqrt(d*x^2 + x*e + c))/(c^2*x^2)]
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{c + dx^2 + ex}\, \sqrt{(a + bx)^2}}{x^3}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)**2)**(1/2)*(d*x**2+e*x+c)**(1/2)/x**3,x)
```

```
[Out] Integral(sqrt(c + d*x**2 + e*x)*sqrt((a + b*x)**2)/x**3, x)
```

**Giac [B]** Leaf count of result is larger than twice the leaf count of optimal. 450 vs. 2(166) = 332.
time = 5.10, size = 450, normalized size = 2.09

$$-b\sqrt{d}\log\left(\left|-2\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)d-\sqrt{d}e\right|\right)\mathrm{sgn}(bx+a)+\frac{\left(4acd\,\mathrm{sgn}(bx+a)+4bce\,\mathrm{sgn}(bx+a)-ae^2\mathrm{sgn}(bx+a)\right)\arctan\left(-\frac{\sqrt{d}x-\sqrt{dx^2+xe+c}}{\sqrt{-c}}\right)}{4\sqrt{-c}\,c}+\frac{4\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^3acd\,\mathrm{sgn}(bx+a)+4\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^3bce\,\mathrm{sgn}(bx+a)+8\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^2bc^2\sqrt{d}\,\mathrm{sgn}(bx+a)+8\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^2ac\sqrt{d}\,e\,\mathrm{sgn}(bx+a)+4\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)ac^2d\,\mathrm{sgn}(bx+a)+\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^3ae^2\mathrm{sgn}(bx+a)-4\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)bc^2e\,\mathrm{sgn}(bx+a)-8bc^3\sqrt{d}\,\mathrm{sgn}(bx+a)+\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)ace^2\mathrm{sgn}(bx+a)}{4\left(\left(\sqrt{d}x-\sqrt{dx^2+xe+c}\right)^2-c\right)^2c}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((b*x+a)^2)^(1/2)*(d*x^2+e*x+c)^(1/2)/x^3,x, algorithm="giac")
```

```
[Out] -b*sqrt(d)*log(abs(-2*(sqrt(d)*x - sqrt(d*x^2 + x*e + c))*d - sqrt(d)*e))*s
gn(b*x + a) + 1/4*(4*a*c*d*sgn(b*x + a) + 4*b*c*e*sgn(b*x + a) - a*e^2*sgn(
b*x + a))*arctan(-(sqrt(d)*x - sqrt(d*x^2 + x*e + c))/sqrt(-c))/(sqrt(-c)*c
) + 1/4*(4*(sqrt(d)*x - sqrt(d*x^2 + x*e + c))^3*a*c*d*sgn(b*x + a) + 4*(sq
rt(d)*x - sqrt(d*x^2 + x*e + c))^3*b*c*e*sgn(b*x + a) + 8*(sqrt(d)*x - sqrt
(d*x^2 + x*e + c))^2*b*c^2*sqrt(d)*sgn(b*x + a) + 8*(sqrt(d)*x - sqrt(d*x^2
 + x*e + c))^2*a*c*sqrt(d)*e*sgn(b*x + a) + 4*(sqrt(d)*x - sqrt(d*x^2 + x*e
 + c))*a*c^2*d*sgn(b*x + a) + (sqrt(d)*x - sqrt(d*x^2 + x*e + c))^3*a*e^2*s
gn(b*x + a) - 4*(sqrt(d)*x - sqrt(d*x^2 + x*e + c))*b*c^2*e*sgn(b*x + a) -
8*b*c^3*sqrt(d)*sgn(b*x + a) + (sqrt(d)*x - sqrt(d*x^2 + x*e + c))*a*c*e^2*
sgn(b*x + a))/(((sqrt(d)*x - sqrt(d*x^2 + x*e + c))^2 - c)^2*c)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{{\left(a+b\,x\right)}^2}\,\sqrt{d\,x^2+e\,x+c}}{x^3} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x^3,x)
[Out] int((((a + b*x)^2)^(1/2)*(c + e*x + d*x^2)^(1/2))/x^3, x)
```

## 3.52 $\int \frac{x^2\sqrt{a+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=452

$$-\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2}+\frac{(af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2\sqrt{c}\,f^3}-\frac{\left(e\left(e-\sqrt{e^2-4df}\right)(af^2+c(e^2-2d\ldots\right.}{\ldots}$$

```
[Out] 1/2*(a*f^2+2*c*(-d*f+e^2))*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))/f^3/c^(1/2)-1
/2*(-f*x+2*e)*(c*x^2+a)^(1/2)/f^2-1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2
)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2
)))^(1/2))*(-2*d*f*(a*f^2+c*(-d*f+e^2))+e*(a*f^2+c*(-2*d*f+e^2))*(e-(-4*d*f
+e^2)^(1/2)))/f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*
f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(
1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(-
2*d*f*(a*f^2+c*(-d*f+e^2))+e*(a*f^2+c*(-2*d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))
/f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))
)^(1/2)
```

**Rubi** [A]
time = 1.26, antiderivative size = 452, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.222$, Rules used = {1083, 1094, 223, 212, 1048, 739}

$$\frac{\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)(af^2+2c(e^2-df))}{2\sqrt{c}\,f^3}-\frac{\left(e\left(e-\sqrt{e^2-4df}\right)(af^2+c(e^2-2df))-2df(af^2+c(e^2-df))\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(e\left(\sqrt{e^2-4df}+e\right)(af^2+c(e^2-2df))-2df(af^2+c(e^2-df))\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{a+cx^2}\,(2e-fx)}{2f^2}$$

Antiderivative was successfully verified.

```
[In] Int[(x^2*Sqrt[a + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] -1/2*((2*e - f*x)*Sqrt[a + c*x^2])/f^2 + ((a*f^2 + 2*c*(e^2 - d*f))*ArcTanh
[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/(2*Sqrt[c]*f^3) - ((e*(e - Sqrt[e^2 - 4*d*f]
)*(a*f^2 + c*(e^2 - 2*d*f)) - 2*d*f*(a*f^2 + c*(e^2 - d*f)))*ArcTanh[(2*a*f
 - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*
Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f^3*Sqrt[e^2 - 4*d*f]*Sqrt[
2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + ((e*(e + Sqrt[e^2 - 4*d
*f])*(a*f^2 + c*(e^2 - 2*d*f)) - 2*d*f*(a*f^2 + c*(e^2 - d*f)))*ArcTanh[(2*
a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f +
 e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f^3*Sqrt[e^2 - 4*d*f]*Sq
rt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1083

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) + (e_.)*(x_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(
p + q + 1)*x)*(a + c*x^2)^p*((d + e*x + f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)
*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3)), Int[
(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) -
c*(C*e)*(2*p + 2*q + 3)) + (p + q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2))
+ f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e
)*(2*p + 2*q + 3)) + (p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*
(q + 1) - c*(C*e)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C
*(e^2 - 4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x^2, x]
, x], x] /; FreeQ[{a, c, d, e, f, A, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && Gt
Q[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !
IGtQ[q, 0]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
```

```
*c, 0]
```

Rubi steps

$$\int \frac{x^2\sqrt{a+cx^2}}{d+ex+fx^2}\,dx = -\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2} - \frac{\int \frac{acdf-ce(2cd-af)x-c(af^2+2c(e^2-df))x^2}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2cf^2}$$

$$= -\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2} - \frac{\int \frac{acdf^2+cd(af^2+2c(e^2-df))+(-cef(2cd-af)+ce(af^2+2c(e^2-df)))x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2cf^3} +$$

$$= -\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{(af^2+2c(e^2-df))\,\mathrm{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{2f^3} + \frac{(}{}$$

$$= -\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{(af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2\sqrt{c}\,f^3} - \frac{\left(e\left(e-\sqrt{e^2}\right.\right.}{}$$

$$= -\frac{(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{(af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2\sqrt{c}\,f^3} - \frac{\left(e\left(e-\sqrt{e^2}\right.\right.}{}$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.52, size = 490, normalized size = 1.08

$$\frac{f(-2e+fx)\sqrt{a+cx^2}-\frac{(2ce^2-2cdf+af^2)\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)}{\sqrt{c}}+2\,\mathrm{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ace^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2acdef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2ef^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}de^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2c^{3/2}d^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+2a\sqrt{c}\,df^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-ce^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+2cdef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-aef^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{2f^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x^2*Sqrt[a + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (f*(-2*e + f*x)*Sqrt[a + c*x^2] - ((2*c*e^2 - 2*c*d*f + a*f^2)*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2]])/Sqrt[c] + 2*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c
*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c*e^3*Log[-(Sqrt[c]
*x) + Sqrt[a + c*x^2] - #1] - 2*a*c*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2
] - #1] + a^2*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(3/2)*d*
e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*c^(3/2)*d^2*f*Log[-(Sqr
t[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*d*f^2*Log[-(Sqrt[c]*x) + S
```

```
qrt[a + c*x^2] - #1]*#1 - c*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1
^2 + 2*c*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - a*e*f^2*Log[
-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*
#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(2*f^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1301 vs. $2(403) = 806$.
time = 0.17, size = 1302, normalized size = 2.88

| method | result |
|---|---|
| default | $\frac{\frac{x\sqrt{c x^2+a}}{2}+\frac{a \ln\left(x\sqrt{c}+\sqrt{c x^2+a}\right)}{2\sqrt{c}}}{f} + \frac{\left(-e\sqrt{-4df+e^2}+2df-e^2\right)\left(\frac{\sqrt{4\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)^2 c -}}{}\right.}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(1/2*x*(c*x^2+a)^(1/2)+1/2*a/c^(1/2)*ln(x*c^(1/2)+(c*x^2+a)^(1/2)))+1/2
*(-e*(-4*d*f+e^2)^(1/2)+2*d*f-e^2)/f^2/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-
1/2*c^(1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*
(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2
*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)
/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2
)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(
-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^
2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/
```

```
(-4*d*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4
*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f
*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c
^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-
4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2\sqrt{a+cx^2}}{d+ex+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*(c*x**2+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**2*sqrt(a + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2\,\sqrt{c\,x^2+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2*(a + c*x^2)^(1/2))/(d + e*x + f*x^2),x)
```

```
[Out] int((x^2*(a + c*x^2)^(1/2))/(d + e*x + f*x^2), x)
```

# 3.53 $\int \frac{x\sqrt{a+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=395

$$\frac{\sqrt{a+cx^2}}{f}-\frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2}-\frac{\left(2cdef-\left(e-\sqrt{e^2-4df}\right)\left(af^2+c(e^2-df)\right)\right)\tanh^{-1}\left(\frac{\cdots}{\sqrt{2}\,\cdots}\right.}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-\cdots\right.}}$$

```
[Out] -e*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))*c^(1/2)/f^2+(c*x^2+a)^(1/2)/f-1/2*arc
tanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^
2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*c*d*e*f-(a*f^2+c*(-d*f+e^2)
)*(e-(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*
d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)
^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)
))^(1/2))*(2*c*d*e*f-(a*f^2+c*(-d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/f^2*2^(1/
2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 0.64, antiderivative size = 395, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.240$, Rules used = {1034, 1094, 223, 212, 1048, 739}

$$-\frac{\left(2cdef-\left(e-\sqrt{e^2-4df}\right)\left(af^2+c(e^2-df)\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(2cdef-\left(\sqrt{e^2-4df}+e\right)\left(af^2+c(e^2-df)\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2}+\frac{\sqrt{a+cx^2}}{f}$$

Antiderivative was successfully verified.

```
[In] Int[(x*Sqrt[a + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] Sqrt[a + c*x^2]/f - (Sqrt[c]*e*ArcTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/f^2 -
((2*c*d*e*f - (e - Sqrt[e^2 - 4*d*f])*(a*f^2 + c*(e^2 - d*f)))*ArcTanh[(2*a
*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f -
e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqr
t[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + ((2*c*d*e*f - (e + Sq
rt[e^2 - 4*d*f])*(a*f^2 + c*(e^2 - d*f)))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2
- 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]
*Sqrt[a + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 -
2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rubi steps

$$\int \frac{x\sqrt{a+cx^2}}{d+ex+fx^2}\,dx = \frac{\sqrt{a+cx^2}}{f} + \frac{\int \frac{-(cd-af)x-cex^2}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{f}$$

$$= \frac{\sqrt{a+cx^2}}{f} + \frac{\int \frac{cde+(ce^2+f(-cd+af))x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{f^2} - \frac{(ce)\int \frac{1}{\sqrt{a+cx^2}}\,dx}{f^2}$$

$$= \frac{\sqrt{a+cx^2}}{f} - \frac{(ce)\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{f^2} + \frac{\left(2cdef - \left(e - \sqrt{e^2-4df}\right)\right.}{}$$

$$= \frac{\sqrt{a+cx^2}}{f} - \frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2} - \frac{\left(2cdef - \left(e - \sqrt{e^2-4df}\right)(af^2+c(e\right.}{}$$

$$= \frac{\sqrt{a+cx^2}}{f} - \frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2} - \frac{\left(2cdef - \left(e - \sqrt{e^2-4df}\right)(af^2+c(e\right.}{\sqrt{2}\,f^2\sqrt{e^2}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.37, size = 379, normalized size = 0.96

$$\frac{f\sqrt{a+cx^2}+\sqrt{c}\,e\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)-\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ace^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-acdf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}de\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-ce^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+cdf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-af^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*Sqrt[a + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (f*Sqrt[a + c*x^2] + Sqrt[c]*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]] - RootSu
m[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f
*#1^4 & , (a*c*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - a*c*d*f*Log[-
(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + a^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c
*x^2] - #1] + 2*c^(3/2)*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - c
*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + c*d*f*Log[-(Sqrt[c]*x)
 + Sqrt[a + c*x^2] - #1]*#1^2 - a*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
#1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3)
 & ])/f^2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1244 vs. $2(350) = 700$.
time = 0.14, size = 1245, normalized size = 3.15

| method | result |
|---|---|
| default | $\dfrac{\left(e+\sqrt{-4df+e^2}\right)\left(\dfrac{\sqrt{4\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)^2 c-\frac{4c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{2}{}}}{2}\right.}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(1/2*(4*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2
)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+
(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e
^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+
(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(
-4*d*f+e^2)^(1/2))/f)))+1/2*(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(1
/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(-4*d*
f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+
(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^
2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-
1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c
```

```
*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(
1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x\sqrt{a + cx^2}}{d + ex + fx^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x**2+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x*sqrt(a + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x\,\sqrt{c\,x^2+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(a + c*x^2)^(1/2))/(d + e*x + f*x^2),x)
```

```
[Out] int((x*(a + c*x^2)^(1/2))/(d + e*x + f*x^2), x)
```

# 3.54 $\int \frac{\sqrt{a+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=298

$$\frac{\sqrt{c}\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f}-\frac{\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2\cdots}\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-\cdots\right.}}\right.}{\sqrt{2}\,f\sqrt{e^2-4df}}$$

```
[Out] arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))*c^(1/2)/f-1/2*arctanh(1/2*(2*a*f-c*x*(e-
(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*
f+e^2)^(1/2)))^(1/2))*(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)/f*
2^(1/2)/(-4*d*f+e^2)^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)
))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/
2))*(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)/f*2^(1/2)/(-4*d*f+e^
2)^(1/2)
```

**Rubi** [A]
time = 0.25, antiderivative size = 298, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.208$, Rules used = {1005, 223, 212, 1048, 739}

$$-\frac{\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}}+\frac{\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}}+\frac{\sqrt{c}\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + c*x^2]/(d + e*x + f*x^2),x]
```

```
[Out] (Sqrt[c]*ArcTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/f - (Sqrt[2*a*f^2 + c*(e^2 -
 2*d*f - e*Sqrt[e^2 - 4*d*f])]*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x
)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c
*x^2])])/(Sqrt[2]*f*Sqrt[e^2 - 4*d*f]) + (Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e
*Sqrt[e^2 - 4*d*f])]*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]
*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(
Sqrt[2]*f*Sqrt[e^2 - 4*d*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1005

```
Int[Sqrt[(a_) + (c_.)*(x_)^2]/((d_) + (e_.)*(x_) + (f_.)*(x_)^2), x_Symbol]
 :> Dist[c/f, Int[1/Sqrt[a + c*x^2], x], x] - Dist[1/f, Int[(c*d - a*f + c*
e*x)/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)), x], x] /; FreeQ[{a, c, d, e, f},
x] && NeQ[e^2 - 4*d*f, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{\sqrt{a+cx^2}}{d+ex+fx^2}\,dx = -\frac{\int \frac{cd-af+cex}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{f} + \frac{c\int \frac{1}{\sqrt{a+cx^2}}\,dx}{f}$$

$$= \frac{c\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{f} + \frac{\left(2f(cd-af)-ce\left(e+\sqrt{e^2-4df}\right)\right)\int \frac{}{\left(e+\sqrt{\right.}\right.}}{f\sqrt{e^2-4df}}$$

$$= \frac{\sqrt{c}\,\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f} - \frac{\left(2f(cd-af)-ce\left(e+\sqrt{e^2-4df}\right)\right)\text{Subst}\left(\int \frac{}{4af^2+c(}\right.}{f\sqrt{e^2-4}}$$

$$= \frac{\sqrt{c}\,\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f} - \frac{\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\,\tanh^{-1}\left(\frac{}{\sqrt{2}\sqrt{}}\right.}{\sqrt{2}\,f\sqrt{e^2-}}$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.32, size = 264, normalized size = 0.89

$$\frac{-\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)+\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ace\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}d\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2a\sqrt{c}\,f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-ce\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{f}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + c*x^2]/(d + e*x + f*x^2),x]
```

```
[Out] (-(Sqrt[c]*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]]) + RootSum[a^2*f + 2*a*Sqrt[
c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c*e*Lo
g[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqr
t[a + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
#1]*#1 - c*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e +
4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/f
```

**Maple [B]** Leaf count of result is larger than twice the leaf count of optimal. 1211 vs. 2(259) = 518.
time = 0.13, size = 1212, normalized size = 4.07

| method | result |
|---|---|
| default | $-\frac{\frac{\sqrt{4\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)^2 c-\frac{4c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{2\sqrt{-4df+e^2}\,ce+}{f^2}}}{2}}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(
-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e+(-4*d*f+e^2)^(1/2))/f
*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^
(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(
1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1
```

```
/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2
*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4
*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/
(-4*d*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4
*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f
*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c
^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-
4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 1196 vs. 2(263) = 526.
time = 106.71, size = 2400, normalized size = 8.05

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] [-1/4*(sqrt(2)*f*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 - f^2*e^2)*sqrt
(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a
*c*e^2 + sqrt(2)*(4*c*d*f*e - c*e^3 - (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(
4*d*f^5 - f^4*e^2)))*sqrt(c*x^2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d
*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) -
```

```
2*(4*a*d*f^3 - a*f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) - sqrt(2)*
f*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*
f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2 - sqrt(2
)*(4*c*d*f*e - c*e^3 - (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e
^2)))*sqrt(c*x^2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 - f^2*e^2)
*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) - 2*(4*a*d*f^3 -
a*f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) + sqrt(2)*f*sqrt((2*c*d*f
 - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2))
)/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2 + sqrt(2)*(4*c*d*f*e -
c*e^3 + (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))*sqrt(c*x^
2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/
(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) + 2*(4*a*d*f^3 - a*f^2*e^2)*sqrt
(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) - sqrt(2)*f*sqrt((2*c*d*f - 2*a*f^2 - c*
e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^
2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2 - sqrt(2)*(4*c*d*f*e - c*e^3 + (4*d*f^
3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))*sqrt(c*x^2 + a)*sqrt((2*
c*d*f - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*
e^2)))/(4*d*f^3 - f^2*e^2)) + 2*(4*a*d*f^3 - a*f^2*e^2)*sqrt(-c^2*e^2/(4*d*
f^5 - f^4*e^2)))/x) - 2*sqrt(c)*log(-2*c*x^2 - 2*sqrt(c*x^2 + a)*sqrt(c)*x
- a))/f, -1/4*(sqrt(2)*f*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 - f^2*e
^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x
*e - 2*a*c*e^2 + sqrt(2)*(4*c*d*f*e - c*e^3 - (4*d*f^3*e - f^2*e^3)*sqrt(-c
^2*e^2/(4*d*f^5 - f^4*e^2)))*sqrt(c*x^2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^
2 + (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*
e^2)) - 2*(4*a*d*f^3 - a*f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) -
sqrt(2)*f*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e
^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2
- sqrt(2)*(4*c*d*f*e - c*e^3 - (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5
 - f^4*e^2)))*sqrt(c*x^2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*d*f^3 -
f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) - 2*(4*a*
d*f^3 - a*f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) + sqrt(2)*f*sqrt(
(2*c*d*f - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f
^4*e^2)))/(4*d*f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2 + sqrt(2)*(4*c*
d*f*e - c*e^3 + (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))*s
qrt(c*x^2 + a)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-
c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) + 2*(4*a*d*f^3 - a*f^2*e
^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/x) - sqrt(2)*f*sqrt((2*c*d*f - 2*a*
f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))/(4*d*
f^3 - f^2*e^2))*log((4*c^2*d*x*e - 2*a*c*e^2 - sqrt(2)*(4*c*d*f*e - c*e^3 +
 (4*d*f^3*e - f^2*e^3)*sqrt(-c^2*e^2/(4*d*f^5 - f^4*e^2)))*sqrt(c*x^2 + a)*
sqrt((2*c*d*f - 2*a*f^2 - c*e^2 - (4*d*f^3 - f^2*e^2)*sqrt(-c^2*e^2/(4*d*f^
5 - f^4*e^2)))/(4*d*f^3 - f^2*e^2)) + 2*(4*a*d*f^3 - a*f^2*e^2)*sqrt(-c^2*e
^2/(4*d*f^5 - f^4*e^2)))/x) + 4*sqrt(-c)*arctan(sqrt(-c)*x/sqrt(c*x^2 + a))
)/f]
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + cx^2}}{d + ex + fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(sqrt(a + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(1/2)/(d + e*x + f*x^2),x)
```

```
[Out] int((a + c*x^2)^(1/2)/(d + e*x + f*x^2), x)
```

## 3.55 $\int \frac{\sqrt{a+cx^2}}{x(d+ex+fx^2)} \, dx$

**Optimal**. Leaf size=358

$$\frac{\left(2aef+(cd-af)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}-$$

```
[Out] -arctanh((c*x^2+a)^(1/2)/a^(1/2))*a^(1/2)/d+1/2*arctanh(1/2*(2*a*f-c*x*(e-(
-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f
+e^2)^(1/2)))^(1/2))*(2*a*e*f+(-a*f+c*d)*(e-(-4*d*f+e^2)^(1/2)))/d*2^(1/2)/
(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*a
rctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*
f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*a*e*f+(-a*f+c*d)*(e+(-4*d
*f+e^2)^(1/2)))/d*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*
f+e^2)^(1/2)))^(1/2)
```

**Rubi [A]**
time = 0.83, antiderivative size = 358, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 9, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {6860, 272, 52, 65, 214, 1034, 1048, 739, 212}

$$\frac{\left(\left(e-\sqrt{e^2-4df}\right)(cd-af)+2aef\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\left(\left(\sqrt{e^2-4df}+e\right)(cd-af)+2aef\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{a}\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + c*x^2]/(x*(d + e*x + f*x^2)),x]
```

```
[Out] ((2*a*e*f + (c*d - a*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sq
rt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4
*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e
^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - ((2*a*e*f + (c*d - a*f)*(e + Sqrt[e^2
 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a
*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d
*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]) -
 (Sqrt[a]*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/d
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
```

```
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+cx^2}}{x\left(d+ex+fx^2\right)}\,dx=\int\left(\frac{\sqrt{a+cx^2}}{dx}+\frac{(-e-fx)\sqrt{a+cx^2}}{d\left(d+ex+fx^2\right)}\right)dx$$

$$=\frac{\int\frac{\sqrt{a+cx^2}}{x}\,dx}{d}+\frac{\int\frac{(-e-fx)\sqrt{a+cx^2}}{d+ex+fx^2}\,dx}{d}$$

$$=-\frac{\sqrt{a+cx^2}}{d}+\frac{\text{Subst}\left(\int\frac{\sqrt{a+cx}}{x}\,dx,x,x^2\right)}{2d}+\frac{\int\frac{-aef+f(cd-af)x}{\sqrt{a+cx^2}\left(d+ex+fx^2\right)}\,dx}{df}$$

$$=\frac{a\text{Subst}\left(\int\frac{1}{x\sqrt{a+cx}}\,dx,x,x^2\right)}{2d}-\frac{\left(2aef+(cd-af)\left(e-\sqrt{e^2-4df}\right)\right)\int\frac{}{\left(e-\right.}}{d\sqrt{e^2-4df}}$$

$$=\frac{a\text{Subst}\left(\int\frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\,dx,x,\sqrt{a+cx^2}\right)}{cd}+\frac{\left(2aef+(cd-af)\left(e-\sqrt{e^2-4df}\right)\right)\text{S}}{}$$

$$=\frac{\left(2aef+(cd-af)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-\right.}}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{\right.}}}\right.}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 0.37, size = 299, normalized size = 0.84

$$\frac{2\sqrt{a}\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)+\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-acd\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2a\sqrt{c}\,e\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-af\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{d}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + c*x^2]/(x*(d + e*x + f*x^2)),x]
```

```
[Out] (2*Sqrt[a]*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/Sqrt[a]] + RootSum[a^2*f +
 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & ,
 (-(a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]) + a^2*f*Log[-(Sqrt[c]*x
) + Sqrt[a + c*x^2] - #1] + 2*a*Sqrt[c]*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2
] - #1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - a*f*Log[-(
Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1
 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/d
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1315 vs. $2(315) = 630$.
time = 0.13, size = 1316, normalized size = 3.68

| method | result |
|---|---|
| default | $2f\left(\frac{\sqrt{4\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)^2c-\frac{4c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{2\sqrt{-4df+e^2}\,ce+}{f^2}}}{2}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(1/2)/x/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*
(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-
```

```
4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-
4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4
*d*f+e^2)^(1/2))/f)))+2*f/(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(1/2*(
4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(-4*d*f+e^
2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*
(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2)
)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))-4*f/(-e+(-4*d*
f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*((c*x^2+a)^(1/2)-a^(1/2)*ln((2*a+2*a^(
1/2)*(c*x^2+a)^(1/2))/x))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(c*x^2 + a)/((f*x^2 + x*e + d)*x), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 1127 vs. $2(322) = 644$.
time = 12.36, size = 2266, normalized size = 6.33

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] [-1/4*(sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d^3*f - d^2*e^2)*sqr
t(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2))*log((2*a*c*d*x*e + sq
```

```
rt(2)*(4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)
)*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d
^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2)) - a^2*e^2 - (4*a*d^3*f - a*d^2*e^2)*
sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/x) - sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f
 + a*e^2 + (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f
 - d^2*e^2))*log((2*a*c*d*x*e - sqrt(2)*(4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)
*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2))*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d
^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2)) -
a^2*e^2 - (4*a*d^3*f - a*d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/x) -
sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f - d^2*e^2)*sqrt(-a^2*
e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2))*log((2*a*c*d*x*e + sqrt(2)*(
4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2))*sqrt(
-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f -
d^4*e^2)))/(4*d^3*f - d^2*e^2)) - a^2*e^2 + (4*a*d^3*f - a*d^2*e^2)*sqrt(-a
^2*e^2/(4*d^5*f - d^4*e^2)))/x) + sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^
2 - (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*
e^2))*log((2*a*c*d*x*e - sqrt(2)*(4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-
a^2*e^2/(4*d^5*f - d^4*e^2))*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f -
d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2)) - a^2*e^2
 + (4*a*d^3*f - a*d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/x) - 2*sqrt(
a)*log(-(c*x^2 - 2*sqrt(c*x^2 + a)*sqrt(a) + 2*a)/x^2))/d, -1/4*(sqrt(2)*d*
sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5
*f - d^4*e^2)))/(4*d^3*f - d^2*e^2))*log((2*a*c*d*x*e + sqrt(2)*(4*d^4*f -
d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2))*sqrt(-(2*c*d^2
- 2*a*d*f + a*e^2 + (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))
/(4*d^3*f - d^2*e^2)) - a^2*e^2 - (4*a*d^3*f - a*d^2*e^2)*sqrt(-a^2*e^2/(4*
d^5*f - d^4*e^2)))/x) - sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d^3
*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2))*log(
(2*a*c*d*x*e - sqrt(2)*(4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4
*d^5*f - d^4*e^2))*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*d^3*f - d^2*e^2)*s
qrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2)) - a^2*e^2 - (4*a*d^
3*f - a*d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/x) - sqrt(2)*d*sqrt(-(
2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^
4*e^2)))/(4*d^3*f - d^2*e^2))*log((2*a*c*d*x*e + sqrt(2)*(4*d^4*f - d^3*e^2
)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2))*sqrt(-(2*c*d^2 - 2*a*d
*f + a*e^2 - (4*d^3*f - d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3
*f - d^2*e^2)) - a^2*e^2 + (4*a*d^3*f - a*d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f -
 d^4*e^2)))/x) + sqrt(2)*d*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f - d^
2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2))*log((2*a*c*
d*x*e - sqrt(2)*(4*d^4*f - d^3*e^2)*sqrt(c*x^2 + a)*sqrt(-a^2*e^2/(4*d^5*f
- d^4*e^2))*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*d^3*f - d^2*e^2)*sqrt(-a^
2*e^2/(4*d^5*f - d^4*e^2)))/(4*d^3*f - d^2*e^2)) - a^2*e^2 + (4*a*d^3*f - a
*d^2*e^2)*sqrt(-a^2*e^2/(4*d^5*f - d^4*e^2)))/x) - 4*sqrt(-a)*arctan(sqrt(-
a)/sqrt(c*x^2 + a)))/d]
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a+cx^2}}{x\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(1/2)/x/(f*x**2+e*x+d),x)
```

```
[Out] Integral(sqrt(a + c*x**2)/(x*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+a}}{x\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(1/2)/(x*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + c*x^2)^(1/2)/(x*(d + e*x + f*x^2)), x)
```

## 3.56 $\int \frac{\sqrt{a+cx^2}}{x^2(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=382

$$-\frac{\sqrt{a+cx^2}}{dx}-\frac{f\left(2cd^2+a\left(e^2-2df+e\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}\right.}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

```
[Out] e*arctanh((c*x^2+a)^(1/2)/a^(1/2))*a^(1/2)/d^2-(c*x^2+a)^(1/2)/d/x-1/2*f*ar
ctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f
^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*c*d^2+a*(e^2-2*d*f+e*(-4*d
*f+e^2)^(1/2)))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*
d*f+e^2)^(1/2)))^(1/2)+1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))
*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
)*(2*c*d^2+a*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/
2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]

time = 0.90, antiderivative size = 382, normalized size of antiderivative = 1.00, number of steps used = 18, number of rules used = 12, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.444$, Rules used = {6860, 283, 223, 212, 272, 52, 65, 214, 1034, 1094, 1048, 739}

$$-\frac{f\left(a\left(e\sqrt{e^2-4df}-2df+e^2\right)+2cd^2\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(a\left(-e\sqrt{e^2-4df}-2df+e^2\right)+2cd^2\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d^2}-\frac{\sqrt{a+cx^2}}{dx}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + c*x^2]/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] -(Sqrt[a + c*x^2]/(d*x)) - (f*(2*c*d^2 + a*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*
f]))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 +
c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^2*Sqrt
[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + (f*(
2*c*d^2 + a*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sq
rt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4
*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*
(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]) + (Sqrt[a]*e*ArcTanh[Sqrt[a + c*x^2]/
Sqrt[a]])/d^2
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
```

b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b, c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] && !(IGtQ[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) && !ILtQ[m + n + 2, 0] && IntLinearQ[a, b, c, d, m, n, x]

Rule 65

Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) + d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Denominator[m]] && IntLinearQ[a, b, c, d, m, n, x]

Rule 212

Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (GtQ[a, 0] || LtQ[b, 0])

Rule 214

Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]

Rule 223

Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x], x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] && !GtQ[a, 0]

Rule 272

Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]

Rule 283

Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*x)^(m + 1)*((a + b*x^n)^p/(c*(m + 1))), x] - Dist[b*n*(p/(c^n*(m + 1))), Int[(c*x)^(m + n)*(a + b*x^n)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && IGtQ[n, 0] && GtQ[p, 0] && LtQ[m, -1] && !ILtQ[(m + n*p + n + 1)/n, 0] && IntBinomialQ[a, b, c, n, m, p, x]

Rule 739

Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ

```
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+cx^2}}{x^2\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{\sqrt{a+cx^2}}{dx^2} - \frac{e\sqrt{a+cx^2}}{d^2x} + \frac{\left(e^2-df+efx\right)\sqrt{a+cx^2}}{d^2\left(d+ex+fx^2\right)}\right)dx$$

$$= \frac{\int \frac{\left(e^2-df+efx\right)\sqrt{a+cx^2}}{d+ex+fx^2}\,dx}{d^2} + \frac{\int \frac{\sqrt{a+cx^2}}{x^2}\,dx}{d} - \frac{e\int \frac{\sqrt{a+cx^2}}{x}\,dx}{d^2}$$

$$= \frac{e\sqrt{a+cx^2}}{d^2} - \frac{\sqrt{a+cx^2}}{dx} + \frac{c\int \frac{1}{\sqrt{a+cx^2}}\,dx}{d} - \frac{e\text{Subst}\left(\int \frac{\sqrt{a+cx}}{x}\,dx, x, x^2\right)}{2d^2}$$

$$= -\frac{\sqrt{a+cx^2}}{dx} - \frac{c\int \frac{1}{\sqrt{a+cx^2}}\,dx}{d} + \frac{c\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{d} - \frac{(ae)\text{S}}{}$$

$$= -\frac{\sqrt{a+cx^2}}{dx} + \frac{\sqrt{c}\,\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{d} - \frac{c\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{d}$$

$$= -\frac{\sqrt{a+cx^2}}{dx} + \frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d^2} + \frac{\left(f\left(2cd^2+a\left(e^2-2df-e\sqrt{e^2-\right.\right.\right.}{}$$

$$= -\frac{\sqrt{a+cx^2}}{dx} - \frac{f\left(2cd^2+a\left(e^2-2df+e\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{}{\sqrt{2}\sqrt{2af^2+\right.}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df\right.}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.40, size = 336, normalized size = 0.88

$$-\frac{d\sqrt{a+cx^2}+2\sqrt{a}\,ex\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)+x\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{a^2ef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}d^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+2a\sqrt{c}\,e^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2a\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-aef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{d^2x}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + c*x^2]/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] -((d*Sqrt[a + c*x^2] + 2*Sqrt[a]*e*x*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/
Sqrt[a]] + x*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2
*Sqrt[c]*e*#1^3 + f*#1^4 & , (a^2*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
#1] + 2*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 2*a*Sqrt[
c]*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*d*f*Log[-(
```

```
Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - a*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c
*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2
*f*#1^3) & ])/(d^2*x))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1414 vs. $2(337) = 674$.

time = 0.13, size = 1415, normalized size = 3.70

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1415 |
| risch | Expression too large to display | 2536 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(1/2)/x^2/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*
f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(
1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*
(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*
(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*
(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f)))+4*f^2/(-e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/
2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c
*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(
-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f
*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1
/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+
(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e
^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))-4*f/(
-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(-1/a/x*(c*x^2+a)^(3/2)+2*c/a
*(1/2*x*(c*x^2+a)^(1/2)+1/2*a/c^(1/2)*ln(x*c^(1/2)+(c*x^2+a)^(1/2))))-16*f^
```

```
2*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2*((c*x^2+a)^(1/2)-a^(
1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(c*x^2 + a)/((f*x^2 + x*e + d)*x^2), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 2545 vs. $2(346) = 692$.
time = 80.30, size = 5102, normalized size = 13.36

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] [-1/4*(sqrt(2)*d^2*x*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d
*f)*e^2 + (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4
+ (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5
*f - d^4*e^2))*log((4*a*c*d*f*x*e^3 - 2*a^2*f*e^4 + 4*(c^2*d^3*f - 2*a*c*d^
2*f^2)*x*e + sqrt(2)*((4*d^6*f*e - d^5*e^3)*sqrt(c*x^2 + a)*sqrt(-(a^2*e^6
+ 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2
)/(4*d^9*f - d^8*e^2)) + sqrt(c*x^2 + a)*(a*d*e^5 + (c*d^3 - 6*a*d^2*f)*e^3
 - 4*(c*d^4*f - 2*a*d^3*f^2)*e))*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c
*d^2 - 4*a*d*f)*e^2 + (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a
^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e
^2)))/(4*d^5*f - d^4*e^2)) - 2*(a*c*d^2*f - 2*a^2*d*f^2)*e^2 + 2*(4*a*d^5*f
^2 - a*d^4*f*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 -
 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/x) - sqrt(2)*d^2*x
*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2 + (4*d^5*f -
 d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d
^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2))*log((
4*a*c*d*f*x*e^3 - 2*a^2*f*e^4 + 4*(c^2*d^3*f - 2*a*c*d^2*f^2)*x*e - sqrt(2)
*((4*d^6*f*e - d^5*e^3)*sqrt(c*x^2 + a)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2
*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2
)) + sqrt(c*x^2 + a)*(a*d*e^5 + (c*d^3 - 6*a*d^2*f)*e^3 - 4*(c*d^4*f - 2*a*
d^3*f^2)*e))*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2
+ (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d
^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4
*e^2)) - 2*(a*c*d^2*f - 2*a^2*d*f^2)*e^2 + 2*(4*a*d^5*f^2 - a*d^4*f*e^2)*sq
```

```
rt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2
*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/x) - sqrt(2)*d^2*x*sqrt((2*c*d^3*f - 2
*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2 - (4*d^5*f - d^4*e^2)*sqrt(-(a^2
*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2
)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2))*log((4*a*c*d*f*x*e^3 - 2*
a^2*f*e^4 + 4*(c^2*d^3*f - 2*a*c*d^2*f^2)*x*e + sqrt(2)*((4*d^6*f*e - d^5*e
^3)*sqrt(c*x^2 + a)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4
 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)) - sqrt(c*x^2 + a)
*(a*d*e^5 + (c*d^3 - 6*a*d^2*f)*e^3 - 4*(c*d^4*f - 2*a*d^3*f^2)*e))*sqrt((2
*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2 - (4*d^5*f - d^4*e^2
)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4
*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2)) - 2*(a*c*d^2*
f - 2*a^2*d*f^2)*e^2 - 2*(4*a*d^5*f^2 - a*d^4*f*e^2)*sqrt(-(a^2*e^6 + 2*(a*
c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^
9*f - d^8*e^2)))/x) + sqrt(2)*d^2*x*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 -
 (c*d^2 - 4*a*d*f)*e^2 - (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 -
2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^
8*e^2)))/(4*d^5*f - d^4*e^2))*log((4*a*c*d*f*x*e^3 - 2*a^2*f*e^4 + 4*(c^2*d
^3*f - 2*a*c*d^2*f^2)*x*e - sqrt(2)*((4*d^6*f*e - d^5*e^3)*sqrt(c*x^2 + a)*
sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a
^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)) - sqrt(c*x^2 + a)*(a*d*e^5 + (c*d^3 -
 6*a*d^2*f)*e^3 - 4*(c*d^4*f - 2*a*d^3*f^2)*e))*sqrt((2*c*d^3*f - 2*a*d^2*f
^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2 - (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2
*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(
4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2)) - 2*(a*c*d^2*f - 2*a^2*d*f^2)*e^2
 - 2*(4*a*d^5*f^2 - a*d^4*f*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e
^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)))/x)
- 2*sqrt(a)*x*e*log(-(c*x^2 + 2*sqrt(c*x^2 + a)*sqrt(a) + 2*a)/x^2) + 4*sqr
t(c*x^2 + a)*d)/(d^2*x), -1/4*(sqrt(2)*d^2*x*sqrt((2*c*d^3*f - 2*a*d^2*f^2
- a*e^4 - (c*d^2 - 4*a*d*f)*e^2 + (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^6 + 2*(a
*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d
^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2))*log((4*a*c*d*f*x*e^3 - 2*a^2*f*e^4 +
 4*(c^2*d^3*f - 2*a*c*d^2*f^2)*x*e + sqrt(2)*((4*d^6*f*e - d^5*e^3)*sqrt(c*
x^2 + a)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^
3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e^2)) + sqrt(c*x^2 + a)*(a*d*e^5 +
 (c*d^3 - 6*a*d^2*f)*e^3 - 4*(c*d^4*f - 2*a*d^3*f^2)*e))*sqrt((2*c*d^3*f -
2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4*a*d*f)*e^2 + (4*d^5*f - d^4*e^2)*sqrt(-(a^
2*e^6 + 2*(a*c*d^2 - 2*a^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^
2)*e^2)/(4*d^9*f - d^8*e^2)))/(4*d^5*f - d^4*e^2)) - 2*(a*c*d^2*f - 2*a^2*d
*f^2)*e^2 + 2*(4*a*d^5*f^2 - a*d^4*f*e^2)*sqrt(-(a^2*e^6 + 2*(a*c*d^2 - 2*a
^2*d*f)*e^4 + (c^2*d^4 - 4*a*c*d^3*f + 4*a^2*d^2*f^2)*e^2)/(4*d^9*f - d^8*e
^2)))/x) - sqrt(2)*d^2*x*sqrt((2*c*d^3*f - 2*a*d^2*f^2 - a*e^4 - (c*d^2 - 4
*a*d*f)*e^2 + (4*d^5*f - d^4*e^2)*sqrt(-(a^2*e^...
```

**Sympy** [F]

time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + cx^2}}{x^2 (d + ex + fx^2)} dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(1/2)/x**2/(f*x**2+e*x+d),x)
```

```
[Out] Integral(sqrt(a + c*x**2)/(x**2*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2 + a}}{x^2\,\left(f\,x^2 + e\,x + d\right)} dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(1/2)/(x^2*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + c*x^2)^(1/2)/(x^2*(d + e*x + f*x^2)), x)
```

# 3.57 $\int \frac{\sqrt{a+cx^2}}{x^3\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=507

$$-\frac{\sqrt{a+cx^2}}{2dx^2}+\frac{e\sqrt{a+cx^2}}{d^2x}+\frac{f\left(cd^2\left(e+\sqrt{e^2-4df}\right)+a\left(e^3-3def+e^2\sqrt{e^2-4df}-df\sqrt{e^2-4df}\right)\right)\tan}{\sqrt{2}\,d^3\sqrt{e^2-4df}\,\sqrt{2af^2+c\left(e^2-2\right.}}$$

```
[Out] -1/2*c*arctanh((c*x^2+a)^(1/2)/a^(1/2))/d/a^(1/2)-(-d*f+e^2)*arctanh((c*x^2
+a)^(1/2)/a^(1/2))*a^(1/2)/d^3-1/2*(c*x^2+a)^(1/2)/d/x^2+e*(c*x^2+a)^(1/2)/
d^2/x+1/2*f*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a
)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(c*d^2*(e+(-4*d
*f+e^2)^(1/2))+a*(e^3-3*d*e*f+e^2*(-4*d*f+e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2)
))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2
)))^(1/2)-1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x
^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(c*d^2*(e-(
-4*d*f+e^2)^(1/2))+a*(e^3-3*d*e*f-e^2*(-4*d*f+e^2)^(1/2)+d*f*(-4*d*f+e^2)^(
1/2)))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^
(1/2)))^(1/2)
```

**Rubi** [A]
time = 1.18, antiderivative size = 507, normalized size of antiderivative = 1.00, number of steps used = 22, number of rules used = 13, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.482$, Rules used = {6860, 272, 43, 65, 214, 283, 223, 212, 52, 1034, 1094, 1048, 739}

$$-\frac{\sqrt{a}\left(e^2-df\right)\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d^3}+\frac{e\sqrt{a+cx^2}}{d^2x}+\frac{f\left(a\left(e^2\sqrt{e^2-4df}-df\sqrt{e^2-4df}-3def+e^3\right)+cd^2\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(a\left(-e^2\sqrt{e^2-4df}+df\sqrt{e^2-4df}-3def+e^3\right)+cd^2\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{a+cx^2}}{2dx^2}-\frac{c\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{2\sqrt{a}\,d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + c*x^2]/(x^3*(d + e*x + f*x^2)),x]
```

```
[Out] -1/2*Sqrt[a + c*x^2]/(d*x^2) + (e*Sqrt[a + c*x^2])/(d^2*x) + (f*(c*d^2*(e +
 Sqrt[e^2 - 4*d*f]) + a*(e^3 - 3*d*e*f + e^2*Sqrt[e^2 - 4*d*f] - d*f*Sqrt[e
^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2
*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]
*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])
]) - (f*(c*d^2*(e - Sqrt[e^2 - 4*d*f]) + a*(e^3 - 3*d*e*f - e^2*Sqrt[e^2 -
4*d*f] + d*f*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])
*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a +
 c*x^2])])/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e
*Sqrt[e^2 - 4*d*f])]) - (c*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/(2*Sqrt[a]*d)
- (Sqrt[a]*(e^2 - d*f)*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/d^3
```

Rule 43

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + 1))), x] - Dist[d*(n/(b*(m + 1))), I
nt[(a + b*x)^(m + 1)*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b, c, d, n}, x]
 && NeQ[b*c - a*d, 0] && ILtQ[m, -1] &&  !IntegerQ[n] && GtQ[n, 0]
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 283

```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*
x)^(m + 1)*((a + b*x^n)^p/(c*(m + 1))), x] - Dist[b*n*(p/(c^n*(m + 1))), In
t[(c*x)^(m + n)*(a + b*x^n)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && IGtQ[
n, 0] && GtQ[p, 0] && LtQ[m, -1] &&  !ILtQ[(m + n*p + n + 1)/n, 0] && IntBi
nomialQ[a, b, c, n, m, p, x]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+cx^2}}{x^3\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{\sqrt{a+cx^2}}{dx^3}-\frac{e\sqrt{a+cx^2}}{d^2x^2}+\frac{\left(e^2-df\right)\sqrt{a+cx^2}}{d^3x}+\frac{\left(-e\left(e^2-2df\right)-f\left(e^2-\right.\right.}{d^3\left(d+ex-\right.}\right.$$

$$=\frac{\int \frac{\left(-e\left(e^2-2df\right)-f\left(e^2-df\right)x\right)\sqrt{a+cx^2}}{d+ex+fx^2}\,dx}{d^3}+\frac{\int \frac{\sqrt{a+cx^2}}{x^3}\,dx}{d}-\frac{e\int \frac{\sqrt{a+cx^2}}{x^2}\,dx}{d^2}+$$

$$=-\frac{\left(e^2-df\right)\sqrt{a+cx^2}}{d^3}+\frac{e\sqrt{a+cx^2}}{d^2x}+\frac{\text{Subst}\left(\int \frac{\sqrt{a+cx}}{x^2}\,dx,x,x^2\right)}{2d}-\frac{(ce)\int}{}$$

$$=-\frac{\sqrt{a+cx^2}}{2dx^2}+\frac{e\sqrt{a+cx^2}}{d^2x}+\frac{c\text{Subst}\left(\int \frac{1}{x\sqrt{a+cx}}\,dx,x,x^2\right)}{4d}+\frac{(ce)\int \frac{1}{\sqrt{a+c}}}{d^2}$$

$$=-\frac{\sqrt{a+cx^2}}{2dx^2}+\frac{e\sqrt{a+cx^2}}{d^2x}-\frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{d^2}+\frac{\text{Subst}\left(\int \frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\,dx\right.}{2d}$$

$$=-\frac{\sqrt{a+cx^2}}{2dx^2}+\frac{e\sqrt{a+cx^2}}{d^2x}-\frac{c\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{2\sqrt{a}\,d}-\frac{\sqrt{a}\left(e^2-df\right)\tanh^{-1}\left(\right.}{d^3}$$

$$=-\frac{\sqrt{a+cx^2}}{2dx^2}+\frac{e\sqrt{a+cx^2}}{d^2x}+\frac{f\left(cd^2\left(e+\sqrt{e^2-4df}\right)+a\left(e^3-3def+e^2\sqrt{e^2}\right.\right.}{\sqrt{2}\,d^3}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.81, size = 533, normalized size = 1.05

$$\frac{\frac{d(-d+2ex)\sqrt{a+cx^2}}{x^2}+\frac{2cd^2\tanh^{-1}\left(\frac{\sqrt{c}x-\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}}-4\sqrt{a}\left(e^2-df\right)\tanh^{-1}\left(\frac{-\sqrt{c}x+\sqrt{a+cx^2}}{\sqrt{a}}\right)-2\text{RootSum}\left[a^2f+2a\sqrt{c}e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}e\#1^3+f\#1^4\&,\frac{-acd^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)-a^2e^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)+a^2df^2\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)-2c^{3/2}d^2e\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1-2a\sqrt{c}e^3\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1+4a\sqrt{c}def\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1+cd^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1^2+ae^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1^2-adf^2\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}e+4cd\#1-2af\#1-3\sqrt{c}e\#1^2+2f\#1^3}\&\right]}{2d^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + c*x^2]/(x^3*(d + e*x + f*x^2)),x]
```

```
[Out] ((d*(-d + 2*e*x)*Sqrt[a + c*x^2])/x^2 + (2*c*d^2*ArcTanh[(Sqrt[c]*x - Sqrt[
a + c*x^2])/Sqrt[a]])/Sqrt[a] - 4*Sqrt[a]*(e^2 - d*f)*ArcTanh[(-(Sqrt[c]*x)
 + Sqrt[a + c*x^2])/Sqrt[a]] - 2*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#
1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(a*c*d^2*f*Log[-(Sqrt[c]
```

```
*x) + Sqrt[a + c*x^2] - #1]) - a^2*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]
 - #1] + a^2*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*c^(3/2)*d^2
*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*e^3*Log[-(Sqrt
[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 4*a*Sqrt[c]*d*e*f*Log[-(Sqrt[c]*x) + Sq
rt[a + c*x^2] - #1]*#1 + c*d^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#
1^2 + a*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - a*d*f^2*Log[-
(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#
1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(2*d^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1520 vs. $2(442) = 884$.
time = 0.16, size = 1521, normalized size = 3.00

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1521 |
| risch | Expression too large to display | 2711 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(1/2)/x^3/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 8*f^3/(e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/
2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1
/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(
e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c
*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(
e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f)))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2)
)*(-1/2/a/x^2*(c*x^2+a)^(3/2)+1/2*c/a*((c*x^2+a)^(1/2)-a^(1/2)*ln((2*a+2*a^
(1/2)*(c*x^2+a)^(1/2))/x)))+8*f^3/(-e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^2)^(1
/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2)
)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-
(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(
1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e
```

```
+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+
e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))-16*f
^2*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2*(-1/a/x*(c*x^2+a)^(
3/2)+2*c/a*(1/2*x*(c*x^2+a)^(1/2)+1/2*a/c^(1/2)*ln(x*c^(1/2)+(c*x^2+a)^(1/2
))))+64*f^3*(d*f-e^2)/(-e+(-4*d*f+e^2)^(1/2))^3/(e+(-4*d*f+e^2)^(1/2))^3*((
c*x^2+a)^(1/2)-a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^3/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(sqrt(c*x^2 + a)/((f*x^2 + x*e + d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^3/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + cx^2}}{x^3 \left(d + ex + fx^2\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(1/2)/x**3/(f*x**2+e*x+d),x)
[Out] Integral(sqrt(a + c*x**2)/(x**3*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(1/2)/x^3/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument Value
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+a}}{x^3\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(1/2)/(x^3*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + c*x^2)^(1/2)/(x^3*(d + e*x + f*x^2)), x)
```

# 3.58 $\int \frac{x^2 (a+cx^2)^{3/2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=795

$$-\frac{(8e(af^2+c(e^2-2df))-f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4}-\frac{(4e-3fx)\,(a+cx^2)^{3/2}}{12f^2}+\frac{(3a^2f^4+12acf}{}$$

```
[Out] -1/12*(-3*f*x+4*e)*(c*x^2+a)^(3/2)/f^2+1/8*(3*a^2*f^4+12*a*c*f^2*(-d*f+e^2)
+8*c^2*(d^2*f^2-3*d*e^2*f+e^4))*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))/f^5/c^(1
/2)-1/8*(8*e*(a*f^2+c*(-2*d*f+e^2))-f*(3*a*f^2+4*c*(-d*f+e^2))*x)*(c*x^2+a)
^(1/2)/f^4-1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^
2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(a^2*f^4*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2))+2*a*c*f^2*(e^4-4*d*e^2*f+2*d^2*f^2-e^3*(-4*d*
f+e^2)^(1/2)+2*d*e*f*(-4*d*f+e^2)^(1/2))+c^2*(e^6-6*d*e^4*f+9*d^2*e^2*f^2-2
*d^3*f^3-e^5*(-4*d*f+e^2)^(1/2)+4*d*e^3*f*(-4*d*f+e^2)^(1/2)-3*d^2*e*f^2*(-
4*d*f+e^2)^(1/2)))/f^5*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(
-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2))
)*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2
))*(a^2*f^4*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))+2*a*c*f^2*(e^4-4*d*e^2*f+2*d^2
*f^2+e^3*(-4*d*f+e^2)^(1/2)-2*d*e*f*(-4*d*f+e^2)^(1/2))+c^2*(e^6-6*d*e^4*f+
9*d^2*e^2*f^2-2*d^3*f^3+e^5*(-4*d*f+e^2)^(1/2)-4*d*e^3*f*(-4*d*f+e^2)^(1/2)
+3*d^2*e*f^2*(-4*d*f+e^2)^(1/2)))/f^5*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c
*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 2.70, antiderivative size = 795, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.259$, Rules used = {1083, 1082, 1094, 223, 212, 1048, 739}

Antiderivative was successfully verified.

```
[In] Int[(x^2*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] -1/8*((8*e*(a*f^2 + c*(e^2 - 2*d*f)) - f*(3*a*f^2 + 4*c*(e^2 - d*f))*x)*Sqr
t[a + c*x^2])/f^4 - ((4*e - 3*f*x)*(a + c*x^2)^(3/2))/(12*f^2) + ((3*a^2*f^
4 + 12*a*c*f^2*(e^2 - d*f) + 8*c^2*(e^4 - 3*d*e^2*f + d^2*f^2))*ArcTanh[(Sq
rt[c]*x)/Sqrt[a + c*x^2]])/(8*Sqrt[c]*f^5) - ((a^2*f^4*(e^2 - 2*d*f - e*Sqr
t[e^2 - 4*d*f]) + 2*a*c*f^2*(e^4 - 4*d*e^2*f + 2*d^2*f^2 - e^3*Sqrt[e^2 - 4
*d*f] + 2*d*e*f*Sqrt[e^2 - 4*d*f]) + c^2*(e^6 - 6*d*e^4*f + 9*d^2*e^2*f^2 -
```

```
 2*d^3*f^3 - e^5*Sqrt[e^2 - 4*d*f] + 4*d*e^3*f*Sqrt[e^2 - 4*d*f] - 3*d^2*e*
f^2*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt
[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])]
)/(Sqrt[2]*f^5*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2
 - 4*d*f])]) + ((a^2*f^4*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + 2*a*c*f^2*(e
^4 - 4*d*e^2*f + 2*d^2*f^2 + e^3*Sqrt[e^2 - 4*d*f] - 2*d*e*f*Sqrt[e^2 - 4*d
*f]) + c^2*(e^6 - 6*d*e^4*f + 9*d^2*e^2*f^2 - 2*d^3*f^3 + e^5*Sqrt[e^2 - 4*
d*f] - 4*d*e^3*f*Sqrt[e^2 - 4*d*f] + 3*d^2*e*f^2*Sqrt[e^2 - 4*d*f]))*ArcTan
h[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*
d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f^5*Sqrt[e^2 - 4*d*
f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1082

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + c*x^2)^p*((d + e*x
+ f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f
^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^
q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) + (p +
q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2)) + f*(B*e - 2*A*f)*(2*p + 2*q + 3
))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) +
```

```
(p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*(q + 1) - c*(C*e - B*
f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C*(e^2 - 4*d*f)*
(2*p + q + 2) + f*(2*C*d - B*e + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x]
/; FreeQ[{a, c, d, e, f, A, B, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0
] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q
, 0]
```

Rule 1083

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) + (e_.)*(x_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(
p + q + 1)*x)*(a + c*x^2)^p*((d + e*x + f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)
*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3)), Int[
(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) -
c*(C*e)*(2*p + 2*q + 3)) + (p + q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2))
+ f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e
)*(2*p + 2*q + 3)) + (p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*
(q + 1) - c*(C*e)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C
*(e^2 - 4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x^2, x]
, x], x] /; FreeQ[{a, c, d, e, f, A, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && Gt
Q[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !
IGtQ[q, 0]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rubi steps

$$\int \frac{x^2(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx = -\frac{(4e-3fx)\,(a+cx^2)^{3/2}}{12f^2} - \frac{\int \frac{\sqrt{a+cx^2}\,(3acdf-3ce(4cd-af)x-3c(3af^2+4c(e^2-df))x^2)}{d+ex+fx^2}\,dx}{12cf^2}$$

$$= -\frac{(8e(af^2+c(e^2-2df)) - f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4} - \frac{(4e-3fx)\,(a+c}{12f^2}$$

$$= -\frac{(8e(af^2+c(e^2-2df)) - f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4} - \frac{(4e-3fx)\,(a+c}{12f^2}$$

$$= -\frac{(8e(af^2+c(e^2-2df)) - f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4} - \frac{(4e-3fx)\,(a+c}{12f^2}$$

$$= -\frac{(8e(af^2+c(e^2-2df)) - f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4} - \frac{(4e-3fx)\,(a+c}{12f^2}$$

$$= -\frac{(8e(af^2+c(e^2-2df)) - f(3af^2+4c(e^2-df))\,x)\,\sqrt{a+cx^2}}{8f^4} - \frac{(4e-3fx)\,(a+c}{12f^2}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.01, size = 942, normalized size = 1.18

Antiderivative was successfully verified.

```
[In] Integrate[(x^2*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] (f*Sqrt[a + c*x^2]*(a*f^2*(-32*e + 15*f*x) - 2*c*(12*e^3 - 6*e^2*f*x + 4*e*
f*(-6*d + f*x^2) - 3*f^2*x*(-2*d + f*x^2))) - (3*(3*a^2*f^4 + 12*a*c*f^2*(e
^2 - d*f) + 8*c^2*(e^4 - 3*d*e^2*f + d^2*f^2))*Log[-(Sqrt[c]*x) + Sqrt[a +
c*x^2]])/Sqrt[c] + 24*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f
*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c^2*e^5*Log[-(Sqrt[c]*x) + Sqrt[a
+ c*x^2] - #1] - 4*a*c^2*d*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] +
 3*a*c^2*d^2*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*a^2*c*e^3*f
^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 4*a^2*c*d*e*f^3*Log[-(Sqrt[c]
```

```
*x) + Sqrt[a + c*x^2] - #1] + a^3*e*f^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]
- #1] + 2*c^(5/2)*d*e^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 6*c^(
5/2)*d^2*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 2*c^(5/2)*d^3*
f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 4*a*c^(3/2)*d*e^2*f^2*Log
[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 4*a*c^(3/2)*d^2*f^3*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2] - #1]*#1 + 2*a^2*Sqrt[c]*d*f^4*Log[-(Sqrt[c]*x) + Sq
rt[a + c*x^2] - #1]*#1 - c^2*e^5*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#
1^2 + 4*c^2*d*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - 3*c^2*d
^2*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - 2*a*c*e^3*f^2*Log[
-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + 4*a*c*d*e*f^3*Log[-(Sqrt[c]*x)
+ Sqrt[a + c*x^2] - #1]*#1^2 - a^2*e*f^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]
 - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1
^3) & ])/(24*f^5)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2366 vs. $2(722) = 1444$.
time = 0.15, size = 2367, normalized size = 2.98

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2367 |
| risch | Expression too large to display | 9198 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(1/4*x*(c*x^2+a)^(3/2)+3/4*a*(1/2*x*(c*x^2+a)^(1/2)+1/2*a/c^(1/2)*ln(x*
c^(1/2)+(c*x^2+a)^(1/2))))+1/2*(-e*(-4*d*f+e^2)^(1/2)+2*d*f-e^2)/f^2/(-4*d*
f+e^2)^(1/2)*(1/3*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(
1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+e^2)^(1/2))/f)/c*((x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c
*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/
2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+
e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c
*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f
+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-
1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+
((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f
^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(
```

```
((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e
^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))))+1/2*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d*f+e^2)^(1/2)*(1/3*((x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*
c*(e-(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e
-(-4*d*f+e^2)^(1/2))/f)/c*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d
*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2
*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-
e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/
f^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(
4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(-4*d*f+e^
2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*
(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2)
)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out

time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Timed out
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2\,{\left(c\,x^2+a\right)}^{3/2}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x)
```

```
[Out] int((x^2*(a + c*x^2)^(3/2))/(d + e*x + f*x^2), x)
```

## 3.59 $\int \frac{x\left(a+cx^2\right)^{3/2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=553

$$\frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3}+\frac{(a+cx^2)^{3/2}}{3f}-\frac{\sqrt{c}\,e(3af^2+2c(e^2-2df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2f^4}-$$

```
[Out] 1/3*(c*x^2+a)^(3/2)/f-1/2*e*(3*a*f^2+2*c*(-2*d*f+e^2))*arctanh(x*c^(1/2)/(c
*x^2+a)^(1/2))*c^(1/2)/f^4+1/2*(2*a*f^2+2*c*(-d*f+e^2)-c*e*f*x)*(c*x^2+a)^(
1/2)/f^3-1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+
a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*c*d*e*f*(2*
a*f^2+c*(-2*d*f+e^2))-(a^2*f^4+2*a*c*f^2*(-d*f+e^2)+c^2*(d^2*f^2-3*d*e^2*f+
e^4))*(e-(-4*d*f+e^2)^(1/2)))/f^4*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+
e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(
1/2)))^(1/2))*(2*c*d*e*f*(2*a*f^2+c*(-2*d*f+e^2))-(a^2*f^4+2*a*c*f^2*(-d*f+
e^2)+c^2*(d^2*f^2-3*d*e^2*f+e^4))*(e+(-4*d*f+e^2)^(1/2)))/f^4*2^(1/2)/(-4*d
*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 1.54, antiderivative size = 553, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.280$, Rules used = {1034, 1082, 1094, 223, 212, 1048, 739}

$$-\frac{\left(2cdef(2af^2+c(e^2-2df))-\left(e-\sqrt{e^2-4df}\right)(a^2f^4+2acf^2(e^2-df)+c^2(d^2f^2-3de^2f+e^4))\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}f^4\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(2cdef(2af^2+c(e^2-2df))-\left(\sqrt{e^2-4df}+e\right)(a^2f^4+2acf^2(e^2-df)+c^2(d^2f^2-3de^2f+e^4))\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}f^4\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{c}\,e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)(3af^2+2c(e^2-2df))}{2f^4}+\frac{\sqrt{a+cx^2}\,(2(af^2+c(e^2-df))-cefx)}{2f^3}+\frac{(a+cx^2)^{3/2}}{3f}$$

Antiderivative was successfully verified.

```
[In] Int[(x*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] ((2*(a*f^2 + c*(e^2 - d*f)) - c*e*f*x)*Sqrt[a + c*x^2])/(2*f^3) + (a + c*x^
2)^(3/2)/(3*f) - (Sqrt[c]*e*(3*a*f^2 + 2*c*(e^2 - 2*d*f))*ArcTanh[(Sqrt[c]*
x)/Sqrt[a + c*x^2]])/(2*f^4) - ((2*c*d*e*f*(2*a*f^2 + c*(e^2 - 2*d*f)) - (e
 - Sqrt[e^2 - 4*d*f])*(a^2*f^4 + 2*a*c*f^2*(e^2 - d*f) + c^2*(e^4 - 3*d*e^2
*f + d^2*f^2)))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt
[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[
2]*f^4*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f
])]) + ((2*c*d*e*f*(2*a*f^2 + c*(e^2 - 2*d*f)) - (e + Sqrt[e^2 - 4*d*f])*(a
^2*f^4 + 2*a*c*f^2*(e^2 - d*f) + c^2*(e^4 - 3*d*e^2*f + d^2*f^2)))*ArcTanh[
(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*
```

```
f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f^4*Sqrt[e^2 - 4*d*f]
*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1082

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + c*x^2)^p*((d + e*x
+ f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f
^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^
q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) + (p +
q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2)) + f*(B*e - 2*A*f)*(2*p + 2*q + 3
```

```
))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) +
(p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*(q + 1) - c*(C*e - B*
f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C*(e^2 - 4*d*f)*
(2*p + q + 2) + f*(2*C*d - B*e + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x]
/; FreeQ[{a, c, d, e, f, A, B, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0
] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q
, 0]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rubi steps

$$\int \frac{x(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx = \frac{(a+cx^2)^{3/2}}{3f} + \frac{\int \frac{\sqrt{a+cx^2}\,(-3(cd-af)x-3cex^2)}{d+ex+fx^2}\,dx}{3f}$$

$$= \frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3} + \frac{(a+cx^2)^{3/2}}{3f} - \frac{\int \frac{-3ac^2def+3c(ace^2f+2(cd-af)(\cdots}{\sqrt{a+\cdots}}}{\cdots}$$

$$= \frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3} + \frac{(a+cx^2)^{3/2}}{3f} - \frac{\int \frac{-3ac^2def^2-3c^2de(3af^2+2c(e^2-\cdots}{\cdots}}{\cdots}$$

$$= \frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3} + \frac{(a+cx^2)^{3/2}}{3f} - \frac{(ce(3af^2+2c(e^2-2df)))\,\mathrm{S}\cdots}{\cdots}$$

$$= \frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3} + \frac{(a+cx^2)^{3/2}}{3f} - \frac{\sqrt{c}\,e(3af^2+2c(e^2-2df))\cdots}{2f^4\cdots}$$

$$= \frac{(2(af^2+c(e^2-df))-cefx)\sqrt{a+cx^2}}{2f^3} + \frac{(a+cx^2)^{3/2}}{3f} - \frac{\sqrt{c}\,e(3af^2+2c(e^2-2df))\cdots}{2f^4\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order

3 in optimal.
time = 0.72, size = 755, normalized size = 1.37

$$\frac{f\sqrt{a+cx^2}\,(8af^2+c(6e^2-6df-3efx+2f^2x^2))+3\sqrt{c}\,e(3af^2+2c(e^2-2df))\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)-6\,\mathrm{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ac^2e^4\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-3ac^2de^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+ac^2d^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2a^2ce^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2a^2cdf^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^3f^4\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{5/2}de^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-4c^{5/2}d^2ef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+4ac^{3/2}def^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-c^2e^4\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+3c^2de^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-c^2d^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-2ace^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+2acdf^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-a^2f^4\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{6f^4}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x]
```

```
[Out] (f*Sqrt[a + c*x^2]*(8*a*f^2 + c*(6*e^2 - 6*d*f - 3*e*f*x + 2*f^2*x^2)) + 3*
Sqrt[c]*e*(3*a*f^2 + 2*c*(e^2 - 2*d*f))*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]]
 - 6*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]
*e*#1^3 + f*#1^4 & , (a*c^2*e^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] -
3*a*c^2*d*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + a*c^2*d^2*f^2*Lo
g[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*a^2*c*e^2*f^2*Log[-(Sqrt[c]*x) +
 Sqrt[a + c*x^2] - #1] - 2*a^2*c*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
 #1] + a^3*f^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(5/2)*d*e^3*L
og[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 4*c^(5/2)*d^2*e*f*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2] - #1]*#1 + 4*a*c^(3/2)*d*e*f^2*Log[-(Sqrt[c]*x) + Sq
rt[a + c*x^2] - #1]*#1 - c^2*e^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#
1^2 + 3*c^2*d*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - c^2*d^2
*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - 2*a*c*e^2*f^2*Log[-(Sq
rt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + 2*a*c*d*f^3*Log[-(Sqrt[c]*x) + Sqrt
[a + c*x^2] - #1]*#1^2 - a^2*f^4*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#
1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])
/(6*f^4)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2293 vs. $2(499) = 998$.
time = 0.14, size = 2294, normalized size = 4.15

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2294 |
| risch | Expression too large to display | 7259 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(1/3*((x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+(-4
*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+
e^2)^(1/2))/f)/c*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1
/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*d*f
```

```
+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d
*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/
2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*
((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*d*
f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(
1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*
(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*
(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*
(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f))))+1/2*(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/
f*(1/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2)^(3/2)-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x-1/2/f*(-e+(
-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/c*((x-1/2/f*(-e+(-4*d*f+e^2
)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+
1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*(-(
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))
^2/f^2)/c^(3/2)*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e
^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2
*c*d*f+c*e^2)/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*
d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*
ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^
(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-
1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2
*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-
(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/
f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4
*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2))))))
```

**Maxima [F(-2)]**

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x(a+cx^2)^{\frac{3}{2}}}{d+ex+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x*(a + c*x**2)**(3/2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x\,{\left(c\,x^2+a\right)}^{3/2}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(a + c*x^2)^(3/2))/(d + e*x + f*x^2),x)
[Out] int((x*(a + c*x^2)^(3/2))/(d + e*x + f*x^2), x)
```

# 3.60 $\int \frac{(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=484

$$-\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2}+\frac{\sqrt{c}\,(3af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2f^3}-\frac{\left(ce\left(e-\sqrt{e^2-4df}\right)(2af^2+c$$

```
[Out] 1/2*(3*a*f^2+2*c*(-d*f+e^2))*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))*c^(1/2)/f^3
-1/2*c*(-f*x+2*e)*(c*x^2+a)^(1/2)/f^2-1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f
+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^
(1/2)))^(1/2))*(-2*f*(2*a*c*d*f^2-a^2*f^3+c^2*d*(-d*f+e^2))+c*e*(2*a*f^2+c*
(-2*d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f
^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+
(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*
f+e^2)^(1/2)))^(1/2))*(-2*f*(2*a*c*d*f^2-a^2*f^3+c^2*d*(-d*f+e^2))+c*e*(2*a
*f^2+c*(-2*d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)
/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 2.76, antiderivative size = 482, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {993, 1094, 223, 212, 1048, 739}

$$\frac{\left(-2a^2f^4-ce\left(e-\sqrt{e^2-4df}\right)(2af^2+c(e^2-2df))+4acdf^3+2c^2df(e^2-df)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}f^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\left(-2a^2f^4-ce\left(\sqrt{e^2-4df}+e\right)(2af^2+c(e^2-2df))+4acdf^3+2c^2df(e^2-df)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}f^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\sqrt{c}\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)(3af^2+2c(e^2-df))}{2f^3}-\frac{c\sqrt{a+cx^2}\,(2e-fx)}{2f^2}$$

Antiderivative was successfully verified.

```
[In] Int[(a + c*x^2)^(3/2)/(d + e*x + f*x^2),x]
```

```
[Out] -1/2*(c*(2*e - f*x)*Sqrt[a + c*x^2])/f^2 + (Sqrt[c]*(3*a*f^2 + 2*c*(e^2 - d
*f))*ArcTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/(2*f^3) + ((4*a*c*d*f^3 - 2*a^2*
f^4 + 2*c^2*d*f*(e^2 - d*f) - c*e*(e - Sqrt[e^2 - 4*d*f])*(2*a*f^2 + c*(e^2
 - 2*d*f)))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a
*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*f
^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])])
 - ((4*a*c*d*f^3 - 2*a^2*f^4 + 2*c^2*d*f*(e^2 - d*f) - c*e*(e + Sqrt[e^2 -
4*d*f])*(2*a*f^2 + c*(e^2 - 2*d*f)))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d
*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt
[a + c*x^2])])/(Sqrt[2]*f^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f
 + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 993

```
Int[((a_.) + (c_.)*(x_)^2)^(p_)*((d_.) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(-c)*(e*(2*p + q) - 2*f*(p + q)*x)*(a + c*x^2)^(p - 1)*((d
 + e*x + f*x^2)^(q + 1)/(2*f^2*(p + q)*(2*p + 2*q + 1))), x] - Dist[1/(2*f^
2*(p + q)*(2*p + 2*q + 1)), Int[(a + c*x^2)^(p - 2)*(d + e*x + f*x^2)^q*Sim
p[(-a)*c*e^2*(1 - p)*(2*p + q) + a*(p + q)*(-2*a*f^2*(2*p + 2*q + 1) + c*(2
*d*f - e^2*(2*p + q))) + (2*(c*d - a*f)*(c*e)*(1 - p)*(2*p + q) + 4*a*c*e*f
*(1 - p)*(p + q))*x + (p*c^2*e^2*(1 - p) - c*(p + q)*(2*a*f^2*(4*p + 2*q -
1) + c*(2*d*f*(1 - 2*p) + e^2*(3*p + q - 1))))*x^2, x], x], x] /; FreeQ[{a,
 c, d, e, f, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 1] && NeQ[p + q, 0] &&
 NeQ[2*p + 2*q + 1, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rubi steps

$$\int \frac{(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx = -\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2} - \frac{\int \frac{af(cd-2af)-ce(2cd-af)x-c(3af^2+2c(e^2-df))x^2}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2f^2}$$

$$= -\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2} - \frac{\int \frac{af^2(cd-2af)+cd(3af^2+2c(e^2-df))+(-cef(2cd-af)+ce(3af^2+2c(e^2-df)}{\sqrt{a+cx^2}\,(d+ex+fx^2)}}{2f^3}$$

$$= -\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{(c(3af^2+2c(e^2-df)))\,\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{2f^3}$$

$$= -\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{\sqrt{c}\,(3af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2f^3} + \frac{\left(2f(af\right.}{}$$

$$= -\frac{c(2e-fx)\sqrt{a+cx^2}}{2f^2} + \frac{\sqrt{c}\,(3af^2+2c(e^2-df))\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2f^3} - \frac{\left(ce\left(e-\right.\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.57, size = 542, normalized size = 1.12

$$\frac{cf(-2e+fx)\sqrt{a+cx^2}+\sqrt{c}\,(-2ce^2+2cdf-3af^2)\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)+2\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ac^2e^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2ac^2def\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2a^2cef^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{5/2}de^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2c^{5/2}d^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+4ac^{3/2}df^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2a^2\sqrt{c}\,f^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-c^2e^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+2c^2def\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-2acef^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{2f^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + c*x^2)^(3/2)/(d + e*x + f*x^2),x]
```

```
[Out] (c*f*(-2*e + f*x)*Sqrt[a + c*x^2] + Sqrt[c]*(-2*c*e^2 + 2*c*d*f - 3*a*f^2)*
Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]] + 2*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 +
4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c^2*e^3*Log[-(Sq
rt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*a*c^2*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a
+ c*x^2] - #1] + 2*a^2*c*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2
*c^(5/2)*d*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*c^(5/2)*d^2*
f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 4*a*c^(3/2)*d*f^2*Log[-(Sqr
t[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*a^2*Sqrt[c]*f^3*Log[-(Sqrt[c]*x) + S
qrt[a + c*x^2] - #1]*#1 - c^2*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*
#1^2 + 2*c^2*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - 2*a*c*e*
```

```
f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1
- 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(2*f^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2260 vs. $2(435) = 870$.
time = 0.14, size = 2261, normalized size = 4.67

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2261 |
| risch | Expression too large to display | 5464 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/(-4*d*f+e^2)^(1/2)*(1/3*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d
*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f*(1/4*(2
*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+e^2)^(1/2))/f)/c*((x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)
+1/8*(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e+(-4*d*f
+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e
+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(
e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e+(-4*d*f+e^2)^(1/2)
)/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))
/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*
2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+
(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))
)+1/(-4*d*f+e^2)^(1/2)*(1/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4
*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f*(1/
4*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/c*((x-
1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+
(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2)^(1/2)+1/8*(2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(
```

```
e-(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*
(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)
))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1
/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*
(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(
-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*
d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f
^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e
^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e
-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/
2/f*(-e+(-4*d*f+e^2)^(1/2))))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(a+cx^2)^{\frac{3}{2}}}{d+ex+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral((a + c*x**2)**(3/2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{{\left(c\,x^2+a\right)}^{3/2}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(3/2)/(d + e*x + f*x^2),x)
```

```
[Out] int((a + c*x^2)^(3/2)/(d + e*x + f*x^2), x)
```

## 3.61 $\int \frac{(a+cx^2)^{3/2}}{x(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=496

$$\frac{a\sqrt{a+cx^2}}{d}+\frac{(cd-af)\sqrt{a+cx^2}}{df}-\frac{c^{3/2}e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2}-\frac{\left(2ef(c^2d^2-a^2f^2)-(c^2de^2-f(cd-af)\right.}{\sqrt{2}\,d}$$

```
[Out] -c^(3/2)*e*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))/f^2-a^(3/2)*arctanh((c*x^2+a)
^(1/2)/a^(1/2))/d+a*(c*x^2+a)^(1/2)/d+(-a*f+c*d)*(c*x^2+a)^(1/2)/d/f-1/2*ar
ctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f
^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*e*f*(-a^2*f^2+c^2*d^2)-(c^
2*d*e^2-f*(-a*f+c*d)^2)*(e-(-4*d*f+e^2)^(1/2)))/d/f^2*2^(1/2)/(-4*d*f+e^2)^
(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2
*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*
d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*e*f*(-a^2*f^2+c^2*d^2)-(c^2*d*e^2-f*(-
a*f+c*d)^2)*(e+(-4*d*f+e^2)^(1/2)))/d/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f
^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 1.38, antiderivative size = 496, normalized size of antiderivative = 1.00, number of steps used = 17, number of rules used = 11, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.407$, Rules used = {6860, 272, 52, 65, 214, 1034, 1094, 223, 212, 1048, 739}

$$-\frac{a^{3/2}\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d}-\frac{\left(2ef(c^2d^2-a^2f^2)-\left(e-\sqrt{e^2-4df}\right)(c^2de^2-f(cd-af)^2)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,df^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(2ef(c^2d^2-a^2f^2)-\left(\sqrt{e^2-4df}+e\right)(c^2de^2-f(cd-af)^2)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,df^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{c^{3/2}e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2}+\frac{\sqrt{a+cx^2}\,(cd-af)}{df}+\frac{a\sqrt{a+cx^2}}{d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + c*x^2)^(3/2)/(x*(d + e*x + f*x^2)),x]
```

```
[Out] (a*Sqrt[a + c*x^2])/d + ((c*d - a*f)*Sqrt[a + c*x^2])/(d*f) - (c^(3/2)*e*Ar
cTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/f^2 - ((2*e*f*(c^2*d^2 - a^2*f^2) - (c^
2*d*e^2 - f*(c*d - a*f)^2)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e -
 Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2
- 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2]*d*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2
 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + ((2*e*f*(c^2*d^2 - a^2*f^2) -
(c^2*d*e^2 - f*(c*d - a*f)^2)*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(
e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e
^2 - 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2]*d*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*
f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]) - (a^(3/2)*ArcTanh[Sqrt[a + c
*x^2]/Sqrt[a]])/d
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{(a+cx^2)^{3/2}}{x(d+ex+fx^2)}\,dx = \int \left(\frac{(a+cx^2)^{3/2}}{dx} + \frac{(-e-fx)(a+cx^2)^{3/2}}{d(d+ex+fx^2)}\right) dx$$

$$= \frac{\int \frac{(a+cx^2)^{3/2}}{x}\,dx}{d} + \frac{\int \frac{(-e-fx)(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx}{d}$$

$$= -\frac{(a+cx^2)^{3/2}}{3d} + \frac{\text{Subst}\left(\int \frac{(a+cx)^{3/2}}{x}\,dx, x, x^2\right)}{2d} + \frac{\int \frac{(-3aef+3f(cd-af)x)\sqrt{a+cx^2}}{d+ex+fx^2}\,dx}{3df}$$

$$= \frac{(cd-af)\sqrt{a+cx^2}}{df} + \frac{a\text{Subst}\left(\int \frac{\sqrt{a+cx}}{x}\,dx, x, x^2\right)}{2d} + \frac{\int \frac{-3a^2ef^2-3f(cd-af)^2x-3c^2a}{\sqrt{a+cx^2}\,(d+ex+fx^2}}{3df^2}$$

$$= \frac{a\sqrt{a+cx^2}}{d} + \frac{(cd-af)\sqrt{a+cx^2}}{df} + \frac{a^2\text{Subst}\left(\int \frac{1}{x\sqrt{a+cx}}\,dx, x, x^2\right)}{2d} + \int \frac{3c^2d^2e}{}$$

$$= \frac{a\sqrt{a+cx^2}}{d} + \frac{(cd-af)\sqrt{a+cx^2}}{df} + \frac{a^2\text{Subst}\left(\int \frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\,dx, x, \sqrt{a+cx^2}\right)}{cd} - \frac{(c^2e}{}$$

$$= \frac{a\sqrt{a+cx^2}}{d} + \frac{(cd-af)\sqrt{a+cx^2}}{df} - \frac{c^{3/2}e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2} - \frac{a^{3/2}\tanh^{-1}\left(\right.}{d}$$

$$= \frac{a\sqrt{a+cx^2}}{d} + \frac{(cd-af)\sqrt{a+cx^2}}{df} - \frac{c^{3/2}e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{f^2} - \frac{\left(2ef(c^2d^2-\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.56, size = 552, normalized size = 1.11

Antiderivative was successfully verified.

```
[In] Integrate[(a + c*x^2)^(3/2)/(x*(d + e*x + f*x^2)),x]
```

```
[Out] (c*d*f*Sqrt[a + c*x^2] + 2*a^(3/2)*f^2*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2]
)/Sqrt[a]] + c^(3/2)*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]] + RootSum[a^2*
```

```
f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4
& , (-(a*c^2*d*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]) + a*c^2*d^2*f*
Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*a^2*c*d*f^2*Log[-(Sqrt[c]*x) +
 Sqrt[a + c*x^2] - #1] + a^3*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] -
 2*c^(5/2)*d^2*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + 2*a^2*Sqrt[c
]*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + c^2*d*e^2*Log[-(Sqrt[
c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - c^2*d^2*f*Log[-(Sqrt[c]*x) + Sqrt[a +
c*x^2] - #1]*#1^2 + 2*a*c*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1
^2 - a^2*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e +
4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(d*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2378 vs. $2(443) = 886$.

time = 0.13, size = 2379, normalized size = 4.80

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2379 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(3/2)/x/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(1/3*((x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1
/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+(-4*d
*f+e^2)^(1/2))/f*(1/4*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+e^
2)^(1/2))/f)/c*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2
))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2
*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c
*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*d*f+e
^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*((
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/
2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e
+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e
+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^
2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))
/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+
```

```
(-4*d*f+e^2)^(1/2))/f))))+2*f/(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(1
/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(3/2)-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/c*((x-1/2/f*(-e+(-4*d*f+e^2)^(1
/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*
(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*(-(-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2/f
^2)/c^(3/2)*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^
(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d
*f+c*e^2)/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+
e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((
-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2
)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e
^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(
1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f
^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(
e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2
))))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(1/3*(c*x^2+a)^(3
/2)+a*((c*x^2+a)^(1/2)-a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x)))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] integrate((c*x^2 + a)^(3/2)/((f*x^2 + x*e + d)*x), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(a+cx^2)^{\frac{3}{2}}}{x\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(3/2)/x/(f*x**2+e*x+d),x)
```

```
[Out] Integral((a + c*x**2)**(3/2)/(x*(d + e*x + f*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{{\left(c\,x^2+a\right)}^{3/2}}{x\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(3/2)/(x*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + c*x^2)^(3/2)/(x*(d + e*x + f*x^2)), x)
```

## 3.62 $\int \frac{(a+cx^2)^{3/2}}{x^2(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=604

$$-\frac{ae\sqrt{a+cx^2}}{d^2}+\frac{3cx\sqrt{a+cx^2}}{2d}+\frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2}-\frac{(a+cx^2)^{3/2}}{dx}+\frac{3a\sqrt{c}\,\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2d}+\frac{\sqrt{c}\,(2\cdots}{\cdots}$$

```
[Out] -(c*x^2+a)^(3/2)/d/x+a^(3/2)*e*arctanh((c*x^2+a)^(1/2)/a^(1/2))/d^2+3/2*a*a
rctanh(x*c^(1/2)/(c*x^2+a)^(1/2))*c^(1/2)/d+1/2*(-3*a*f+2*c*d)*arctanh(x*c^
(1/2)/(c*x^2+a)^(1/2))*c^(1/2)/d/f-a*e*(c*x^2+a)^(1/2)/d^2+3/2*c*x*(c*x^2+a
)^(1/2)/d+1/2*(-c*d*x+2*a*e)*(c*x^2+a)^(1/2)/d^2-1/2*arctanh(1/2*(2*a*f-c*x
*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-
4*d*f+e^2)^(1/2)))^(1/2))*(4*a*c*d^2*f^2+c^2*d^2*(e^2-2*d*f-e*(-4*d*f+e^2)^
(1/2))+a^2*f^2*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))/d^2/f*2^(1/2)/(-4*d*f+e^2)
^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(
2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2
*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(4*a*c*d^2*f^2+a^2*f^2*(e^2-2*d*f-e*(-4*
d*f+e^2)^(1/2))+c^2*d^2*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))/d^2/f*2^(1/2)/(-4
*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 1.72, antiderivative size = 604, normalized size of antiderivative = 1.00, number of steps used = 21, number of rules used = 14, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.518$, Rules used = {6860, 283, 201, 223, 212, 272, 52, 65, 214, 1034, 1082, 1094, 1048, 739}

$$\frac{a^{3/2}e\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d^2}-\frac{\left(a^2f^2\left(e\sqrt{e^2-4df}-2df+e^2\right)+4acd^2f^2+c^2d^2\left(-e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^2f\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(a^2f^2\left(-e\sqrt{e^2-4df}-2df+e^2\right)+4acd^2f^2+c^2d^2\left(e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^2f\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{ae\sqrt{a+cx^2}}{d^2}+\frac{\sqrt{a+cx^2}\,(2ae-cdx)}{2d^2}+\frac{\sqrt{c}\,(2cd-3af)\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2df}-\frac{(a+cx^2)^{3/2}}{dx}+\frac{3cx\sqrt{a+cx^2}}{2d}+\frac{3a\sqrt{c}\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{2d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + c*x^2)^(3/2)/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] -((a*e*Sqrt[a + c*x^2])/d^2) + (3*c*x*Sqrt[a + c*x^2])/(2*d) + ((2*a*e - c*
d*x)*Sqrt[a + c*x^2])/(2*d^2) - (a + c*x^2)^(3/2)/(d*x) + (3*a*Sqrt[c]*ArcT
anh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/(2*d) + (Sqrt[c]*(2*c*d - 3*a*f)*ArcTanh[
(Sqrt[c]*x)/Sqrt[a + c*x^2]])/(2*d*f) - ((4*a*c*d^2*f^2 + c^2*d^2*(e^2 - 2*
d*f - e*Sqrt[e^2 - 4*d*f]) + a^2*f^2*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]))*A
rcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2
 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^2*f*Sqrt[e^2
 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + ((4*a*c*
d^2*f^2 + a^2*f^2*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + c^2*d^2*(e^2 - 2*d*
```

```
f + e*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sq
rt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2]
)])/(Sqrt[2]*d^2*f*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt
[e^2 - 4*d*f])]) + (a^(3/2)*e*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/d^2
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 201

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^p/(n*p
+ 1)), x] + Dist[a*n*(p/(n*p + 1)), Int[(a + b*x^n)^(p - 1), x], x] /; Free
Q[{a, b}, x] && IGtQ[n, 0] && GtQ[p, 0] && (IntegerQ[2*p] || (EqQ[n, 2] &&
IntegerQ[4*p]) || (EqQ[n, 2] && IntegerQ[3*p]) || LtQ[Denominator[p + 1/n],
 Denominator[p]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 283

```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*
x)^(m + 1)*((a + b*x^n)^p/(c*(m + 1))), x] - Dist[b*n*(p/(c^n*(m + 1))), In
t[(c*x)^(m + n)*(a + b*x^n)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && IGtQ[
n, 0] && GtQ[p, 0] && LtQ[m, -1] &&  !ILtQ[(m + n*p + n + 1)/n, 0] && IntBi
nomialQ[a, b, c, n, m, p, x]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1082

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + c*x^2)^p*((d + e*x
+ f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f
^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^
q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) + (p +
q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2)) + f*(B*e - 2*A*f)*(2*p + 2*q + 3
))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) +
```

```
(p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*(q + 1) - c*(C*e - B*
f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C*(e^2 - 4*d*f)*
(2*p + q + 2) + f*(2*C*d - B*e + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x]
/; FreeQ[{a, c, d, e, f, A, B, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0
] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q
, 0]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{(a+cx^2)^{3/2}}{x^2(d+ex+fx^2)}dx = \int \left(\frac{(a+cx^2)^{3/2}}{dx^2} - \frac{e(a+cx^2)^{3/2}}{d^2x} + \frac{(e^2-df+efx)(a+cx^2)^{3/2}}{d^2(d+ex+fx^2)}\right)dx$$

$$= \frac{\int \frac{(e^2-df+efx)(a+cx^2)^{3/2}}{d+ex+fx^2}dx}{d^2} + \frac{\int \frac{(a+cx^2)^{3/2}}{x^2}dx}{d} - \frac{e\int \frac{(a+cx^2)^{3/2}}{x}dx}{d^2}$$

$$= \frac{e(a+cx^2)^{3/2}}{3d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{(3c)\int \sqrt{a+cx^2}\,dx}{d} - \frac{e\text{Subst}\left(\int \frac{(a+cx)^{3/2}}{x}dx, x, x\right.}{2d^2}$$

$$= \frac{3cx\sqrt{a+cx^2}}{2d} + \frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{(3ac)\int \frac{1}{\sqrt{a+cx^2}}dx}{2d}$$

$$= -\frac{ae\sqrt{a+cx^2}}{d^2} + \frac{3cx\sqrt{a+cx^2}}{2d} + \frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{(3ac)}{}$$

$$= -\frac{ae\sqrt{a+cx^2}}{d^2} + \frac{3cx\sqrt{a+cx^2}}{2d} + \frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{3a\sqrt{}}{}$$

$$= -\frac{ae\sqrt{a+cx^2}}{d^2} + \frac{3cx\sqrt{a+cx^2}}{2d} + \frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{3a\sqrt{}}{}$$

$$= -\frac{ae\sqrt{a+cx^2}}{d^2} + \frac{3cx\sqrt{a+cx^2}}{2d} + \frac{(2ae-cdx)\sqrt{a+cx^2}}{2d^2} - \frac{(a+cx^2)^{3/2}}{dx} + \frac{3a\sqrt{}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.64, size = 497, normalized size = 0.82

$$-\frac{2a^{3/2}efx\tanh^{-1}\left(\frac{\sqrt{c}x-\sqrt{a+cx^2}}{\sqrt{a}}\right)+d\left(af\sqrt{a+cx^2}+c^{3/2}dx\log\left(-\sqrt{c}x+\sqrt{a+cx^2}\right)\right)+x\text{RootSum}\left[a^2f+2a\sqrt{c}e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}e\#1^3+f\#1^4\&,\frac{-ac^2d^2e\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)+a^3ef^2\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)-2c^{5/2}d^3\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1+4ac^{3/2}d^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1+2a^2\sqrt{c}e^2f\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1-2a^2\sqrt{c}df^2\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1+c^2d^2e\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1^2-a^2ef^2\log\left(-\sqrt{c}x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}e+4cd\#1-2af\#1-3\sqrt{c}e\#1^2+2f\#1^3}\&\right]}{d^2fx}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + c*x^2)^(3/2)/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] -((2*a^(3/2)*e*f*x*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/Sqrt[a]] + d*(a*f*
Sqrt[a + c*x^2] + c^(3/2)*d*x*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]]) + x*Root
```

```
Sum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 +
 f*#1^4 & , (-(a*c^2*d^2*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]) + a^3*
e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*c^(5/2)*d^3*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2] - #1]*#1 + 4*a*c^(3/2)*d^2*f*Log[-(Sqrt[c]*x) + Sqrt
[a + c*x^2] - #1]*#1 + 2*a^2*Sqrt[c]*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^
2] - #1]*#1 - 2*a^2*Sqrt[c]*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*
#1 + c^2*d^2*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - a^2*e*f^2*Lo
g[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*
f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(d^2*f*x))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2493 vs. $2(529) = 1058$.

time = 0.15, size = 2494, normalized size = 4.13

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2494 |
| risch | Expression too large to display | 5069 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(3/2)/x^2/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*(1/3*((x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))
/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+
(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d
*f+e^2)^(1/2))/f)/c*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)
^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*
d*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*
d*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1
/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4
*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)
^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*
c^(1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c
-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*
e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d
*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2
-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((
(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d
*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(
1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/
```

```
2*(e+(-4*d*f+e^2)^(1/2))/f))))+4*f^2/(-e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)
^(1/2)*(1/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2
*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x-1/2/f*
(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/c*((x-1/2/f*(-e+(-4*d*
f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/
2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*
c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(
1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*
d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e
-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(1/2)*(e-(-4*d*f+e^2)^(1/2
))/f*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))
))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/
f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)/f^2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*l
n(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1
/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^
2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2))))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(-1/a/x*
(c*x^2+a)^(5/2)+4*c/a*(1/4*x*(c*x^2+a)^(3/2)+3/4*a*(1/2*x*(c*x^2+a)^(1/2)+1
/2*a/c^(1/2)*ln(x*c^(1/2)+(c*x^2+a)^(1/2)))))-16*f^2*e/(-e+(-4*d*f+e^2)^(1/
2))^2/(e+(-4*d*f+e^2)^(1/2))^2*(1/3*(c*x^2+a)^(3/2)+a*((c*x^2+a)^(1/2)-a^(1
/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x)))
```

**<u>Maxima</u> [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^2/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate((c*x^2 + a)^(3/2)/((f*x^2 + x*e + d)*x^2), x)
```

**<u>Fricas</u> [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^2/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(a+cx^2)^{\frac{3}{2}}}{x^2\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(3/2)/x**2/(f*x**2+e*x+d),x)
```

```
[Out] Integral((a + c*x**2)**(3/2)/(x**2*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^2/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{{\left(c\,x^2+a\right)}^{3/2}}{x^2\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(3/2)/(x^2*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + c*x^2)^(3/2)/(x^2*(d + e*x + f*x^2)), x)
```

# 3.63 $\int \frac{(a+cx^2)^{3/2}}{x^3(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=668

$$\frac{3c\sqrt{a+cx^2}}{2d}+\frac{a(e^2-df)\sqrt{a+cx^2}}{d^3}-\frac{3cex\sqrt{a+cx^2}}{2d^2}-\frac{(2(cd^2+a(e^2-df))-cdex)\sqrt{a+cx^2}}{2d^3}-\frac{(a+cx^2)^{3/}}{2dx^2}$$

```
[Out] -1/2*(c*x^2+a)^(3/2)/d/x^2+e*(c*x^2+a)^(3/2)/d^2/x-a^(3/2)*(-d*f+e^2)*arcta
nh((c*x^2+a)^(1/2)/a^(1/2))/d^3-3/2*c*arctanh((c*x^2+a)^(1/2)/a^(1/2))*a^(1
/2)/d+3/2*c*(c*x^2+a)^(1/2)/d+a*(-d*f+e^2)*(c*x^2+a)^(1/2)/d^3-3/2*c*e*x*(c
*x^2+a)^(1/2)/d^2-1/2*(2*c*d^2+2*a*(-d*f+e^2)-c*d*e*x)*(c*x^2+a)^(1/2)/d^3+
1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/
(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(c^2*d^3*(e-(-4*d*f+e^2
)^(1/2))+2*a*c*d^2*f*(e+(-4*d*f+e^2)^(1/2))+a^2*f*(e^3-3*d*e*f+e^2*(-4*d*f+
e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2)))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2
+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*arctanh(1/2*(2*a*f-c*x*(e+(-
4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+
e^2)^(1/2)))^(1/2))*(2*a*c*d^2*f*(e-(-4*d*f+e^2)^(1/2))+c^2*d^3*(e+(-4*d*f+
e^2)^(1/2))+a^2*f*(e^3-3*d*e*f-e^2*(-4*d*f+e^2)^(1/2)+d*f*(-4*d*f+e^2)^(1/2
)))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/
2)))^(1/2)
```

**Rubi** [A]
time = 2.09, antiderivative size = 668, normalized size of antiderivative = 1.00, number of steps used = 26, number of rules used = 15, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.556$, Rules used = {6860, 272, 43, 52, 65, 214, 283, 201, 223, 212, 1034, 1082, 1094, 1048, 739}

$$-\frac{a^{3/2}(e^2-df)\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{d^3}+\frac{\left(a^2f\left(e^2\sqrt{e^2-4df}-df\sqrt{e^2-4df}-3def+e^3\right)+2acd^2f\left(\sqrt{e^2-4df}+e\right)+c^2d^3\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\left(a^2f\left(-e^2\sqrt{e^2-4df}+df\sqrt{e^2-4df}-3def+e^3\right)+2acd^2f\left(e-\sqrt{e^2-4df}\right)+c^2d^3\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{a\sqrt{a+cx^2}(e^2-df)}{d^3}+\frac{e(a+cx^2)^{3/2}}{d^2x}-\frac{3cex\sqrt{a+cx^2}}{2d^2}-\frac{\sqrt{a+cx^2}(2(a(e^2-df)+cd^2)-cdex)}{2d^3}-\frac{(a+cx^2)^{3/2}}{2dx^2}+\frac{3c\sqrt{a+cx^2}}{2d}-\frac{3\sqrt{a}c\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{2d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + c*x^2)^(3/2)/(x^3*(d + e*x + f*x^2)),x]
```

```
[Out] (3*c*Sqrt[a + c*x^2])/(2*d) + (a*(e^2 - d*f)*Sqrt[a + c*x^2])/d^3 - (3*c*e*
x*Sqrt[a + c*x^2])/(2*d^2) - ((2*(c*d^2 + a*(e^2 - d*f)) - c*d*e*x)*Sqrt[a
+ c*x^2])/(2*d^3) - (a + c*x^2)^(3/2)/(2*d*x^2) + (e*(a + c*x^2)^(3/2))/(d^
2*x) + ((c^2*d^3*(e - Sqrt[e^2 - 4*d*f]) + 2*a*c*d^2*f*(e + Sqrt[e^2 - 4*d*
f]) + a^2*f*(e^3 - 3*d*e*f + e^2*Sqrt[e^2 - 4*d*f] - d*f*Sqrt[e^2 - 4*d*f])
)*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(
e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^3*Sqrt[e^
2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - ((2*a*c
```

```
*d^2*f*(e - Sqrt[e^2 - 4*d*f]) + c^2*d^3*(e + Sqrt[e^2 - 4*d*f]) + a^2*f*(e
^3 - 3*d*e*f - e^2*Sqrt[e^2 - 4*d*f] + d*f*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a
*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f +
e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqr
t[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]) - (3*Sqrt[a]*c*ArcTanh[
Sqrt[a + c*x^2]/Sqrt[a]])/(2*d) - (a^(3/2)*(e^2 - d*f)*ArcTanh[Sqrt[a + c*x
^2]/Sqrt[a]])/d^3
```

Rule 43

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + 1))), x] - Dist[d*(n/(b*(m + 1))), I
nt[(a + b*x)^(m + 1)*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b, c, d, n}, x]
 && NeQ[b*c - a*d, 0] && ILtQ[m, -1] &&  !IntegerQ[n] && GtQ[n, 0]
```

Rule 52

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^n/(b*(m + n + 1))), x] + Dist[n*((b*c - a*d)/(
b*(m + n + 1))), Int[(a + b*x)^m*(c + d*x)^(n - 1), x], x] /; FreeQ[{a, b,
c, d}, x] && NeQ[b*c - a*d, 0] && GtQ[n, 0] && NeQ[m + n + 1, 0] &&  !(IGtQ
[m, 0] && ( !IntegerQ[n] || (GtQ[m, 0] && LtQ[m - n, 0]))) &&  !ILtQ[m + n
+ 2, 0] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 201

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^p/(n*p
+ 1)), x] + Dist[a*n*(p/(n*p + 1)), Int[(a + b*x^n)^(p - 1), x], x] /; Free
Q[{a, b}, x] && IGtQ[n, 0] && GtQ[p, 0] && (IntegerQ[2*p] || (EqQ[n, 2] &&
IntegerQ[4*p]) || (EqQ[n, 2] && IntegerQ[3*p]) || LtQ[Denominator[p + 1/n],
 Denominator[p]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 283

```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*
x)^(m + 1)*((a + b*x^n)^p/(c*(m + 1))), x] - Dist[b*n*(p/(c^n*(m + 1))), In
t[(c*x)^(m + n)*(a + b*x^n)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && IGtQ[
n, 0] && GtQ[p, 0] && LtQ[m, -1] &&  !ILtQ[(m + n*p + n + 1)/n, 0] && IntBi
nomialQ[a, b, c, n, m, p, x]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1034

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] + Dist[1/(2*f*(p + q + 1)), Int[(a + c*x^2)^(p -
 1)*(d + e*x + f*x^2)^q*Simp[a*h*e*p - a*(h*e - 2*g*f)*(p + q + 1) - 2*h*p*
(c*d - a*f)*x - (h*c*e*p + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /;
FreeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && N
eQ[p + q + 1, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1082

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*((-c)*e*(2*p + q + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + c*x^2)^p*((d + e*x
+ f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f
^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + c*x^2)^(p - 1)*(d + e*x + f*x^2)^
q*Simp[p*((-a)*e)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) + (p +
q + 1)*(a*c*(C*(2*d*f - e^2*(2*p + q + 2)) + f*(B*e - 2*A*f)*(2*p + 2*q + 3
))) + (2*p*(c*d - a*f)*(C*(c*e)*(q + 1) - c*(C*e - B*f)*(2*p + 2*q + 3)) +
(p + q + 1)*(C*e*f*p*(-4*a*c)))*x + (p*(c*e)*(C*(c*e)*(q + 1) - c*(C*e - B*
f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(-4*a*c) - c^2*(C*(e^2 - 4*d*f)*
(2*p + q + 2) + f*(2*C*d - B*e + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x]
/; FreeQ[{a, c, d, e, f, A, B, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0
] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q
, 0]
```

Rule 1094

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2],
 x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a + b*x + c*x^2)*Sqrt
[d + f*x^2]), x], x] /; FreeQ[{a, b, c, d, f, A, B, C}, x] && NeQ[b^2 - 4*a
*c, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{(a+cx^2)^{3/2}}{x^3(d+ex+fx^2)}\,dx = \int \left( \frac{(a+cx^2)^{3/2}}{dx^3} - \frac{e(a+cx^2)^{3/2}}{d^2x^2} + \frac{(e^2-df)(a+cx^2)^{3/2}}{d^3x} + \frac{(-e(e^2-2df)-f(e\ldots}{d^3(d+\ldots} \right.$$

$$= \frac{\int \frac{(-e(e^2-2df)-f(e^2-df)x)(a+cx^2)^{3/2}}{d+ex+fx^2}\,dx}{d^3} + \frac{\int \frac{(a+cx^2)^{3/2}}{x^3}\,dx}{d} - \frac{e\int \frac{(a+cx^2)^{3/2}}{x^2}\,dx}{d^2} + \frac{(e^2-d\ldots}{}$$

$$= -\frac{(e^2-df)(a+cx^2)^{3/2}}{3d^3} + \frac{e(a+cx^2)^{3/2}}{d^2x} + \frac{\text{Subst}\left(\int \frac{(a+cx)^{3/2}}{x^2}\,dx, x, x^2\right)}{2d} - \frac{(3ce)\int\ldots}{}$$

$$= -\frac{3cex\sqrt{a+cx^2}}{2d^2} - \frac{(2(cd^2+a(e^2-df))-cdex)\sqrt{a+cx^2}}{2d^3} - \frac{(a+cx^2)^{3/2}}{2dx^2} + \frac{e(\ldots}{}$$

$$= \frac{3c\sqrt{a+cx^2}}{2d} + \frac{a(e^2-df)\sqrt{a+cx^2}}{d^3} - \frac{3cex\sqrt{a+cx^2}}{2d^2} - \frac{(2(cd^2+a(e^2-df))-\ldots}{2d^3}$$

$$= \frac{3c\sqrt{a+cx^2}}{2d} + \frac{a(e^2-df)\sqrt{a+cx^2}}{d^3} - \frac{3cex\sqrt{a+cx^2}}{2d^2} - \frac{(2(cd^2+a(e^2-df))-\ldots}{2d^3}$$

$$= \frac{3c\sqrt{a+cx^2}}{2d} + \frac{a(e^2-df)\sqrt{a+cx^2}}{d^3} - \frac{3cex\sqrt{a+cx^2}}{2d^2} - \frac{(2(cd^2+a(e^2-df))-\ldots}{2d^3}$$

$$= \frac{3c\sqrt{a+cx^2}}{2d} + \frac{a(e^2-df)\sqrt{a+cx^2}}{d^3} - \frac{3cex\sqrt{a+cx^2}}{2d^2} - \frac{(2(cd^2+a(e^2-df))-\ldots}{2d^3}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.79, size = 617, normalized size = 0.92

Antiderivative was successfully verified.

```
[In] Integrate[(a + c*x^2)^(3/2)/(x^3*(d + e*x + f*x^2)),x]
```

```
[Out] ((a*d*(-d + 2*e*x)*Sqrt[a + c*x^2])/x^2 + 6*Sqrt[a]*c*d^2*ArcTanh[(Sqrt[c]*
x - Sqrt[a + c*x^2])/Sqrt[a]] - 4*a^(3/2)*(e^2 - d*f)*ArcTanh[(-(Sqrt[c]*x)
```

```
 + Sqrt[a + c*x^2])/Sqrt[a]] - 2*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#
1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c^2*d^3*Log[-(Sqrt[c]*x
) + Sqrt[a + c*x^2] - #1] - 2*a^2*c*d^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2
] - #1] - a^3*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + a^3*d*f^2*Lo
g[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 4*a*c^(3/2)*d^2*e*Log[-(Sqrt[c]*x)
 + Sqrt[a + c*x^2] - #1]*#1 - 2*a^2*Sqrt[c]*e^3*Log[-(Sqrt[c]*x) + Sqrt[a +
 c*x^2] - #1]*#1 + 4*a^2*Sqrt[c]*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
 #1]*#1 - c^2*d^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + 2*a*c*d^2
*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + a^2*e^2*f*Log[-(Sqrt[c]*
x) + Sqrt[a + c*x^2] - #1]*#1^2 - a^2*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x
^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f
*#1^3) & ])/(2*d^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2613 vs. $2(584) = 1168$.
time = 0.17, size = 2614, normalized size = 3.91

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2614 |
| risch | Expression too large to display | 4599 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+a)^(3/2)/x^3/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 8*f^3/(e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^2)^(1/2)*(1/3*((x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e+(
-4*d*f+e^2)^(1/2))/f*(1/4*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*
f+e^2)^(1/2))/f)/c*((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^
(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f
^2-2*c*d*f+c*e^2)/f^2)^(1/2)+1/8*(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d
*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e+(-4*d
*f+e^2)^(1/2))/f+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4
*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(
1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/
2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x+1/2*(e+(-4*
d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c
^(1/2)*(e+(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e+(-4*d*f+e^2)^(1/2))/f+c*(x+1/
2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-
c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^
2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*((-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-
c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((
```

```
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*
f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(
1/2))*(-1/2/a/x^2*(c*x^2+a)^(5/2)+3/2*c/a*(1/3*(c*x^2+a)^(3/2)+a*((c*x^2+a)
^(1/2)-a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x))))+8*f^3/(-e+(-4*d*f+e
^2)^(1/2))^3/(-4*d*f+e^2)^(1/2)*(1/3*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c
-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f
+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(3/2)-1/2*c*(e-(-4*d*f+e^2)^(1/
2))/f*(1/4*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/
f)/c*((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1
/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)/f^2)^(1/2)+1/8*(2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/
f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2/f^2)/c^(3/2)*ln((-1/2*c*(e-(-4*d*f+e^2)^(1
/2))/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))
)+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)))+1/2*(-(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*(1/2*(4*(x-1/2/f*(-e+(-4*d*
f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-1/2*c^(
1/2)*(e-(-4*d*f+e^2)^(1/2))/f*ln((-1/2*c*(e-(-4*d*f+e^2)^(1/2))/f+c*(x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-
c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))-1/2*(-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)
/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/
2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*
(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/
2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))))-16*f^2*e/(-e+(-4*d*f+e^2)^(1/2))^2
/(e+(-4*d*f+e^2)^(1/2))^2*(-1/a/x*(c*x^2+a)^(5/2)+4*c/a*(1/4*x*(c*x^2+a)^(3
/2)+3/4*a*(1/2*x*(c*x^2+a)^(1/2)+1/2*a/c^(1/2)*ln(x*c^(1/2)+(c*x^2+a)^(1/2)
))))+64*f^3*(d*f-e^2)/(-e+(-4*d*f+e^2)^(1/2))^3/(e+(-4*d*f+e^2)^(1/2))^3*(1
/3*(c*x^2+a)^(3/2)+a*((c*x^2+a)^(1/2)-a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(
1/2))/x)))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^3/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate((c*x^2 + a)^(3/2)/((f*x^2 + x*e + d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^3/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+a)**(3/2)/x**3/(f*x**2+e*x+d),x)
[Out] Timed out
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+a)^(3/2)/x^3/(f*x^2+e*x+d),x, algorithm="giac")
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(c\,x^2+a\right)^{3/2}}{x^3\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + c*x^2)^(3/2)/(x^3*(d + e*x + f*x^2)),x)
[Out] int((a + c*x^2)^(3/2)/(x^3*(d + e*x + f*x^2)), x)
```

## 3.64 $\int \frac{x^3}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=380

$$\frac{\sqrt{a+cx^2}}{cf}-\frac{e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f^2}-\frac{\left(2def-(e^2-df)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c}{\sqrt{2}\sqrt{2af^2+c\left(e^2-\right.}}\right.}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{\right.}}}$$

```
[Out] -e*arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))/f^2/c^(1/2)+(c*x^2+a)^(1/2)/c/f-1/2*a
rctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*
f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*d*e*f-(-d*f+e^2)*(e-(-4*d
*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*
d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2
^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*
(2*d*e*f-(-d*f+e^2)*(e+(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/
(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]

time = 0.71, antiderivative size = 380, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 6, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.222$, Rules used = {6860, 223, 212, 267, 1048, 739}

$$-\frac{\left(2def-(e^2-df)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(2def-(e^2-df)\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f^2}+\frac{\sqrt{a+cx^2}}{cf}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] Sqrt[a + c*x^2]/(c*f) - (e*ArcTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]])/(Sqrt[c]*f
^2) - ((2*d*e*f - (e^2 - d*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(
e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e
^2 - 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^
2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + ((2*d*e*f - (e^2 - d*f)*(e +
Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*
Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])]])/(S
qrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4
*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 267

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(a + b*x^n)
^(p + 1)/(b*n*(p + 1)), x] /; FreeQ[{a, b, m, n, p}, x] && EqQ[m, n - 1] &&
 NeQ[p, -1]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^3}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(-\frac{e}{f^2\sqrt{a+cx^2}} + \frac{x}{f\sqrt{a+cx^2}} + \frac{de+(e^2-df)\,x}{f^2\sqrt{a+cx^2}\,(d+ex+fx^2)}\right) dx$$

$$= \frac{\int \frac{de+(e^2-df)x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{f^2} - \frac{e\int \frac{1}{\sqrt{a+cx^2}}\,dx}{f^2} + \frac{\int \frac{x}{\sqrt{a+cx^2}}\,dx}{f}$$

$$= \frac{\sqrt{a+cx^2}}{cf} - \frac{e\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{f^2} + \frac{\left(2def-(e^2-df)\left(e\right.\right.}{}$$

$$= \frac{\sqrt{a+cx^2}}{cf} - \frac{e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f^2} - \frac{\left(2def-(e^2-df)\left(e-\sqrt{e^2-4\right.\right.}{}$$

$$= \frac{\sqrt{a+cx^2}}{cf} - \frac{e\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f^2} - \frac{\left(2def-(e^2-df)\left(e-\sqrt{e^2-4\right.\right.}{\sqrt{2}\,f^2\sqrt{}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.38, size = 312, normalized size = 0.82

$$\frac{f\sqrt{a+cx^2}+\sqrt{c}\,e\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)-c\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ae^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-adf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2\sqrt{c}\,de\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-e^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{cf^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] (f*Sqrt[a + c*x^2] + Sqrt[c]*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]] - c*Root
Sum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 +
 f*#1^4 & , (a*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - a*d*f*Log[-(S
qrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*Sqrt[c]*d*e*Log[-(Sqrt[c]*x) + Sqrt[a
 + c*x^2] - #1]*#1 - e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + d*
f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 -
2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(c*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 712 vs. $2(335) = 670$.
time = 0.14, size = 713, normalized size = 1.88

| method | result |
|---|---|
| default | $\frac{\sqrt{c\,x^{2}+a}}{cf}-\frac{e\ln\left(x\sqrt{c}+\sqrt{c\,x^{2}+a}\right)}{f^{2}\sqrt{c}}-\frac{\left(e^{3}-3def+e^{2}\sqrt{-4df+e^{2}}-df\sqrt{-4df+e^{2}}\right)\sqrt{2}\,\ln\left(\frac{\sqrt{-4df}\cdots}{\cdots}\right.}{\cdots}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] (c*x^2+a)^(1/2)/c/f-e/f^2*ln(x*c^(1/2)+(c*x^2+a)^(1/2))/c^(1/2)-1/2*(e^3-3*
d*e*f+e^2*(-4*d*f+e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2))/f^3/(-4*d*f+e^2)^(1/2)
*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*
(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f
^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e
+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))
-1/2*(-d*f*(-4*d*f+e^2)^(1/2)+e^2*(-4*d*f+e^2)^(1/2)+3*d*e*f-e^3)/f^3/(-4*d
*f+e^2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)
^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f
+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+
2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+
(-4*d*f+e^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^3}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(x**3/(sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^3/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.65 $\int \frac{x^2}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=344

$$\frac{\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f}-\frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\,\sqrt{}}\right.}{\sqrt{2}\,f\sqrt{e^2-4df}\,\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

```
[Out] arctanh(x*c^(1/2)/(c*x^2+a)^(1/2))/f/c^(1/2)-1/2*arctanh(1/2*(2*a*f-c*x*(e-
(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*
f+e^2)^(1/2)))^(1/2))*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f*2^(1/2)/(-4*d*f+e^
2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*arctanh(1/2
*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2
-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*d*f-e*(e+(-4*d*f+e^2)^(1/2)))/f*2^(
1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]

time = 0.35, antiderivative size = 344, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.185$, Rules used = {1095, 223, 212, 1048, 739}

$$-\frac{\left(-e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\left(2df-e\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ArcTanh[(Sqrt[c]*x)/Sqrt[a + c*x^2]]/(Sqrt[c]*f) - ((e^2 - 2*d*f - e*Sqrt[e
^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*
a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*
f*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])])
- ((2*d*f - e*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4
*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sq
rt[a + c*x^2])])/(Sqrt[2]*f*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f
 + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 223

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Subst[Int[1/(1 - b*x^2), x],
x, x/Sqrt[a + b*x^2]] /; FreeQ[{a, b}, x] &&  !GtQ[a, 0]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1095

```
Int[((A_.) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + f*x^2], x], x] + Dis
t[1/c, Int[(A*c - a*C - b*C*x)/((a + b*x + c*x^2)*Sqrt[d + f*x^2]), x], x]
/; FreeQ[{a, b, c, d, f, A, C}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rubi steps

$$\int \frac{x^2}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \frac{\int \frac{1}{\sqrt{a+cx^2}}\,dx}{f} + \frac{\int \frac{-d-ex}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{f}$$

$$= \frac{\text{Subst}\left(\int \frac{1}{1-cx^2}\,dx, x, \frac{x}{\sqrt{a+cx^2}}\right)}{f} + \frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\int \frac{}{\left(e-\sqrt{\ldots}\right.}}{f\sqrt{e^2-\ldots}}$$

$$= \frac{\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f} - \frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\text{Subst}\left(\int \frac{}{4af^2+c\left(e-\ldots\right.}\right.}{f\sqrt{e^2\ldots}}$$

$$= \frac{\tanh^{-1}\left(\frac{\sqrt{c}\,x}{\sqrt{a+cx^2}}\right)}{\sqrt{c}\,f} - \frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{}{\sqrt{2}\sqrt{2a\ldots}}\right.}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{2af^2+\ldots}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.35, size = 226, normalized size = 0.66

$$\frac{-\frac{\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}\right)}{\sqrt{c}} + \text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{ae\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-e\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{f}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] (-(Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]]/Sqrt[c]) + RootSum[a^2*f + 2*a*Sqrt[
c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*e*Log[
-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[
a + c*x^2] - #1]*#1 - e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*S
qrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 662 vs. 2(300) = 600.
time = 0.12, size = 663, normalized size = 1.93

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{\ln\left(x\sqrt{c}+\sqrt{c\,x^2+a}\right)}{f\sqrt{c}} - \frac{\left(-e\sqrt{-4df+e^2}+2df-e^2\right)\sqrt{2}\,\ln\left(\frac{\sqrt{-4df+e^2}\,ce+2a\,f^2-2cdf+c\,e^2}{f^2}-\frac{c\left(e+\sqrt{-4df\right.}}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*ln(x*c^(1/2)+(c*x^2+a)^(1/2))/c^(1/2)-1/2*(-e*(-4*d*f+e^2)^(1/2)+2*d*f-
e^2)/f^2/(-4*d*f+e^2)^(1/2)*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-
c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*
f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f))-1/2*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d*
f+e^2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^
(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+
e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2
*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(
-4*d*f+e^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2}{\sqrt{a + cx^2}\,(d + ex + fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(x**2/(sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^2/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.66 $\int \frac{x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=294

$$\frac{\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}-\frac{\left(e+\sqrt{e^2-4df}\right)\text{ta}}{\sqrt{2}\sqrt{}}$$

```
[Out] 1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/
(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(e-(-4*d*f+e^2)^(1/2))*
2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/
2)-1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/
2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(e+(-4*d*f+e^2)^(1/2
))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^
(1/2)
```

**Rubi** [A]
time = 0.17, antiderivative size = 294, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.120$, Rules used = {1048, 739, 212}

$$\frac{\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\left(\sqrt{e^2-4df}+e\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}$$

Antiderivative was successfully verified.

```
[In] Int[x/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((e - Sqrt[e^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqr
t[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])
])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 -
4*d*f])]) - ((e + Sqrt[e^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d
*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt
[a + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e
*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = -\left(\left(-1-\frac{e}{\sqrt{e^2-4df}}\right)\int \frac{1}{\left(e+\sqrt{e^2-4df}+2fx\right)\sqrt{a+cx^2}}\,dx\right)$$

$$= \left(-1+\frac{e}{\sqrt{e^2-4df}}\right)\text{Subst}\left(\int \frac{1}{4af^2+c\left(e-\sqrt{e^2-4df}\right)^2-x^2}\,dx,\right.$$

$$= -\frac{\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.31, size = 156, normalized size = 0.53

$$\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-a\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[x/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1
^3 + f*#1^4 & , (-(a*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]) + Log[-(Sqrt
```

```
[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3
*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ]
```

**Maple [B]** Leaf count of result is larger than twice the leaf count of optimal. 621 vs. $2(257) = 514$.
time = 0.12, size = 622, normalized size = 2.12

| method | result |
|---|---|
| default | $-\frac{\left(e+\sqrt{-4df+e^2}\right)\sqrt{2}\,\ln\left(\frac{\frac{\sqrt{-4df+e^2}\,ce+2af^2-2cdf+ce^2}{f^2}-\frac{c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\cdots}{\cdots}\right.}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*2^(1/2)/(((-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2
)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-1/2*(-e+(-4*d*f+e^2)^(1/2)
)/(-4*d*f+e^2)^(1/2)/f*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(
e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1
/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
```

```
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 4789 vs. $2(261) = 522$.
time = 0.96, size = 4789, normalized size = 16.29

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt(2)*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*c^2*d^3*f - 8*a*c*d^2*f^2
 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-a^2*e
^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4
+ 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4
 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^
4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 -
 6*a*c*d*f + a^2*f^2)*e^2))*log((4*a*c*d^2*x*e - 2*a^2*d*e^2 + sqrt(2)*(4*a
^2*d*f*e^2 - a^2*e^4 + (8*c^3*d^5*f - 24*a*c^2*d^4*f^2 + 24*a^2*c*d^3*f^3 -
 8*a^3*d^2*f^4 - a^2*c*e^6 - (3*a*c^2*d^2 - 8*a^2*c*d*f + a^3*f^2)*e^4 - 2*
(c^3*d^4 - 9*a*c^2*d^3*f + 11*a^2*c*d^2*f^2 - 3*a^3*d*f^3)*e^2)*sqrt(-a^2*e
^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4
+ 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4
 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^
4)*e^2)))*sqrt(c*x^2 + a)*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 + (4*c^2*d^3*f -
 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^
2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*
a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f +
a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*
d*f^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^
4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)) + 2*(4*a*c^2*d^4*f - 8*a^2*c*d^3*
f^2 + 4*a^3*d^2*f^3 - a^2*c*d*e^4 - (a*c^2*d^3 - 6*a^2*c*d^2*f + a^3*d*f^2)
*e^2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 -
16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f
 + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3
*c*d*f^3 + a^4*f^4)*e^2)))/x) - 1/4*sqrt(2)*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^
2 + (4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c
*d*f + a^2*f^2)*e^2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2
*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2
- 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d
^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*
a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2))*log((4*a*c*d^2*
x*e - 2*a^2*d*e^2 - sqrt(2)*(4*a^2*d*f*e^2 - a^2*e^4 + (8*c^3*d^5*f - 24*a*
c^2*d^4*f^2 + 24*a^2*c*d^3*f^3 - 8*a^3*d^2*f^4 - a^2*c*e^6 - (3*a*c^2*d^2 -
```

```
 8*a^2*c*d*f + a^3*f^2)*e^4 - 2*(c^3*d^4 - 9*a*c^2*d^3*f + 11*a^2*c*d^2*f^2
 - 3*a^3*d*f^3)*e^2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2
*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2
- 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d
^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))*sqrt(c*x^2 + a)*sqrt(-(2*c*d^2 -
2*a*d*f + a*e^2 + (4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c
^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^
4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 -
 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f
+ 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*
c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)) +
 2*(4*a*c^2*d^4*f - 8*a^2*c*d^3*f^2 + 4*a^3*d^2*f^3 - a^2*c*d*e^4 - (a*c^2*
d^3 - 6*a^2*c*d^2*f + a^3*d*f^2)*e^2)*sqrt(-a^2*e^2/(4*c^4*d^5*f - 16*a*c^3
*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^
6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3
*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))/x) + 1/4*sqrt(2)
*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*
f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-a^2*e^2/(4*c^4*d
^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f
^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4
 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))/(
4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f
+ a^2*f^2)*e^2))*log((4*a*c*d^2*x*e - 2*a^2*d*e^2 + sqrt(2)*(4*a^2*d*f*e^2
- a^2*e^4 - (8*c^3*d^5*f - 24*a*c^2*d^4*f^2 + 24*a^2*c*d^3*f^3 - 8*a^3*d^2*
f^4 - a^2*c*e^6 - (3*a*c^2*d^2 - 8*a^2*c*d*f + a^3*f^2)*e^4 - 2*(c^3*d^4 -
9*a*c^2*d^3*f + 11*a^2*c*d^2*f^2 - 3*a^3*d*f^3)*e^2)*sqrt(-a^2*e^2/(4*c^4*d
^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f
^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4
 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2)))*s
qrt(c*x^2 + a)*sqrt(-(2*c*d^2 - 2*a*d*f + a*e^2 - (4*c^2*d^3*f - 8*a*c*d^2*
f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-a^
2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f
^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*
e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^...
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(x/(sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
[Out] int(x/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.67 $\int \frac{1}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=266

$$-\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right)}{\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}+\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{2\cdots}{\sqrt{2}\sqrt{2af^2+c\,(\cdots}}\right)}{\sqrt{e^2-4df}\sqrt{2af^2+\cdots}}$$

```
[Out] -f*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(
2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*2^(1/2)/(-4*d*f+e^2)^(1/
2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+f*arctanh(1/2*(2*a*f-
c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e
*(-4*d*f+e^2)^(1/2)))^(1/2))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d
*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]

time = 0.10, antiderivative size = 266, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.125$, Rules used = {999, 739, 212}

$$\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -((Sqrt[2]*f*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*
a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[e^2
 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])])) + (Sqrt[2
]*f*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c
*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[e^2 - 4*d*f]
*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 999

```
Int[1/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f_.)*(x_)^2]), x_Sym
bol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[2*(c/q), Int[1/((b - q + 2*c*x)
*Sqrt[d + f*x^2]), x], x] - Dist[2*(c/q), Int[1/((b + q + 2*c*x)*Sqrt[d + f
*x^2]), x], x]] /; FreeQ[{a, b, c, d, f}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ
[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{1}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \frac{(2f)\int \frac{1}{\left(e-\sqrt{e^2-4df}+2fx\right)\sqrt{a+cx^2}}\,dx}{\sqrt{e^2-4df}} - \frac{(2f)\int \frac{1}{\left(e+\sqrt{e^2-4df}+2fx\right)}$$

$$= -\frac{(2f)\text{Subst}\left(\int \frac{1}{4af^2+c\left(e-\sqrt{e^2-4df}\right)^2-x^2}\,dx, x, \frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{a+cx^2}}\right.}{\sqrt{e^2-4df}}$$

$$= -\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right.}{\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.31, size = 131, normalized size = 0.49

$$-2\sqrt{c}\,\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -2*Sqrt[c]*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*S
qrt[c]*e*#1^3 + f*#1^4 & , (Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1)/(a
*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 588 vs. $2(232) = 464$.
time = 0.12, size = 589, normalized size = 2.21

| method | result |
|---|---|
| default | $\frac{\sqrt{2}\,\ln\left(\frac{\sqrt{-4df+e^2}\,ce+2af^2-2cdf+ce^2}{f^2}-\frac{c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{\sqrt{2}\,\sqrt{\frac{\sqrt{-4df+e^2}\,c}{f}}}{}\right.}{\sqrt{-4df}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/(-4*d*f+e^2)^(1/2)*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-
4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4
*d*f+e^2)^(1/2))/f))-1/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c
*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))
)+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(
4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 4761 vs. $2(236) = 472$.
time = 1.01, size = 4761, normalized size = 17.90

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt(2)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*c^2*d^3*f - 8*a*c*d^2*f^2
 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-c^2*e
^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4
+ 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4
 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^
4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 -
 6*a*c*d*f + a^2*f^2)*e^2))*log((4*c^2*d*f*x*e - 2*a*c*f*e^2 + sqrt(2)*sqrt
(c*x^2 + a)*((c^2*d - a*c*f)*e^3 - 4*(c^2*d^2*f - a*c*d*f^2)*e - ((a*c^2*d
+ a^2*c*f)*e^5 + (c^3*d^3 - 5*a*c^2*d^2*f - 5*a^2*c*d*f^2 + a^3*f^3)*e^3 -
4*(c^3*d^4*f - a*c^2*d^3*f^2 - a^2*c*d^2*f^3 + a^3*d*f^4)*e)*sqrt(-c^2*e^2/
(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4
*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 -
(c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*
e^2)))*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a
^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-c^2*e^2/(4*
c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4*a^
4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 - (c^
4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*e^2
)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c
*d*f + a^2*f^2)*e^2)) + 2*(4*a*c^2*d^3*f^2 - 8*a^2*c*d^2*f^3 + 4*a^3*d*f^4
- a^2*c*f*e^4 - (a*c^2*d^2*f - 6*a^2*c*d*f^2 + a^3*f^3)*e^2)*sqrt(-c^2*e^2/
(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2*f^4 + 4
*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2)*e^4 -
(c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a^4*f^4)*
e^2)))/x) + 1/4*sqrt(2)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*c^2*d^3*f - 8*
a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*
sqrt(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3
*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3
*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f
^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 -
 (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2))*log((4*c^2*d*f*x*e - 2*a*c*f*e^2 - s
qrt(2)*sqrt(c*x^2 + a)*((c^2*d - a*c*f)*e^3 - 4*(c^2*d^2*f - a*c*d*f^2)*e -
 ((a*c^2*d + a^2*c*f)*e^5 + (c^3*d^3 - 5*a*c^2*d^2*f - 5*a^2*c*d*f^2 + a^3*
f^3)*e^3 - 4*(c^3*d^4*f - a*c^2*d^3*f^2 - a^2*c*d^2*f^3 + a^3*d*f^4)*e)*sqr
t(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*
d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*
```

```
f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3
+ a^4*f^4)*e^2)))*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 + (4*c^2*d^3*f - 8*a*c*d^
2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2)*sqrt(-
c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*d^2
*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*f^2
)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3 + a
^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*
d^2 - 6*a*c*d*f + a^2*f^2)*e^2)) + 2*(4*a*c^2*d^3*f^2 - 8*a^2*c*d^2*f^3 + 4
*a^3*d*f^4 - a^2*c*f*e^4 - (a*c^2*d^2*f - 6*a^2*c*d*f^2 + a^3*f^3)*e^2)*sqr
t(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 16*a^3*c*
d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f + a^3*c*
f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*c*d*f^3
+ a^4*f^4)*e^2)))/x) - 1/4*sqrt(2)*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 - (4*c^2
*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2
*f^2)*e^2)*sqrt(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f
^3 - 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^
2*d*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 1
2*a^3*c*d*f^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*d^2*f^2 + 4*a^2*d*f^3
- a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*e^2))*log((4*c^2*d*f*x*e - 2*a*
c*f*e^2 + sqrt(2)*sqrt(c*x^2 + a)*((c^2*d - a*c*f)*e^3 - 4*(c^2*d^2*f - a*c
*d*f^2)*e + ((a*c^2*d + a^2*c*f)*e^5 + (c^3*d^3 - 5*a*c^2*d^2*f - 5*a^2*c*d
*f^2 + a^3*f^3)*e^3 - 4*(c^3*d^4*f - a*c^2*d^3*f^2 - a^2*c*d^2*f^3 + a^3*d*
f^4)*e)*sqrt(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3
- 16*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d
*f + a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a
^3*c*d*f^3 + a^4*f^4)*e^2)))*sqrt((2*c*d*f - 2*a*f^2 - c*e^2 - (4*c^2*d^3*f
 - 8*a*c*d^2*f^2 + 4*a^2*d*f^3 - a*c*e^4 - (c^2*d^2 - 6*a*c*d*f + a^2*f^2)*
e^2)*sqrt(-c^2*e^2/(4*c^4*d^5*f - 16*a*c^3*d^4*f^2 + 24*a^2*c^2*d^3*f^3 - 1
6*a^3*c*d^2*f^4 + 4*a^4*d*f^5 - a^2*c^2*e^6 - 2*(a*c^3*d^2 - 4*a^2*c^2*d*f
+ a^3*c*f^2)*e^4 - (c^4*d^4 - 12*a*c^3*d^3*f + 22*a^2*c^2*d^2*f^2 - 12*a^3*
c*d*f^3 + a^4*f^4)*e^2)))/(4*c^2*d^3*f - 8*a*c*...
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\sqrt{a + cx^2}\,(d + ex + fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(1/(sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/((a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.68 $\int \frac{1}{x\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=330

$$\frac{f\left(e+\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}-\frac{f\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\,\ldots}$$

```
[Out] -arctanh((c*x^2+a)^(1/2)/a^(1/2))/d/a^(1/2)+1/2*f*arctanh(1/2*(2*a*f-c*x*(e
-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d
*f+e^2)^(1/2)))^(1/2))*(e+(-4*d*f+e^2)^(1/2))/d*2^(1/2)/(-4*d*f+e^2)^(1/2)/
(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*f*arctanh(1/2*(2*a*f
-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+
e*(-4*d*f+e^2)^(1/2)))^(1/2))*(e-(-4*d*f+e^2)^(1/2))/d*2^(1/2)/(-4*d*f+e^2)
^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 0.54, antiderivative size = 330, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.259$, Rules used = {6860, 272, 65, 214, 1048, 739, 212}

$$\frac{f\left(\sqrt{e^2-4df}+e\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}\,d}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] (f*(e + Sqrt[e^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(S
qrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2
])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^
2 - 4*d*f])]) - (f*(e - Sqrt[e^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e + Sqrt[e^2
 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])
]*Sqrt[a + c*x^2])]])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2
*d*f + e*Sqrt[e^2 - 4*d*f])]) - ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]]/(Sqrt[a]*d
)
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
```

```
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx\sqrt{a+cx^2}} + \frac{-e-fx}{d\sqrt{a+cx^2}\,(d+ex+fx^2)}\right)dx$$

$$= \frac{\int \frac{1}{x\sqrt{a+cx^2}}\,dx}{d} + \frac{\int \frac{-e-fx}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{d}$$

$$= \frac{\text{Subst}\left(\int \frac{1}{x\sqrt{a+cx}}\,dx, x, x^2\right)}{2d} - \frac{\left(f\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\right)\int \frac{}{\left(e+\sqrt{e^2-\cdots\right.}}}{d}$$

$$= \frac{\text{Subst}\left(\int \frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\,dx, x, \sqrt{a+cx^2}\right)}{cd} + \frac{\left(f\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\right)\text{Subst}\left(\cdots\right.}{}$$

$$= \frac{f\left(1+\frac{e}{\sqrt{e^2-4df}}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df\cdots}\right.}}\right.}{\sqrt{2}\,d\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.34, size = 236, normalized size = 0.72

$$\frac{\frac{2\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}} - \text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{af\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2\sqrt{c}\,e\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{-a\sqrt{c}\,e-4cd\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]}{d}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((2*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/Sqrt[a]])/Sqrt[a] - RootSum[a^2*f
 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 &
 , (a*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*Sqrt[c]*e*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2] - #1]*#1 - f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1
]*#1^2)/(-(a*Sqrt[c]*e) - 4*c*d*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3
) & ])/d
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 680 vs. 2(287) = 574.
time = 0.13, size = 681, normalized size = 2.06

| method | result |
|---|---|
| default | $-\frac{2f\sqrt{2}\,\ln\left(\frac{\frac{\sqrt{-4df+e^2}\,ce+2a\,f^2-2cdf+c\,e^2}{f^2}-\frac{c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{\sqrt{2}\,\sqrt{\frac{\sqrt{-4df+}}{}}}{}}{}\right.}{\left(e+\sqrt{-4df+\right.}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*2^(1/2)/(((-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-
2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))
/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*
(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)
/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-2*f/(-e+(-4*d*f+e^2)^(1/2))/
(-4*d*f+e^2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)
/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-
4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/
2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f
*(-e+(-4*d*f+e^2)^(1/2))))+4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2
))/a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(1/(sqrt(c*x^2 + a)*(f*x^2 + x*e + d)*x), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 6674 vs. $2(291) = 582$.
time = 184.04, size = 13360, normalized size = 40.48

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(1/x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")

[Out] [1/4*(sqrt(2)*a*d*sqrt(-(2*c*d^2*f^2 - 2*a*d*f^3 + c*e^4 - (4*c*d*f - a*f^2 )*e^2 + (4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^4 - (c^2*d ^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2*d*f - a*c*f^ 2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d^9*f - 16*a*c ^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5*f^5 - a^2*c^ 2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4*d^4*f^4)*e^2 )))/(4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^4 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2))*log((4*c^2*d*f^2*x*e^3 - 2*a*c*f^2*e^4 - 4*(2*c^2*d^2*f^3 - a*c*d*f^4)*x*e + sqrt(2)*(c^2*d*e^6 - (6*c^2*d^2*f - a*c *d*f^2)*e^4 + 4*(2*c^2*d^3*f^2 - a*c*d^2*f^3)*e^2 + (8*c^3*d^7*f^2 - 24*a*c ^2*d^6*f^3 + 24*a^2*c*d^5*f^4 - 8*a^3*d^4*f^5 + a*c^2*d^3*e^6 + (c^3*d^5 - 8*a*c^2*d^4*f + 3*a^2*c*d^3*f^2)*e^4 - 2*(3*c^3*d^6*f - 11*a*c^2*d^5*f^2 + 9*a^2*c*d^4*f^3 - a^3*d^3*f^4)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2*d*f - a*c*f^2 )*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d^9*f - 16*a*c^ 3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5*f^5 - a^2*c^2 *d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4*d^4*f^4)*e^2) ))*sqrt(c*x^2 + a)*sqrt(-(2*c*d^2*f^2 - 2*a*d*f^3 + c*e^4 - (4*c*d*f - a*f^ 2)*e^2 + (4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^4 - (c^2* d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2*d*f - a*c*f ^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d^9*f - 16*a* c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5*f^5 - a^2*c ^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4*d^4*f^4)*e^ 2)))/(4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^4 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)) + 2*(2*a*c*d*f^3 - a^2*f^4)*e^2 - 2*(4*a *c^2*d^5*f^3 - 8*a^2*c*d^4*f^4 + 4*a^3*d^3*f^5 - a^2*c*d^2*f^2*e^4 - (a*c^2 *d^4*f^2 - 6*a^2*c*d^3*f^3 + a^3*d^2*f^4)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2*d* f - a*c*f^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d^9* f - 16*a*c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5*f^ 5 - a^2*c^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4*d^ 4*f^4)*e^2)))/x) - sqrt(2)*a*d*sqrt(-(2*c*d^2*f^2 - 2*a*d*f^3 + c*e^4 - (4* c*d*f - a*f^2)*e^2 + (4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2 *e^4 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2 *d*f - a*c*f^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d ^9*f - 16*a*c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5 *f^5 - a^2*c^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^ 4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4 *d^4*f^4)*e^2)))/(4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^4

```
 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2))*log((4*c^2*d*f^2*x*e^3 - 2*a
*c*f^2*e^4 - 4*(2*c^2*d^2*f^3 - a*c*d*f^4)*x*e - sqrt(2)*(c^2*d*e^6 - (6*c^
2*d^2*f - a*c*d*f^2)*e^4 + 4*(2*c^2*d^3*f^2 - a*c*d^2*f^3)*e^2 + (8*c^3*d^7
*f^2 - 24*a*c^2*d^6*f^3 + 24*a^2*c*d^5*f^4 - 8*a^3*d^4*f^5 + a*c^2*d^3*e^6
+ (c^3*d^5 - 8*a*c^2*d^4*f + 3*a^2*c*d^3*f^2)*e^4 - 2*(3*c^3*d^6*f - 11*a*c
^2*d^5*f^2 + 9*a^2*c*d^4*f^3 - a^3*d^3*f^4)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^2*
d*f - a*c*f^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*d^
9*f - 16*a*c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^5*
f^5 - a^2*c^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e^4
 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^4*
d^4*f^4)*e^2)))*sqrt(c*x^2 + a)*sqrt(-(2*c*d^2*f^2 - 2*a*d*f^3 + c*e^4 - (4
*c*d*f - a*f^2)*e^2 + (4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^
2*e^4 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)*sqrt(-(c^2*e^6 - 2*(2*c^
2*d*f - a*c*f^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2)/(4*c^4*
d^9*f - 16*a*c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 + 4*a^4*d^
5*f^5 - a^2*c^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d^4*f^2)*e
^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5*f^3 + a^
4*d^4*f^4)*e^2)))/(4*c^2*d^5*f - 8*a*c*d^4*f^2 + 4*a^2*d^3*f^3 - a*c*d^2*e^
4 - (c^2*d^4 - 6*a*c*d^3*f + a^2*d^2*f^2)*e^2)) + 2*(2*a*c*d*f^3 - a^2*f^4)
*e^2 - 2*(4*a*c^2*d^5*f^3 - 8*a^2*c*d^4*f^4 + 4*a^3*d^3*f^5 - a^2*c*d^2*f^2
*e^4 - (a*c^2*d^4*f^2 - 6*a^2*c*d^3*f^3 + a^3*d^2*f^4)*e^2)*sqrt(-(c^2*e^6
- 2*(2*c^2*d*f - a*c*f^2)*e^4 + (4*c^2*d^2*f^2 - 4*a*c*d*f^3 + a^2*f^4)*e^2
)/(4*c^4*d^9*f - 16*a*c^3*d^8*f^2 + 24*a^2*c^2*d^7*f^3 - 16*a^3*c*d^6*f^4 +
 4*a^4*d^5*f^5 - a^2*c^2*d^4*e^6 - 2*(a*c^3*d^6 - 4*a^2*c^2*d^5*f + a^3*c*d
^4*f^2)*e^4 - (c^4*d^8 - 12*a*c^3*d^7*f + 22*a^2*c^2*d^6*f^2 - 12*a^3*c*d^5
*f^3 + a^4*d^4*f^4)*e^2)))/x) + sqrt(2)*a*d*sqr...
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(1/(x*sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\,\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
[Out] int(1/(x*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

## 3.69 $\int \frac{1}{x^2\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=367

$$-\frac{\sqrt{a+cx^2}}{adx}-\frac{f\left(e^2-2df+e\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{2af-c\left(e-\sqrt{e^2-4df}\right)x}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}\sqrt{a+cx^2}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4df}\right)}}$$

```
[Out] e*arctanh((c*x^2+a)^(1/2)/a^(1/2))/d^2/a^(1/2)-(c*x^2+a)^(1/2)/a/d/x-1/2*f*
arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a
*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(e^2-2*d*f+e*(-4*d*f+e^2)^(
1/2))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(
1/2)))^(1/2)+1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(
c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(e^2-2*d
*f-e*(-4*d*f+e^2)^(1/2))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d
*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 0.78, antiderivative size = 367, normalized size of antiderivative = 1.00, number of steps used = 11, number of rules used = 8, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.296$, Rules used = {6860, 270, 272, 65, 214, 1048, 739, 212}

$$-\frac{f\left(e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(-e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{e\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}\,d^2}-\frac{\sqrt{a+cx^2}}{adx}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -(Sqrt[a + c*x^2]/(a*d*x)) - (f*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])*ArcTanh
[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d
*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f
]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + (f*(e^2 - 2*d*f
- e*Sqrt[e^2 - 4*d*f])*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[
2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])
/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2
- 4*d*f])]) + (e*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/(Sqrt[a]*d^2)
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
```

```
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 270

```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*
x)^(m + 1)*((a + b*x^n)^(p + 1)/(a*c*(m + 1))), x] /; FreeQ[{a, b, c, m, n,
 p}, x] && EqQ[(m + 1)/n + p + 1, 0] && NeQ[m, -1]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx^2\sqrt{a+cx^2}} - \frac{e}{d^2x\sqrt{a+cx^2}} + \frac{e^2-df+efx}{d^2\sqrt{a+cx^2}\,(d+ex+fx^2)}\right)$$

$$= \frac{\int \frac{e^2-df+efx}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{d^2} + \frac{\int \frac{1}{x^2\sqrt{a+cx^2}}\,dx}{d} - \frac{e\int \frac{1}{x\sqrt{a+cx^2}}\,dx}{d^2}$$

$$= -\frac{\sqrt{a+cx^2}}{adx} - \frac{e\text{Subst}\left(\int \frac{1}{x\sqrt{a+cx}}\,dx, x, x^2\right)}{2d^2} - \frac{\left(f\left(e^2-2df-e\sqrt{\right.\right.}{}$$

$$= -\frac{\sqrt{a+cx^2}}{adx} - \frac{e\text{Subst}\left(\int \frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\,dx, x, \sqrt{a+cx^2}\right)}{cd^2} + \frac{\left(f\left(e^2-2df-\right.\right.}{}$$

$$= -\frac{\sqrt{a+cx^2}}{adx} - \frac{f\left(e^2-2df+e\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{}{\sqrt{2}\sqrt{2af^2+c}}\right.}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-\right.}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.41, size = 298, normalized size = 0.81

$$-\frac{d\sqrt{a+cx^2}+2\sqrt{a}\,ex\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)+ax\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{aef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2\sqrt{c}\,e^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-ef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{ad^2x}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -((d*Sqrt[a + c*x^2] + 2*Sqrt[a]*e*x*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/
Sqrt[a]] + a*x*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 -
 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
#1] + 2*Sqrt[c]*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*Sqrt[c]
*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - e*f*Log[-(Sqrt[c]*x) + S
qrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e
*#1^2 + 2*f*#1^3) & ])/(a*d^2*x))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 735 vs. 2(322) = 644.
time = 0.13, size = 736, normalized size = 2.01 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

[In] int(1/x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)

[Out] 4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-4*f^2/(-e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))+4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))/a/x*(c*x^2+a)^(1/2)+16*f^2*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2/a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x)

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(1/x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")

[Out] integrate(1/(sqrt(c*x^2 + a)*(f*x^2 + x*e + d)*x^2), x)

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

[In] integrate(1/x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")

[Out] Timed out

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^2\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(1/(x**2*sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^2\,\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x^2*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.70 $\int \frac{1}{x^3\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=457

$$-\frac{\sqrt{a+cx^2}}{2adx^2}+\frac{e\sqrt{a+cx^2}}{ad^2x}+\frac{f\left(2e^3-4def-(e^2-df)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e-\cdots\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2\cdots\right.}}\right.}{\sqrt{2}\,d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e^2-2df-e\sqrt{e^2-4\cdots}\right.}}$$

```
[Out] 1/2*c*arctanh((c*x^2+a)^(1/2)/a^(1/2))/a^(3/2)/d-(-d*f+e^2)*arctanh((c*x^2+
a)^(1/2)/a^(1/2))/d^3/a^(1/2)-1/2*(c*x^2+a)^(1/2)/a/d/x^2+e*(c*x^2+a)^(1/2)
/a/d^2/x+1/2*f*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^
2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*e^3-4*d*e
*f-(-d*f+e^2)*(e-(-4*d*f+e^2)^(1/2)))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f
^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*f*arctanh(1/2*(2*a*f-c*x*(
e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*
d*f+e^2)^(1/2)))^(1/2))*(2*e^3-4*d*e*f-(-d*f+e^2)*(e+(-4*d*f+e^2)^(1/2)))/d
^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^
(1/2)
```

**Rubi** [A]
time = 1.25, antiderivative size = 457, normalized size of antiderivative = 1.00, number of steps used = 15, number of rules used = 9, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {6860, 272, 44, 65, 214, 270, 1048, 739, 212}

$$\frac{c\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{2a^{3/2}d}-\frac{(e^2-df)\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}\,d^3}+\frac{f\left(-(e^2-df)\left(e-\sqrt{e^2-4df}\right)-4def+2e^3\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(-(e^2-df)\left(\sqrt{e^2-4df}+e\right)-4def+2e^3\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d^3\sqrt{e^2-4df}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{e\sqrt{a+cx^2}}{ad^2x}-\frac{\sqrt{a+cx^2}}{2adx^2}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^3*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -1/2*Sqrt[a + c*x^2]/(a*d*x^2) + (e*Sqrt[a + c*x^2])/(a*d^2*x) + (f*(2*e^3
- 4*d*e*f - (e^2 - d*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sq
rt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4
*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*
(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - (f*(2*e^3 - 4*d*e*f - (e^2 - d*f)*(
e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt
[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])]
)/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2
 - 4*d*f])]) + (c*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/(2*a^(3/2)*d) - ((e^2 -
 d*f)*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/(Sqrt[a]*d^3)
```

Rule 44

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Dist[d*((
m + n + 2)/((b*c - a*d)*(m + 1))), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && ILtQ[m, -1] &&  !Int
egerQ[n] && LtQ[n, 0]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 270

```
Int[((c_.)*(x_))^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(c*
x)^(m + 1)*((a + b*x^n)^(p + 1)/(a*c*(m + 1))), x] /; FreeQ[{a, b, c, m, n,
 p}, x] && EqQ[(m + 1)/n + p + 1, 0] && NeQ[m, -1]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
```

```
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^3\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx^3\sqrt{a+cx^2}} - \frac{e}{d^2x^2\sqrt{a+cx^2}} + \frac{e^2-df}{d^3x\sqrt{a+cx^2}} + \frac{-e(e^2-2df)}{d^3\sqrt{a+cx^2}}\right.$$

$$= \frac{\int \frac{-e(e^2-2df)-f(e^2-df)x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{d^3} + \frac{\int \frac{1}{x^3\sqrt{a+cx^2}}\,dx}{d} - \frac{e\int \frac{1}{x^2\sqrt{a+cx^2}}\,dx}{d^2} +$$

$$= \frac{e\sqrt{a+cx^2}}{ad^2x} + \frac{\text{Subst}\left(\int \frac{1}{x^2\sqrt{a+cx}}\,dx, x, x^2\right)}{2d} + \frac{(e^2-df)\,\text{Subst}\left(\int \frac{}{x}\right.}{2d}$$

$$= -\frac{\sqrt{a+cx^2}}{2adx^2} + \frac{e\sqrt{a+cx^2}}{ad^2x} - \frac{c\text{Subst}\left(\int \frac{1}{x\sqrt{a+cx}}\,dx, x, x^2\right)}{4ad} + \frac{(e^2-}{}$$

$$= -\frac{\sqrt{a+cx^2}}{2adx^2} + \frac{e\sqrt{a+cx^2}}{ad^2x} + \frac{f\left(2e^3-4def-(e^2-df)\left(e-\sqrt{e^2-4}\right.\right.}{\sqrt{2}\,d^3\sqrt{e^2-}}$$

$$= -\frac{\sqrt{a+cx^2}}{2adx^2} + \frac{e\sqrt{a+cx^2}}{ad^2x} + \frac{f\left(2e^3-4def-(e^2-df)\left(e-\sqrt{e^2-4}\right.\right.}{\sqrt{2}\,d^3\sqrt{e^2-}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 0.65, size = 422, normalized size = 0.92

$$\frac{\frac{d(-d+2ex)\sqrt{a+cx^2}}{ax^2}-\frac{2cd^2\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)}{a^{3/2}}-\frac{4(e^2-df)\tanh^{-1}\left(\frac{-\sqrt{c}\,x+\sqrt{a+cx^2}}{\sqrt{a}}\right)}{\sqrt{a}}+2\mathrm{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{ae^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-adf^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2\sqrt{c}\,e^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-4\sqrt{c}\,def\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-e^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+df^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{2d^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^3*Sqrt[a + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((d*(-d + 2*e*x)*Sqrt[a + c*x^2])/(a*x^2) - (2*c*d^2*ArcTanh[(Sqrt[c]*x - S
qrt[a + c*x^2])/Sqrt[a]])/a^(3/2) - (4*(e^2 - d*f)*ArcTanh[(-(Sqrt[c]*x) +
Sqrt[a + c*x^2])/Sqrt[a]])/Sqrt[a] + 2*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4
*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*e^2*f*Log[-(Sqrt[
c]*x) + Sqrt[a + c*x^2] - #1] - a*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]
- #1] + 2*Sqrt[c]*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 4*Sqrt[
c]*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - e^2*f*Log[-(Sqrt[c]*
x) + Sqrt[a + c*x^2] - #1]*#1^2 + d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2]
- #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^
3) & ])/(2*d^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 826 vs. $2(400) = 800$.
time = 0.14, size = 827, normalized size = 1.81

| method | result |
|---|---|
| default | $-\frac{8f^3\sqrt{2}\,\ln\left(\frac{\frac{\sqrt{-4df+e^2}\,ce+2a\,f^2-2cdf+c\,e^2}{f^2}-\frac{c\left(e+\sqrt{-4df+e^2}\right)\left(x+\frac{e+\sqrt{-4df+e^2}}{2f}\right)}{f}+\frac{\sqrt{2}\,\sqrt{\sqrt{-4df}}}{}}{}\right)}{\left(e+\sqrt{-4df+\right.}}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -8*f^3/(e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^2)^(1/2)*2^(1/2)/(((-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1
/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x
+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-4*f/(-e+(-4*d*f+e^2)^(1/
```

```
2))/(e+(-4*d*f+e^2)^(1/2))*(-1/2/a/x^2*(c*x^2+a)^(1/2)+1/2*c/a^(3/2)*ln((2*
a+2*a^(1/2)*(c*x^2+a)^(1/2))/x))-8*f^3/(-e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^
2)^(1/2)*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2
)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)
^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)
))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2))))+16*f^2*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^
2/a/x*(c*x^2+a)^(1/2)-64*f^3*(d*f-e^2)/(-e+(-4*d*f+e^2)^(1/2))^3/(e+(-4*d*f
+e^2)^(1/2))^3/a^(1/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x)
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(1/(sqrt(c*x^2 + a)*(f*x^2 + x*e + d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^3\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**3/(f*x**2+e*x+d)/(c*x**2+a)**(1/2),x)
```

```
[Out] Integral(1/(x**3*sqrt(a + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(f*x^2+e*x+d)/(c*x^2+a)^(1/2),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^3\,\sqrt{c\,x^2+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^3*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x^3*(a + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.71 $\int \frac{x^3}{(a+cx^2)^{3/2}(d+ex+fx^2)} dx$

**Optimal**. Leaf size=499

$$-\frac{1}{cf\sqrt{a+cx^2}}-\frac{ex}{af^2\sqrt{a+cx^2}}+\frac{af(cd^2+a(e^2-df))+ce(cd^2+a(e^2-2df))\,x}{af^2\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}}-\frac{\left(2adef-\left(e-\sqrt{e^2-4d\right.\right.}{\sqrt{2}\,\sqrt{}}$$

[Out] -1/c/f/(c*x^2+a)^(1/2)-e*x/a/f^2/(c*x^2+a)^(1/2)+(a*f*(c*d^2+a*(-d*f+e^2))+c*e*(c*d^2+a*(-2*d*f+e^2))*x)/a/f^2/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)-1/2*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*a*d*e*f-(c*d^2+a*(-d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*a*d*e*f-(c*d^2+a*(-d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)

**Rubi** [A]
time = 1.34, antiderivative size = 499, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.259$, Rules used = {6860, 197, 267, 1031, 1048, 739, 212}

$$\frac{cex(a(e^2-2df)+cd^2)+af(a(e^2-df)+cd^2)}{af^2\sqrt{a+cx^2}\left((cd-af)^2+ace^2\right)}-\frac{\left(2adef-\left(e-\sqrt{e^2-4df}\right)\left(a(e^2-df)+cd^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(2adef-\left(\sqrt{e^2-4df}+e\right)\left(a(e^2-df)+cd^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{ex}{af^2\sqrt{a+cx^2}}-\frac{1}{cf\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -(1/(c*f*Sqrt[a + c*x^2])) - (e*x)/(a*f^2*Sqrt[a + c*x^2]) + (a*f*(c*d^2 +
a*(e^2 - d*f)) + c*e*(c*d^2 + a*(e^2 - 2*d*f))*x)/(a*f^2*(a*c*e^2 + (c*d -
a*f)^2)*Sqrt[a + c*x^2]) - ((2*a*d*e*f - (e - Sqrt[e^2 - 4*d*f])*(c*d^2 + a
*(e^2 - d*f)))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[
2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2
]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f
 - e*Sqrt[e^2 - 4*d*f])]) + ((2*a*d*e*f - (e + Sqrt[e^2 - 4*d*f])*(c*d^2 +
a*(e^2 - d*f)))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt
[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[
2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*
f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 197

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^(p + 1)
/a), x] /; FreeQ[{a, b, n, p}, x] && EqQ[1/n + p + 1, 0]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 267

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[(a + b*x^n)
^(p + 1)/(b*n*(p + 1)), x] /; FreeQ[{a, b, m, n, p}, x] && EqQ[m, n - 1] &&
 NeQ[p, -1]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1031

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + c*x^2)^(p + 1)*((d + e*x + f*x^2)^
(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)))*(g*c*(2*a*c*e) + ((-a
)*h)*(2*c^2*d - c*(2*a*f)) + c*(g*(2*c^2*d - c*(2*a*f)) - h*(-2*a*c*e))*x),
 x] + Dist[1/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^
(p + 1)*(d + e*x + f*x^2)^q*Simp[(-2*g*c)*((c*d - a*f)^2 - ((-a)*e)*(c*e))*
(p + 1) + (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(a*f*(p + 1) - c*d*(p + 2))
- e*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d - c*((Plus[2])*a*f)))*(p + q + 2)
- (2*f*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d + (-c)*((Plus[2])*a*f)))*(p + q
 + 2) - (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*((-c)*e*(2*p + q + 4)))*x - c*
f*(2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(2*p + 2*q + 5)*x^2, x], x], x] /; F
reeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && N
eQ[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^3}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx = \int \left(-\frac{e}{f^2(a+cx^2)^{3/2}} + \frac{x}{f(a+cx^2)^{3/2}} + \frac{de+(e^2-df)x}{f^2(a+cx^2)^{3/2}(d+ex+fx^2)}\right.$$

$$= \frac{\int \frac{de+(e^2-df)x}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx}{f^2} - \frac{e\int \frac{1}{(a+cx^2)^{3/2}}\,dx}{f^2} + \frac{\int \frac{x}{(a+cx^2)^{3/2}}\,dx}{f}$$

$$= -\frac{1}{cf\sqrt{a+cx^2}} - \frac{ex}{af^2\sqrt{a+cx^2}} + \frac{af(cd^2+a(e^2-df))+ce(cd^2+a(e}{af^2(ace^2+(cd-af)^2)\sqrt{a+}}$$

$$= -\frac{1}{cf\sqrt{a+cx^2}} - \frac{ex}{af^2\sqrt{a+cx^2}} + \frac{af(cd^2+a(e^2-df))+ce(cd^2+a(e}{af^2(ace^2+(cd-af)^2)\sqrt{a+}}$$

$$= -\frac{1}{cf\sqrt{a+cx^2}} - \frac{ex}{af^2\sqrt{a+cx^2}} + \frac{af(cd^2+a(e^2-df))+ce(cd^2+a(e}{af^2(ace^2+(cd-af)^2)\sqrt{a+}}$$

$$= -\frac{1}{cf\sqrt{a+cx^2}} - \frac{ex}{af^2\sqrt{a+cx^2}} + \frac{af(cd^2+a(e^2-df))+ce(cd^2+a(e}{af^2(ace^2+(cd-af)^2)\sqrt{a+}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.61, size = 405, normalized size = 0.81

$$\frac{-a(-cd+af+cex)+c\sqrt{a+cx^2}\,\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-acd^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-a^2e^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2a\sqrt{c}\,de\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+cd^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+ae^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-adf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{c\left(c^2d^2+a^2f^2+ac\left(e^2-2df\right)\right)\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-(a*(-(c*d) + a*f + c*e*x)) + c*Sqrt[a + c*x^2]*RootSum[a^2*f + 2*a*Sqrt[c
]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(a*c*d^2
```

```
*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]) - a^2*e^2*Log[-(Sqrt[c]*x) + Sqr
t[a + c*x^2] - #1] + a^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*a
*Sqrt[c]*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + c*d^2*Log[-(Sqrt
[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + a*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x
^2] - #1]*#1^2 - a*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sq
rt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(c*(c^2*d
^2 + a^2*f^2 + a*c*(e^2 - 2*d*f))*Sqrt[a + c*x^2])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1575 vs. $2(456) = 912$.
time = 0.11, size = 1576, normalized size = 3.16

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1576 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/c/f/(c*x^2+a)^(1/2)-e*x/a/f^2/(c*x^2+a)^(1/2)+1/2*(e^3-3*d*e*f+e^2*(-4*d
*f+e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2))/f^3/(-4*d*f+e^2)^(1/2)*(2/((-4*d*f+e^
2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2
*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f
+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e+(-4*d*f+e^2)^(1/2)
)*f/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)-c*(e+(-4*d*f+e^2)^(1/2))/f)/(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/((x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/((-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f
)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4
*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f
^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(-d*f*(-4*d*f+e^2)^(1/2)+
e^2*(-4*d*f+e^2)^(1/2)+3*d*e*f-e^3)/f^3/(-4*d*f+e^2)^(1/2)*(2/(-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2
*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e-(-4*d*f+e^2)^(1/
2))*f/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/(2*c*(-(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2/f^2)/((x-1/2/f*(-e
+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-
2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-(-4*d*f+e^
```

```
2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/
f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/
2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-
2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 26071 vs. $2(465) = 930$.
time = 79.47, size = 26071, normalized size = 52.25

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] -1/4*(sqrt(2)*(a*c^3*d^2 - 2*a^2*c^2*d*f + a^3*c*f^2 + (c^4*d^2 - 2*a*c^3*d
*f + a^2*c^2*f^2)*x^2 + (a*c^3*x^2 + a^2*c^2)*e^2)*sqrt(-(2*c^3*d^6 - 6*a*c
^2*d^5*f + 6*a^2*c*d^4*f^2 - 2*a^3*d^3*f^3 + a^3*e^6 + 3*(a^2*c*d^2 - 2*a^3
*d*f)*e^4 + 3*(a*c^2*d^4 - 4*a^2*c*d^3*f + 3*a^3*d^2*f^2)*e^2 + (4*c^6*d^7*
f - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5*f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2
*d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d*f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 -
10*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 - 3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a
^3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^5*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5
*f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*d^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*
c*d*f^5 + a^6*f^6)*e^2)*sqrt(-(a^6*e^10 + 2*(3*a^5*c*d^2 - 4*a^6*d*f)*e^8 +
 (15*a^4*c^2*d^4 - 36*a^5*c*d^3*f + 22*a^6*d^2*f^2)*e^6 + 6*(3*a^3*c^3*d^6
- 10*a^4*c^2*d^5*f + 11*a^5*c*d^4*f^2 - 4*a^6*d^3*f^3)*e^4 + 9*(a^2*c^4*d^8
 - 4*a^3*c^3*d^7*f + 6*a^4*c^2*d^6*f^2 - 4*a^5*c*d^5*f^3 + a^6*d^4*f^4)*e^2
)/(4*c^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10*d^11*f^3 - 880*a^3*c^9
*d^10*f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*f^6 + 3696*a^6*c^6*d^7*
f^7 - 3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 - 880*a^9*c^3*d^4*f^10 +
264*a^10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*d*f^13 - a^6*c^6*e^14 -
```

```
 2*(3*a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e^12 - 3*(5*a^4*c^8*d^4
- 28*a^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5*d*f^3 + 5*a^8*c^4*f^4)
*e^10 - 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c^7*d^4*f^2 - 38*a^6*c^6
*d^3*f^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^9*c^3*f^6)*e^8 - 5*(3*a
^2*c^10*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2 - 408*a^5*c^7*d^5*f^3
+ 530*a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^8*c^4*d^2*f^6 - 40*a^9*
c^3*d*f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 20*a^2*c^10*d^9*f + 125*
a^3*c^9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*d^6*f^4 - 952*a^6*c^6*d
^5*f^5 + 770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7 + 125*a^9*c^3*d^2*f^8 -
20*a^10*c^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 - 36*a*c^11*d^11*f + 306*
a^2*c^10*d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4*c^8*d^8*f^4 - 5832*a^5*
c^7*d^7*f^5 + 6972*a^6*c^6*d^6*f^6 - 5832*a^7*c^5*d^5*f^7 + 3375*a^8*c^4*d^
4*f^8 - 1300*a^9*c^3*d^3*f^9 + 306*a^10*c^2*d^2*f^10 - 36*a^11*c*d*f^11 + a
^12*f^12)*e^2)))/(4*c^6*d^7*f - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5*f^3 - 80*
a^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d*f^7 - a^3
*c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 - 3*(a*c^5*
d^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^5*c*f^4)*e
^4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*d^3*f^3 +
63*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2))*log((4*a^3*c*d^4*x*e^5
 - 2*a^4*d^3*e^6 + 4*(3*a^2*c^2*d^6 - 4*a^3*c*d^5*f)*x*e^3 + 12*(a*c^3*d^8
- 2*a^2*c^2*d^7*f + a^3*c*d^6*f^2)*x*e + sqrt(2)*(a^5*e^10 + 5*(a^4*c*d^2 -
 2*a^5*d*f)*e^8 + (9*a^3*c^2*d^4 - 40*a^4*c*d^3*f + 35*a^5*d^2*f^2)*e^6 + 2
*(3*a^2*c^3*d^6 - 27*a^3*c^2*d^5*f + 49*a^4*c*d^4*f^2 - 25*a^5*d^3*f^3)*e^4
 - 24*(a^2*c^3*d^7*f - 3*a^3*c^2*d^6*f^2 + 3*a^4*c*d^5*f^3 - a^5*d^4*f^4)*e
^2 - (8*c^8*d^11*f - 64*a*c^7*d^10*f^2 + 224*a^2*c^6*d^9*f^3 - 448*a^3*c^5*
d^8*f^4 + 560*a^4*c^4*d^7*f^5 - 448*a^5*c^3*d^6*f^6 + 224*a^6*c^2*d^5*f^7 -
 64*a^7*c*d^4*f^8 + 8*a^8*d^3*f^9 - a^5*c^3*e^12 - (5*a^4*c^4*d^2 - 14*a^5*
c^3*d*f + 3*a^6*c^2*f^2)*e^10 - (11*a^3*c^5*d^4 - 60*a^4*c^4*d^3*f + 90*a^5
*c^3*d^2*f^2 - 36*a^6*c^2*d*f^3 + 3*a^7*c*f^4)*e^8 - (13*a^2*c^6*d^6 - 110*
a^3*c^5*d^5*f + 295*a^4*c^4*d^4*f^2 - 340*a^5*c^3*d^3*f^3 + 171*a^6*c^2*d^2
*f^4 - 30*a^7*c*d*f^5 + a^8*f^6)*e^6 - 8*(a*c^7*d^8 - 13*a^2*c^6*d^7*f + 51
*a^3*c^5*d^6*f^2 - 95*a^4*c^4*d^5*f^3 + 95*a^5*c^3*d^4*f^4 - 51*a^6*c^2*d^3
*f^5 + 13*a^7*c*d^2*f^6 - a^8*d*f^7)*e^4 - 2*(c^8*d^10 - 24*a*c^7*d^9*f + 1
32*a^2*c^6*d^8*f^2 - 344*a^3*c^5*d^7*f^3 + 510*a^4*c^4*d^6*f^4 - 456*a^5*c^
3*d^5*f^5 + 244*a^6*c^2*d^4*f^6 - 72*a^7*c*d^3*f^7 + 9*a^8*d^2*f^8)*e^2)*sq
rt(-(a^6*e^10 + 2*(3*a^5*c*d^2 - 4*a^6*d*f)*e^8 + (15*a^4*c^2*d^4 - 36*a^5*
c*d^3*f + 22*a^6*d^2*f^2)*e^6 + 6*(3*a^3*c^3*d^6 - 10*a^4*c^2*d^5*f + 11*a^
5*c*d^4*f^2 - 4*a^6*d^3*f^3)*e^4 + 9*(a^2*c^4*d^8 - 4*a^3*c^3*d^7*f + 6*a^4
*c^2*d^6*f^2 - 4*a^5*c*d^5*f^3 + a^6*d^4*f^4)*e^2)/(4*c^12*d^13*f - 48*a*c^
11*d^12*f^2 + 264*a^2*c^10*d^11*f^3 - 880*a^3*c^9*d^10*f^4 + 1980*a^4*c^8*d
^9*f^5 - 3168*a^5*c^7*d^8*f^6 + 3696*a^6*c^6*d^7*f^7 - 3168*a^7*c^5*d^6*f^8
 + 1980*a^8*c^4*d^5*f^9 - 880*a^9*c^3*d^4*f^10 + 264*a^10*c^2*d^3*f^11 - 48
*a^11*c*d^2*f^12 + 4*a^12*d*f^13 - a^6*c^6*e^14 - 2*(3*a^5*c^7*d^2 - 8*a^6*
c^6*d*f + 3*a^7*c^5*f^2)*e^12 - 3*(5*a^4*c^8*d^4 - 28*a^5*c^7*d^3*f + 46*a^
6*c^6*d^2*f^2 - 28*a^7*c^5*d*f^3 + 5*a^8*c^4*f^4)*e^10 - 20*(a^3*c^9*d^6 -
```

```
9*a^4*c^8*d^5*f + 27*a^5*c^7*d^4*f^2 - 38*a^6*c^6*d^3*f^3 + 27*a^7*c^5*d^2*
f^4 - 9*a^8*c^4*d*f^5 + a^9*c^3*f^6)*e^8 - 5*(3*a^2*c^10*d^8 - 40*a^3*c^9*d
^7*f + 180*a^4*c^8*d^6*f^2 - 408*a^5*c^7*d^5*f^3 + 530*a^6*c^6*d^4*f^4 - 40
8*a^7*c^5*d^3*f^5 + 180*a^8*c^4*d^2*f^6 - 40*a^9*c^3*d*f^7 + 3*a^10*c^2*f^8
)*e^6 - 6*(a*c^11*d^10 - 20*a^2*c^10*d^9*f + 12...
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^3}{(a+cx^2)^{\frac{3}{2}}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
[Out] Integral(x**3/((a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{(c\,x^2+a)^{3/2}\,(f\,x^2+e\,x+d)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
[Out] int(x^3/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.72 $\int \frac{x^2}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=410

$$-\frac{ae+(cd-af)x}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}}-\frac{f\left(2d(cd-af)+ae\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-\right.}}\right.}{\sqrt{2}\sqrt{e^2-4df}\,(ace^2+(cd-af)^2)\sqrt{2af^2+c\left(e^2-2d\right.}}$$

```
[Out] (-a*e-(-a*f+c*d)*x)/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)-1/2*f*arctanh(1/
2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*d*(-a*f+c*d)+a*e*(e-(-4*d*f+e^2)^(
1/2)))/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*
d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^
2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/
2)))^(1/2))*(2*d*(-a*f+c*d)+a*e*(e+(-4*d*f+e^2)^(1/2)))/(a*c*e^2+(-a*f+c*d)
^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))
^(1/2)
```

**Rubi** [A]
time = 0.42, antiderivative size = 410, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.148$, Rules used = {1077, 1048, 739, 212}

$$-\frac{f\left(2d(cd-af)+ae\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2+ace^2)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(2d(cd-af)+ae\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2+ace^2)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{x(cd-af)+ae}{\sqrt{a+cx^2}\,((cd-af)^2+ace^2)}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -((a*e + (c*d - a*f)*x)/((a*c*e^2 + (c*d - a*f)^2)*Sqrt[a + c*x^2])) - (f*(
2*d*(c*d - a*f) + a*e*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sqrt
[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d
*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^
2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + (f*(2*d*(c*d -
a*f) + a*e*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*
f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[
a + c*x^2])]])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a
*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1077

```
Int[((a_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) + (e_.)*(x_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + c*x^2)^(p + 1)*((d + e*x + f*x^2
)^(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)))*((A*c - a*C)*(2*a*c
*e) + c*(A*(2*c^2*d - c*(2*a*f)) + C*(-2*a*(c*d - a*f)))*x), x] + Dist[1/((
-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^(p + 1)*(d + e*
x + f*x^2)^q*Simp[(-2*A*c - 2*a*C)*((c*d - a*f)^2 - ((-a)*e)*(c*e))*(p + 1)
 + (2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(a*f*(p + 1) - c*d*(p + 2)) -
e*((A*c - a*C)*(2*a*c*e))*(p + q + 2) - (2*f*((A*c - a*C)*(2*a*c*e))*(p + q
 + 2) - (2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*((-c)*e*(2*p + q + 4)))*x
 - c*f*(2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(2*p + 2*q + 5)*x^2, x], x
], x] /; FreeQ[{a, c, d, e, f, A, C, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p
, -1] && NeQ[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]
) &&  !IGtQ[q, 0]
```

Rubi steps

$$\int \frac{x^2}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx = -\frac{ae+(cd-af)x}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} + \frac{\int \frac{2acd(cd-af)-2a^2cefx}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2ac\,(ace^2+(cd-af)^2)}$$

$$= -\frac{ae+(cd-af)x}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} + \frac{\left(f\left(2d(cd-af)+ae\left(e-\sqrt{e^2-\right.\right.\right.}{\sqrt{e^2-4df}}$$

$$= -\frac{ae+(cd-af)x}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} - \frac{\left(f\left(2d(cd-af)+ae\left(e-\sqrt{e^2-\right.\right.\right.}{}$$

$$= -\frac{ae+(cd-af)x}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} - \frac{f\left(2d(cd-af)+ae\left(e-\sqrt{e^2-4\right.\right.}{\sqrt{2}\sqrt{e^2-4df}\,(ac}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.53, size = 303, normalized size = 0.74

$$\frac{-ae-cdx+afx+\sqrt{a+cx^2}\,\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{a^2ef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2c^{3/2}d^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+2a\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-aef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{(c^2d^2+a^2f^2+ac(e^2-2df))\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-(a*e) - c*d*x + a*f*x + Sqrt[a + c*x^2]*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1
+ 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a^2*e*f*Log[-(Sq
rt[c]*x) + Sqrt[a + c*x^2] - #1] - 2*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a
+ c*x^2] - #1]*#1 + 2*a*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1
]*#1 - a*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e +
4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/((c^2*d^2 + a^2*f^2
 + a*c*(e^2 - 2*d*f))*Sqrt[a + c*x^2])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1524 vs. 2(372) = 744.
time = 0.12, size = 1525, normalized size = 3.72

| method | result | size |
|---|---|---|

| default | Expression too large to display | 1525 |
|---|---|---|

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*x/a/(c*x^2+a)^(1/2)+1/2*(-e*(-4*d*f+e^2)^(1/2)+2*d*f-e^2)/f^2/(-4*d*f+e
^2)^(1/2)*(2/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2
*c*(e+(-4*d*f+e^2)^(1/2))*f/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*
(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+e^2)^(1/2))/f)/(2*c*((-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2
/f^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2)-2/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2
)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*
(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*
d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*
f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(
e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d*f+e^2)^(1/2)*(2/(-(-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c
*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e-(-4*d*f+e^2)^(1/2))*
f/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/(2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2/f^2)/((x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(-
(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2
)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)
^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/
f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*
d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 24806 vs. $2(375) = 750$.
time = 111.16, size = 24806, normalized size = 60.50

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] 1/4*(sqrt(2)*(a*c^2*d^2 - 2*a^2*c*d*f + a^3*f^2 + (c^3*d^2 - 2*a*c^2*d*f +
a^2*c*f^2)*x^2 + (a*c^2*x^2 + a^2*c)*e^2)*sqrt((2*c^3*d^5*f - 6*a*c^2*d^4*f
^2 + 6*a^2*c*d^3*f^3 - 2*a^3*d^2*f^4 - a^3*f^2*e^4 - (c^3*d^4 + 3*a^2*c*d^2
*f^2 - 4*a^3*d*f^3)*e^2 + (4*c^6*d^7*f - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5*
f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d*
f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 -
3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^5
*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*d
^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2)*sqrt(-(a^6*f^4
*e^6 - 2*(a^3*c^3*d^4*f^2 - 3*a^5*c*d^2*f^4 + 2*a^6*d*f^5)*e^4 + (c^6*d^8 -
 6*a^2*c^4*d^6*f^2 + 4*a^3*c^3*d^5*f^3 + 9*a^4*c^2*d^4*f^4 - 12*a^5*c*d^3*f
^5 + 4*a^6*d^2*f^6)*e^2)/(4*c^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10
*d^11*f^3 - 880*a^3*c^9*d^10*f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*
f^6 + 3696*a^6*c^6*d^7*f^7 - 3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 -
880*a^9*c^3*d^4*f^10 + 264*a^10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*
d*f^13 - a^6*c^6*e^14 - 2*(3*a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e
^12 - 3*(5*a^4*c^8*d^4 - 28*a^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5
*d*f^3 + 5*a^8*c^4*f^4)*e^10 - 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c
^7*d^4*f^2 - 38*a^6*c^6*d^3*f^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^
9*c^3*f^6)*e^8 - 5*(3*a^2*c^10*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2
 - 408*a^5*c^7*d^5*f^3 + 530*a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^
8*c^4*d^2*f^6 - 40*a^9*c^3*d*f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 2
0*a^2*c^10*d^9*f + 125*a^3*c^9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*
d^6*f^4 - 952*a^6*c^6*d^5*f^5 + 770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7 +
 125*a^9*c^3*d^2*f^8 - 20*a^10*c^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 -
36*a*c^11*d^11*f + 306*a^2*c^10*d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4*
c^8*d^8*f^4 - 5832*a^5*c^7*d^7*f^5 + 6972*a^6*c^6*d^6*f^6 - 5832*a^7*c^5*d^
5*f^7 + 3375*a^8*c^4*d^4*f^8 - 1300*a^9*c^3*d^3*f^9 + 306*a^10*c^2*d^2*f^10
 - 36*a^11*c*d*f^11 + a^12*f^12)*e^2)))/(4*c^6*d^7*f - 24*a*c^5*d^6*f^2 + 6
0*a^2*c^4*d^5*f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2*
```

```
f^6 + 4*a^6*d*f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*c
^2*f^2)*e^6 - 3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*c
^2*d*f^3 + a^5*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2
- 92*a^3*c^3*d^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2))
*log((4*a^3*c*d^3*f^3*x*e^3 - 2*a^4*d^2*f^3*e^4 - 4*(c^4*d^7*f - 3*a^2*c^2*
d^5*f^3 + 2*a^3*c*d^4*f^4)*x*e + sqrt(2)*(a^5*f^3*e^7 - (a^2*c^3*d^4*f + a^
3*c^2*d^3*f^2 - 5*a^4*c*d^2*f^3 + 7*a^5*d*f^4)*e^5 + (c^5*d^7 - 2*a*c^4*d^6
*f + 2*a^2*c^3*d^5*f^2 + 12*a^3*c^2*d^4*f^3 - 27*a^4*c*d^3*f^4 + 14*a^5*d^2
*f^5)*e^3 - 4*(c^5*d^8*f - 2*a*c^4*d^7*f^2 - 2*a^2*c^3*d^6*f^3 + 8*a^3*c^2*
d^5*f^4 - 7*a^4*c*d^4*f^5 + 2*a^5*d^3*f^6)*e - (a^5*c^3*f*e^11 + (a^3*c^5*d
^3 + 5*a^4*c^4*d^2*f - 13*a^5*c^3*d*f^2 + 3*a^6*c^2*f^3)*e^9 + (3*a^2*c^6*d
^5 - a^3*c^5*d^4*f - 50*a^4*c^4*d^3*f^2 + 78*a^5*c^3*d^2*f^3 - 33*a^6*c^2*d
*f^4 + 3*a^7*c*f^5)*e^7 + (3*a*c^7*d^7 - 17*a^2*c^6*d^6*f - 33*a^3*c^5*d^5*
f^2 + 195*a^4*c^4*d^4*f^3 - 263*a^5*c^3*d^3*f^4 + 141*a^6*c^2*d^2*f^5 - 27*
a^7*c*d*f^6 + a^8*f^7)*e^5 + (c^8*d^9 - 16*a*c^7*d^8*f + 20*a^2*c^6*d^7*f^2
 + 112*a^3*c^5*d^6*f^3 - 370*a^4*c^4*d^5*f^4 + 464*a^5*c^3*d^4*f^5 - 284*a^
6*c^2*d^3*f^6 + 80*a^7*c*d^2*f^7 - 7*a^8*d*f^8)*e^3 - 4*(c^8*d^10*f - 4*a*c
^7*d^9*f^2 + 28*a^3*c^5*d^7*f^4 - 70*a^4*c^4*d^6*f^5 + 84*a^5*c^3*d^5*f^6 -
 56*a^6*c^2*d^4*f^7 + 20*a^7*c*d^3*f^8 - 3*a^8*d^2*f^9)*e)*sqrt(-(a^6*f^4*e
^6 - 2*(a^3*c^3*d^4*f^2 - 3*a^5*c*d^2*f^4 + 2*a^6*d*f^5)*e^4 + (c^6*d^8 - 6
*a^2*c^4*d^6*f^2 + 4*a^3*c^3*d^5*f^3 + 9*a^4*c^2*d^4*f^4 - 12*a^5*c*d^3*f^5
 + 4*a^6*d^2*f^6)*e^2)/(4*c^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10*d
^11*f^3 - 880*a^3*c^9*d^10*f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*f^
6 + 3696*a^6*c^6*d^7*f^7 - 3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 - 88
0*a^9*c^3*d^4*f^10 + 264*a^10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*d*
f^13 - a^6*c^6*e^14 - 2*(3*a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e^1
2 - 3*(5*a^4*c^8*d^4 - 28*a^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5*d
*f^3 + 5*a^8*c^4*f^4)*e^10 - 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c^7
*d^4*f^2 - 38*a^6*c^6*d^3*f^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^9*
c^3*f^6)*e^8 - 5*(3*a^2*c^10*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2 -
 408*a^5*c^7*d^5*f^3 + 530*a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^8*
c^4*d^2*f^6 - 40*a^9*c^3*d*f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 20*
a^2*c^10*d^9*f + 125*a^3*c^9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*d^
6*f^4 - 952*a^6*c^6*d^5*f^5 + 770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7 + 1
25*a^9*c^3*d^2*f^8 - 20*a^10*c^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 - 36
*a*c^11*d^11*f + 306*a^2*c^10*d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4*c^
8*d^8*f^4 - 5832*a^5*c^7*d^7*f^5 + 6972*a^6*c^6...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2}{(a+cx^2)^{\frac{3}{2}}(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
[Out] Integral(x**2/((a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{{\left(c\,x^2+a\right)}^{3/2}\,\left(f\,x^2+e\,x+d\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
[Out] int(x^2/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.73 $\int \frac{x}{\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=411

$$-\frac{cd-af-cex}{\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}}+\frac{f\left(2cde-(cd-af)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e-\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-2\right.}}\right.}{\sqrt{2}\sqrt{e^2-4df}\left(ace^2+(cd-af)^2\right)\sqrt{2af^2+c\left(e^2-2df\right.}}$$

```
[Out] (c*e*x+a*f-c*d)/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)+1/2*f*arctanh(1/2*(2
*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*
d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*c*d*e-(-a*f+c*d)*(e-(-4*d*f+e^2)^(1/2)
))/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-
e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(
1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))
^(1/2))*(2*c*d*e-(-a*f+c*d)*(e+(-4*d*f+e^2)^(1/2)))/(a*c*e^2+(-a*f+c*d)^2)*
2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/
2)
```

**Rubi** [A]
time = 0.51, antiderivative size = 411, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.160$, Rules used = {1031, 1048, 739, 212}

$$\frac{f\left(2cde-\left(e-\sqrt{e^2-4df}\right)(cd-af)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(2cde-\left(\sqrt{e^2-4df}+e\right)(cd-af)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{-af+cd-cex}{\sqrt{a+cx^2}\left((cd-af)^2+ace^2\right)}$$

Antiderivative was successfully verified.

```
[In] Int[x/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -((c*d - a*f - c*e*x)/((a*c*e^2 + (c*d - a*f)^2)*Sqrt[a + c*x^2])) + (f*(2*
c*d*e - (c*d - a*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e - Sqrt[e
^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f
])]*Sqrt[a + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)
*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - (f*(2*c*d*e - (c*
d - a*f)*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f]
)*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a
+ c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f
^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1031

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + c*x^2)^(p + 1)*((d + e*x + f*x^2)^
(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)))*(g*c*(2*a*c*e) + ((-a
)*h)*(2*c^2*d - c*(2*a*f)) + c*(g*(2*c^2*d - c*(2*a*f)) - h*(-2*a*c*e))*x),
 x] + Dist[1/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^
(p + 1)*(d + e*x + f*x^2)^q*Simp[(-2*g*c)*((c*d - a*f)^2 - ((-a)*e)*(c*e))*
(p + 1) + (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(a*f*(p + 1) - c*d*(p + 2))
- e*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d - c*((Plus[2])*a*f)))*(p + q + 2)
- (2*f*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d + (-c)*((Plus[2])*a*f)))*(p + q
 + 2) - (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*((-c)*e*(2*p + q + 4)))*x - c*
f*(2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(2*p + 2*q + 5)*x^2, x], x], x] /; F
reeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && N
eQ[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{x}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx = -\frac{cd-af-cex}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} + \frac{\int \frac{-2ac^2de-2acf(cd-af)x}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2ac\,(ace^2+(cd-af)^2)}$$

$$= -\frac{cd-af-cex}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} - \frac{\left(f\left(2cde-(cd-af)\left(e-\sqrt{e^2-4\right.\right.\right.}{\sqrt{e^2-4df}}$$

$$= -\frac{cd-af-cex}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} + \frac{\left(f\left(2cde-(cd-af)\left(e-\sqrt{e^2-4\right.\right.\right.}{}$$

$$= -\frac{cd-af-cex}{(ace^2+(cd-af)^2)\sqrt{a+cx^2}} + \frac{f\left(2cde-(cd-af)\left(e-\sqrt{e^2-4d\right.\right.}{\sqrt{2}\sqrt{e^2-4df}\,(ace}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.52, size = 330, normalized size = 0.80

$$\frac{-cd+af+cex-\sqrt{a+cx^2}\,\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-acdf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-2c^{3/2}de\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+cdf\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-af^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{(c^2d^2+a^2f^2+ac(e^2-2df))\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-(c*d) + a*f + c*e*x - Sqrt[a + c*x^2]*RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 +
4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(a*c*d*f*Log[-(Sq
rt[c]*x) + Sqrt[a + c*x^2] - #1]) + a^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x
^2] - #1] - 2*c^(3/2)*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 + c*d
*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 - a*f^2*Log[-(Sqrt[c]*x) +
 Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]
*e*#1^2 + 2*f*#1^3) & ])/((c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f))*Sqrt[a +
c*x^2])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1489 vs. 2(370) = 740.
time = 0.14, size = 1490, normalized size = 3.63

| method | result | size |
|---|---|---|

| | | |
|---|---|---|
| default | Expression too large to display | 1490 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(2/((-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d
*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e+(-4*d*f+e^2)^(1/2))*f/((-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
-c*(e+(-4*d*f+e^2)^(1/2))/f)/(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c
*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/((-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*
c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/
f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)
*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x
+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(
1/2)/f*(2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e
+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+
2*c*(e-(-4*d*f+e^2)^(1/2))*f/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/(2*c*(
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2
))^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f
*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c
*d*f+c*e^2)/f^2)^(1/2)-2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^
2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-
4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+
e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 24872 vs. $2(373) = 746$.
time = 115.30, size = 24872, normalized size = 60.52

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] 1/4*(sqrt(2)*(a*c^2*d^2 - 2*a^2*c*d*f + a^3*f^2 + (c^3*d^2 - 2*a*c^2*d*f +
a^2*c*f^2)*x^2 + (a*c^2*x^2 + a^2*c)*e^2)*sqrt(-(2*c^3*d^4*f^2 - 6*a*c^2*d^
3*f^3 + 6*a^2*c*d^2*f^4 - 2*a^3*d*f^5 + c^3*d^2*e^4 - (4*c^3*d^3*f - 3*a*c^
2*d^2*f^2 - a^3*f^4)*e^2 + (4*c^6*d^7*f - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5
*f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d
*f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 -
 3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^
5*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*
d^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2)*sqrt(-(c^6*d^
4*e^6 - 2*(2*c^6*d^5*f - 3*a*c^5*d^4*f^2 + a^3*c^3*d^2*f^4)*e^4 + (4*c^6*d^
6*f^2 - 12*a*c^5*d^5*f^3 + 9*a^2*c^4*d^4*f^4 + 4*a^3*c^3*d^3*f^5 - 6*a^4*c^
2*d^2*f^6 + a^6*f^8)*e^2)/(4*c^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^1
0*d^11*f^3 - 880*a^3*c^9*d^10*f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8
*f^6 + 3696*a^6*c^6*d^7*f^7 - 3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 -
 880*a^9*c^3*d^4*f^10 + 264*a^10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12
*d*f^13 - a^6*c^6*e^14 - 2*(3*a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*
e^12 - 3*(5*a^4*c^8*d^4 - 28*a^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^
5*d*f^3 + 5*a^8*c^4*f^4)*e^10 - 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*
c^7*d^4*f^2 - 38*a^6*c^6*d^3*f^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a
^9*c^3*f^6)*e^8 - 5*(3*a^2*c^10*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^
2 - 408*a^5*c^7*d^5*f^3 + 530*a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a
^8*c^4*d^2*f^6 - 40*a^9*c^3*d*f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 -
20*a^2*c^10*d^9*f + 125*a^3*c^9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7
*d^6*f^4 - 952*a^6*c^6*d^5*f^5 + 770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7
+ 125*a^9*c^3*d^2*f^8 - 20*a^10*c^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 -
 36*a*c^11*d^11*f + 306*a^2*c^10*d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4
*c^8*d^8*f^4 - 5832*a^5*c^7*d^7*f^5 + 6972*a^6*c^6*d^6*f^6 - 5832*a^7*c^5*d
^5*f^7 + 3375*a^8*c^4*d^4*f^8 - 1300*a^9*c^3*d^3*f^9 + 306*a^10*c^2*d^2*f^1
0 - 36*a^11*c*d*f^11 + a^12*f^12)*e^2)))/(4*c^6*d^7*f - 24*a*c^5*d^6*f^2 +
60*a^2*c^4*d^5*f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2
```

```
*f^6 + 4*a^6*d*f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*
c^2*f^2)*e^6 - 3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*
c^2*d*f^3 + a^5*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2
 - 92*a^3*c^3*d^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2)
)*log((4*c^4*d^4*f^2*x*e^3 - 2*a*c^3*d^3*f^2*e^4 - 4*(2*c^4*d^5*f^3 - 3*a*c
^3*d^4*f^4 + a^3*c*d^2*f^6)*x*e + sqrt(2)*sqrt(c*x^2 + a)*((c^5*d^4 - a^2*c
^3*d^2*f^2)*e^6 - (6*c^5*d^5*f - 3*a*c^4*d^4*f^2 - 6*a^2*c^3*d^3*f^3 + 4*a^
3*c^2*d^2*f^4 - a^5*f^6)*e^4 + 4*(2*c^5*d^6*f^2 - 3*a*c^4*d^5*f^3 - 2*a^2*c
^3*d^4*f^4 + 4*a^3*c^2*d^3*f^5 - a^5*d*f^7)*e^2 + (8*c^8*d^10*f^2 - 64*a*c^
7*d^9*f^3 + 224*a^2*c^6*d^8*f^4 - 448*a^3*c^5*d^7*f^5 + 560*a^4*c^4*d^6*f^6
 - 448*a^5*c^3*d^5*f^7 + 224*a^6*c^2*d^4*f^8 - 64*a^7*c*d^3*f^9 + 8*a^8*d^2
*f^10 + (a^3*c^5*d^2 + a^5*c^3*f^2)*e^10 + (3*a^2*c^6*d^4 - 12*a^3*c^5*d^3*
f + 10*a^4*c^4*d^2*f^2 - 12*a^5*c^3*d*f^3 + 3*a^6*c^2*f^4)*e^8 + (3*a*c^7*d
^6 - 30*a^2*c^6*d^5*f + 77*a^3*c^5*d^4*f^2 - 100*a^4*c^4*d^3*f^3 + 77*a^5*c
^3*d^2*f^4 - 30*a^6*c^2*d*f^5 + 3*a^7*c*f^6)*e^6 + (c^8*d^8 - 24*a*c^7*d^7*
f + 124*a^2*c^6*d^6*f^2 - 296*a^3*c^5*d^5*f^3 + 390*a^4*c^4*d^4*f^4 - 296*a
^5*c^3*d^3*f^5 + 124*a^6*c^2*d^2*f^6 - 24*a^7*c*d*f^7 + a^8*f^8)*e^4 - 2*(3
*c^8*d^9*f - 32*a*c^7*d^8*f^2 + 132*a^2*c^6*d^7*f^3 - 288*a^3*c^5*d^6*f^4 +
 370*a^4*c^4*d^5*f^5 - 288*a^5*c^3*d^4*f^6 + 132*a^6*c^2*d^3*f^7 - 32*a^7*c
*d^2*f^8 + 3*a^8*d*f^9)*e^2)*sqrt(-(c^6*d^4*e^6 - 2*(2*c^6*d^5*f - 3*a*c^5*
d^4*f^2 + a^3*c^3*d^2*f^4)*e^4 + (4*c^6*d^6*f^2 - 12*a*c^5*d^5*f^3 + 9*a^2*
c^4*d^4*f^4 + 4*a^3*c^3*d^3*f^5 - 6*a^4*c^2*d^2*f^6 + a^6*f^8)*e^2)/(4*c^12
*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10*d^11*f^3 - 880*a^3*c^9*d^10*f^4
 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*f^6 + 3696*a^6*c^6*d^7*f^7 - 316
8*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 - 880*a^9*c^3*d^4*f^10 + 264*a^10*
c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*d*f^13 - a^6*c^6*e^14 - 2*(3*a^5
*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e^12 - 3*(5*a^4*c^8*d^4 - 28*a^5*
c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5*d*f^3 + 5*a^8*c^4*f^4)*e^10 - 2
0*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c^7*d^4*f^2 - 38*a^6*c^6*d^3*f^3
+ 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^9*c^3*f^6)*e^8 - 5*(3*a^2*c^10*d
^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2 - 408*a^5*c^7*d^5*f^3 + 530*a^6
*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^8*c^4*d^2*f^6 - 40*a^9*c^3*d*f^7
 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 20*a^2*c^10*d^9*f + 125*a^3*c^9*d
^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*d^6*f^4 - 952*a^6*c^6*d^5*f^5 +
770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7 + 125*a^9*c^3*d^2*f^8 - 20*a^10*c
^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 - 36*a*c^11*d^11*f + 306*a^2*c^10*
d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4*c^8*...
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x}{(a+cx^2)^{\frac{3}{2}}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x/((a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\left(c\,x^2+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.74 $\int \frac{1}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=416

$$\frac{c(ae+(cd-af)x)}{a\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}}-\frac{f\left(2af^2+c\left(e^2-2df+e\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{2af-c\left(e\right.}{\sqrt{2}\sqrt{2af^2+c\left(e^2-\right.}}\right.}{\sqrt{2}\sqrt{e^2-4df}\left(ace^2+(cd-af)^2\right)\sqrt{2af^2+c\left(e^2-2df\right.}}$$

```
[Out] c*(a*e+(-a*f+c*d)*x)/a/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)-1/2*f*arctanh
(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*
(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^
2)^(1/2)))/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*
f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)
^(1/2)))^(1/2))*(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))/(a*c*e^2+(-a*f
+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1
/2)))^(1/2)
```

**Rubi** [A]

time = 0.38, antiderivative size = 416, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {990, 1048, 739, 212}

$$-\frac{f\left(2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{c(x(cd-af)+ae)}{a\sqrt{a+cx^2}\left((cd-af)^2+ace^2\right)}$$

Antiderivative was successfully verified.

```
[In] Int[1/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (c*(a*e + (c*d - a*f)*x))/(a*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[a + c*x^2]) - (
f*(2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e -
 Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2
- 4*d*f])]*Sqrt[a + c*x^2])]])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d -
a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) + (f*(2*a*f^
2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]))*ArcTanh[(2*a*f - c*(e + Sqrt[e^2
 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])
]*Sqrt[a + c*x^2])]])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*S
qrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 990

```
Int[((a_.) + (c_.)*(x_)^2)^(p_)*((d_.) + (e_.)*(x_) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(2*a*c^2*e + c*(2*c^2*d - c*(2*a*f))*x)*(a + c*x^2)^(p + 1
)*((d + e*x + f*x^2)^(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1))),
 x] - Dist[1/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^
(p + 1)*(d + e*x + f*x^2)^q*Simp[2*c*((c*d - a*f)^2 - ((-a)*e)*(c*e))*(p +
1) - (2*c^2*d - c*(2*a*f))*(a*f*(p + 1) - c*d*(p + 2)) - e*(-2*a*c^2*e)*(p
+ q + 2) + (2*f*(2*a*c^2*e)*(p + q + 2) - (2*c^2*d - c*(2*a*f))*((-c)*e*(2*
p + q + 4)))*x + c*f*(2*c^2*d - c*(2*a*f))*(2*p + 2*q + 5)*x^2, x], x], x]
/; FreeQ[{a, c, d, e, f, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ
[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]) &&  !IGtQ[
q, 0]
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{1}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx = \frac{c(ae+(cd-af)x)}{a\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} - \frac{\int \frac{-2ac(af^2+c(e^2-df))-2ac^2efx}{\sqrt{a+cx^2}\,(d+ex+fx^2)}\,dx}{2ac\left(ace^2+(cd-af)^2\right)}$$

$$= \frac{c(ae+(cd-af)x)}{a\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} - \frac{\left(f\left(2af^2+c\left(e^2-2df-e\sqrt{e^2-\right.\right.\right.}{\sqrt{e^2-4df}}$$

$$= \frac{c(ae+(cd-af)x)}{a\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} + \frac{\left(f\left(2af^2+c\left(e^2-2df-e\sqrt{e^2-\right.\right.\right.}{}$$

$$= \frac{c(ae+(cd-af)x)}{a\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} - \frac{f\left(2af^2+c\left(e^2-2df+e\sqrt{e^2-4d\right.\right.}{\sqrt{2}\sqrt{e^2-4df}\,(ac}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.53, size = 346, normalized size = 0.83

$$\frac{c(cdx+a(e-fx)) - a\sqrt{a+cx^2}\,\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{acef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}e^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-2c^{3/2}df\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+2a\sqrt{c}\,f^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-cef\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{a\left(c^2d^2+a^2f^2+ac\left(e^2-2df\right)\right)\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (c*(c*d*x + a*(e - f*x)) - a*Sqrt[a + c*x^2]*RootSum[a^2*f + 2*a*Sqrt[c]*e*
#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c*e*f*Log[-
(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(3/2)*e^2*Log[-(Sqrt[c]*x) + Sqrt
[a + c*x^2] - #1]*#1 - 2*c^(3/2)*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #
1]*#1 + 2*a*Sqrt[c]*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - c*e*f
*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2
*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/(a*(c^2*d^2 + a^2*f^2 + a*c*(e^
2 - 2*d*f))*Sqrt[a + c*x^2])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1456 vs. 2(377) = 754.
time = 0.11, size = 1457, normalized size = 3.50

| method | result | size |
|---|---|---|

| default | Expression too large to display | 1457 |
|---|---|---|

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/(-4*d*f+e^2)^(1/2)*(2/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2
/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/
f^2)^(1/2)+2*c*(e+(-4*d*f+e^2)^(1/2))*f/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c
*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c*(e+(-4*d*f+e^2)^(1/2))/
f)/(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e+(-4*d*f+e
^2)^(1/2))^2/f^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f+e^2)^(
1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2
)*f^2*2^(1/2)/((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln
((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e+(-4*d*f+e^2)^(1/2
))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(((-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-4
*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*((-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)))+1/(-4*d*f+e^2)^(1/2)*(2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e
^2)*f^2/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d
*f+c*e^2)/f^2)^(1/2)+2*c*(e-(-4*d*f+e^2)^(1/2))*f/(-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+
e^2)^(1/2))/f)/(2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2
*(e-(-4*d*f+e^2)^(1/2))^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-
(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*
e^2)/f^2)^(1/2)*ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(
e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))+2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1
/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 26013 vs. $2(383) = 766$.
time = 74.22, size = 26013, normalized size = 62.53

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] -1/4*(sqrt(2)*(a^2*c^2*d^2 - 2*a^3*c*d*f + a^4*f^2 + (a*c^3*d^2 - 2*a^2*c^2
*d*f + a^3*c*f^2)*x^2 + (a^2*c^2*x^2 + a^3*c)*e^2)*sqrt((2*c^3*d^3*f^3 - 6*
a*c^2*d^2*f^4 + 6*a^2*c*d*f^5 - 2*a^3*f^6 - c^3*e^6 + 3*(2*c^3*d*f - a*c^2*
f^2)*e^4 - 3*(3*c^3*d^2*f^2 - 4*a*c^2*d*f^3 + a^2*c*f^4)*e^2 + (4*c^6*d^7*f
 - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5*f^3 - 80*a^3*c^3*d^4*f^4 + 60*a^4*c^2*
d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d*f^7 - a^3*c^3*e^8 - (3*a^2*c^4*d^2 - 1
0*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 - 3*(a*c^5*d^4 - 8*a^2*c^4*d^3*f + 14*a^
3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^5*c*f^4)*e^4 - (c^6*d^6 - 18*a*c^5*d^5*
f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*d^3*f^3 + 63*a^4*c^2*d^2*f^4 - 18*a^5*c
*d*f^5 + a^6*f^6)*e^2)*sqrt(-(c^6*e^10 - 2*(4*c^6*d*f - 3*a*c^5*f^2)*e^8 +
(22*c^6*d^2*f^2 - 36*a*c^5*d*f^3 + 15*a^2*c^4*f^4)*e^6 - 6*(4*c^6*d^3*f^3 -
 11*a*c^5*d^2*f^4 + 10*a^2*c^4*d*f^5 - 3*a^3*c^3*f^6)*e^4 + 9*(c^6*d^4*f^4
- 4*a*c^5*d^3*f^5 + 6*a^2*c^4*d^2*f^6 - 4*a^3*c^3*d*f^7 + a^4*c^2*f^8)*e^2)
/(4*c^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10*d^11*f^3 - 880*a^3*c^9*
d^10*f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*f^6 + 3696*a^6*c^6*d^7*f
^7 - 3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 - 880*a^9*c^3*d^4*f^10 + 2
64*a^10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*d*f^13 - a^6*c^6*e^14 -
2*(3*a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e^12 - 3*(5*a^4*c^8*d^4 -
 28*a^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5*d*f^3 + 5*a^8*c^4*f^4)*
e^10 - 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c^7*d^4*f^2 - 38*a^6*c^6*
d^3*f^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^9*c^3*f^6)*e^8 - 5*(3*a^
2*c^10*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2 - 408*a^5*c^7*d^5*f^3 +
 530*a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^8*c^4*d^2*f^6 - 40*a^9*c
^3*d*f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 20*a^2*c^10*d^9*f + 125*a
^3*c^9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*d^6*f^4 - 952*a^6*c^6*d^
5*f^5 + 770*a^7*c^5*d^4*f^6 - 400*a^8*c^4*d^3*f^7 + 125*a^9*c^3*d^2*f^8 - 2
0*a^10*c^2*d*f^9 + a^11*c*f^10)*e^4 - (c^12*d^12 - 36*a*c^11*d^11*f + 306*a
^2*c^10*d^10*f^2 - 1300*a^3*c^9*d^9*f^3 + 3375*a^4*c^8*d^8*f^4 - 5832*a^5*c
^7*d^7*f^5 + 6972*a^6*c^6*d^6*f^6 - 5832*a^7*c^5*d^5*f^7 + 3375*a^8*c^4*d^4
*f^8 - 1300*a^9*c^3*d^3*f^9 + 306*a^10*c^2*d^2*f^10 - 36*a^11*c*d*f^11 + a^
12*f^12)*e^2)))/(4*c^6*d^7*f - 24*a*c^5*d^6*f^2 + 60*a^2*c^4*d^5*f^3 - 80*a
```

```
^3*c^3*d^4*f^4 + 60*a^4*c^2*d^3*f^5 - 24*a^5*c*d^2*f^6 + 4*a^6*d*f^7 - a^3*
c^3*e^8 - (3*a^2*c^4*d^2 - 10*a^3*c^3*d*f + 3*a^4*c^2*f^2)*e^6 - 3*(a*c^5*d
^4 - 8*a^2*c^4*d^3*f + 14*a^3*c^3*d^2*f^2 - 8*a^4*c^2*d*f^3 + a^5*c*f^4)*e^
4 - (c^6*d^6 - 18*a*c^5*d^5*f + 63*a^2*c^4*d^4*f^2 - 92*a^3*c^3*d^3*f^3 + 6
3*a^4*c^2*d^2*f^4 - 18*a^5*c*d*f^5 + a^6*f^6)*e^2))*log((4*c^4*d*f^3*x*e^5
- 2*a*c^3*f^3*e^6 - 4*(4*c^4*d^2*f^4 - 3*a*c^3*d*f^5)*x*e^3 + 12*(c^4*d^3*f
^5 - 2*a*c^3*d^2*f^6 + a^2*c^2*d*f^7)*x*e + sqrt(2)*(c^5*d*e^9 - (9*c^5*d^2
*f - 5*a*c^4*d*f^2 + a^2*c^3*f^3)*e^7 + (27*c^5*d^3*f^2 - 37*a*c^4*d^2*f^3
+ 17*a^2*c^3*d*f^4 - 3*a^3*c^2*f^5)*e^5 - (31*c^5*d^4*f^3 - 80*a*c^4*d^3*f^
4 + 70*a^2*c^3*d^2*f^5 - 24*a^3*c^2*d*f^6 + 3*a^4*c*f^7)*e^3 + 12*(c^5*d^5*
f^4 - 4*a*c^4*d^4*f^5 + 6*a^2*c^3*d^3*f^6 - 4*a^3*c^2*d^2*f^7 + a^4*c*d*f^8
)*e - (a^3*c^5*d*e^11 + (3*a^2*c^6*d^3 - 13*a^3*c^5*d^2*f + 5*a^4*c^4*d*f^2
 + a^5*c^3*f^3)*e^9 + (3*a*c^7*d^5 - 33*a^2*c^6*d^4*f + 78*a^3*c^5*d^3*f^2
- 50*a^4*c^4*d^2*f^3 - a^5*c^3*d*f^4 + 3*a^6*c^2*f^5)*e^7 + (c^8*d^7 - 27*a
*c^7*d^6*f + 141*a^2*c^6*d^5*f^2 - 263*a^3*c^5*d^4*f^3 + 195*a^4*c^4*d^3*f^
4 - 33*a^5*c^3*d^2*f^5 - 17*a^6*c^2*d*f^6 + 3*a^7*c*f^7)*e^5 - (7*c^8*d^8*f
 - 80*a*c^7*d^7*f^2 + 284*a^2*c^6*d^6*f^3 - 464*a^3*c^5*d^5*f^4 + 370*a^4*c
^4*d^4*f^5 - 112*a^5*c^3*d^3*f^6 - 20*a^6*c^2*d^2*f^7 + 16*a^7*c*d*f^8 - a^
8*f^9)*e^3 + 4*(3*c^8*d^9*f^2 - 20*a*c^7*d^8*f^3 + 56*a^2*c^6*d^7*f^4 - 84*
a^3*c^5*d^6*f^5 + 70*a^4*c^4*d^5*f^6 - 28*a^5*c^3*d^4*f^7 + 4*a^7*c*d^2*f^9
 - a^8*d*f^10)*e)*sqrt(-(c^6*e^10 - 2*(4*c^6*d*f - 3*a*c^5*f^2)*e^8 + (22*c
^6*d^2*f^2 - 36*a*c^5*d*f^3 + 15*a^2*c^4*f^4)*e^6 - 6*(4*c^6*d^3*f^3 - 11*a
*c^5*d^2*f^4 + 10*a^2*c^4*d*f^5 - 3*a^3*c^3*f^6)*e^4 + 9*(c^6*d^4*f^4 - 4*a
*c^5*d^3*f^5 + 6*a^2*c^4*d^2*f^6 - 4*a^3*c^3*d*f^7 + a^4*c^2*f^8)*e^2)/(4*c
^12*d^13*f - 48*a*c^11*d^12*f^2 + 264*a^2*c^10*d^11*f^3 - 880*a^3*c^9*d^10*
f^4 + 1980*a^4*c^8*d^9*f^5 - 3168*a^5*c^7*d^8*f^6 + 3696*a^6*c^6*d^7*f^7 -
3168*a^7*c^5*d^6*f^8 + 1980*a^8*c^4*d^5*f^9 - 880*a^9*c^3*d^4*f^10 + 264*a^
10*c^2*d^3*f^11 - 48*a^11*c*d^2*f^12 + 4*a^12*d*f^13 - a^6*c^6*e^14 - 2*(3*
a^5*c^7*d^2 - 8*a^6*c^6*d*f + 3*a^7*c^5*f^2)*e^12 - 3*(5*a^4*c^8*d^4 - 28*a
^5*c^7*d^3*f + 46*a^6*c^6*d^2*f^2 - 28*a^7*c^5*d*f^3 + 5*a^8*c^4*f^4)*e^10
- 20*(a^3*c^9*d^6 - 9*a^4*c^8*d^5*f + 27*a^5*c^7*d^4*f^2 - 38*a^6*c^6*d^3*f
^3 + 27*a^7*c^5*d^2*f^4 - 9*a^8*c^4*d*f^5 + a^9*c^3*f^6)*e^8 - 5*(3*a^2*c^1
0*d^8 - 40*a^3*c^9*d^7*f + 180*a^4*c^8*d^6*f^2 - 408*a^5*c^7*d^5*f^3 + 530*
a^6*c^6*d^4*f^4 - 408*a^7*c^5*d^3*f^5 + 180*a^8*c^4*d^2*f^6 - 40*a^9*c^3*d*
f^7 + 3*a^10*c^2*f^8)*e^6 - 6*(a*c^11*d^10 - 20*a^2*c^10*d^9*f + 125*a^3*c^
9*d^8*f^2 - 400*a^4*c^8*d^7*f^3 + 770*a^5*c^7*d...
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{(a + cx^2)^{\frac{3}{2}}(d + ex + fx^2)} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/((a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

$$\text{Timed out}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{{\left(c\,x^2+a\right)}^{3/2}\,\left(f\,x^2+e\,x+d\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/((a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.75 $\int \frac{1}{x\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=526

$$\frac{1}{ad\sqrt{a+cx^2}}-\frac{a(af^2+c(e^2-df))+c^2dex}{ad\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}}+\frac{f\left(2e(af^2+c(e^2-2df))-\left(e-\sqrt{e^2-4df}\right)(af^2+c(e^2\cdots\right.}{\sqrt{2}\,d\sqrt{e^2-4df}\,\left(ace^2+(cd\cdots\right.}$$

```
[Out] -arctanh((c*x^2+a)^(1/2)/a^(1/2))/a^(3/2)/d+1/a/d/(c*x^2+a)^(1/2)+(-a*(a*f^
2+c*(-d*f+e^2))-c^2*d*e*x)/a/d/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)+1/2*f
*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*
a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*e*(a*f^2+c*(-2*d*f+e^2)
)-(a*f^2+c*(-d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/d/(a*c*e^2+(-a*f+c*d)^2)*2^(
1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-
1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2
)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2))*(2*e*(a*f^2+c*(-2*d*f
+e^2))-(a*f^2+c*(-d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/d/(a*c*e^2+(-a*f+c*d)^2
)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(
1/2)
```

**Rubi** [A]
time = 1.41, antiderivative size = 526, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 9, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {6860, 272, 53, 65, 214, 1031, 1048, 739, 212}

$$-\frac{\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{a^{3/2}d}-\frac{a(af^2+c(e^2-df))+c^2dex}{ad\sqrt{a+cx^2}\left((cd-af)^2+ace^2\right)}+\frac{f\left(2e(af^2+c(e^2-2df))-\left(e-\sqrt{e^2-4df}\right)(af^2+c(e^2-df))\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(2e(af^2+c(e^2-2df))-\left(\sqrt{e^2-4df}+e\right)(af^2+c(e^2-df))\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\left((cd-af)^2+ace^2\right)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{1}{ad\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] 1/(a*d*Sqrt[a + c*x^2]) - (a*(a*f^2 + c*(e^2 - d*f)) + c^2*d*e*x)/(a*d*(a*c
*e^2 + (c*d - a*f)^2)*Sqrt[a + c*x^2]) + (f*(2*e*(a*f^2 + c*(e^2 - 2*d*f))
- (e - Sqrt[e^2 - 4*d*f])*(a*f^2 + c*(e^2 - d*f)))*ArcTanh[(2*a*f - c*(e -
Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 -
 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d -
 a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - (f*(2*e*(
a*f^2 + c*(e^2 - 2*d*f)) - (e + Sqrt[e^2 - 4*d*f])*(a*f^2 + c*(e^2 - d*f)))
*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(e
^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 -
 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt[e^
2 - 4*d*f])]) - ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]]/(a^(3/2)*d)
```

Rule 53

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Dist[d*((
m + n + 2)/((b*c - a*d)*(m + 1))), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&  !(LtQ
[n, -1] && (EqQ[a, 0] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && I
ntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1031

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + c*x^2)^(p + 1)*((d + e*x + f*x^2)^
(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)))*(g*c*(2*a*c*e) + ((-a
)*h)*(2*c^2*d - c*(2*a*f)) + c*(g*(2*c^2*d - c*(2*a*f)) - h*(-2*a*c*e))*x),
 x] + Dist[1/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^
```

```
(p + 1)*(d + e*x + f*x^2)^q*Simp[(-2*g*c)*((c*d - a*f)^2 - ((-a)*e)*(c*e))*
(p + 1) + (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(a*f*(p + 1) - c*d*(p + 2))
- e*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d - c*((Plus[2])*a*f)))*(p + q + 2)
- (2*f*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d + (-c)*((Plus[2])*a*f)))*(p + q
 + 2) - (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*((-c)*e*(2*p + q + 4)))*x - c*
f*(2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(2*p + 2*q + 5)*x^2, x], x], x] /; F
reeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && N
eQ[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{1}{dx\left(a+cx^2\right)^{3/2}} + \frac{-e-fx}{d\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\right)dx$$

$$= \frac{\int \frac{1}{x(a+cx^2)^{3/2}}\,dx}{d} + \frac{\int \frac{-e-fx}{(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx}{d}$$

$$= -\frac{a(af^2+c(e^2-df))+c^2dex}{ad\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} + \frac{\text{Subst}\left(\int \frac{1}{x(a+cx)^{3/2}}\,dx, x, x^2\right)}{2d}$$

$$= \frac{1}{ad\sqrt{a+cx^2}} - \frac{a(af^2+c(e^2-df))+c^2dex}{ad\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} + \frac{\text{Subst}\left(\int \frac{1}{x\sqrt{a-}}\right.}{2d}$$

$$= \frac{1}{ad\sqrt{a+cx^2}} - \frac{a(af^2+c(e^2-df))+c^2dex}{ad\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} + \frac{\text{Subst}\left(\int \frac{1}{-\frac{a}{c}+\frac{x^2}{c}}\right.}{}$$

$$= \frac{1}{ad\sqrt{a+cx^2}} - \frac{a(af^2+c(e^2-df))+c^2dex}{ad\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} + \frac{f\left(2e(af^2+c(e\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.87, size = 542, normalized size = 1.03

$$-\frac{c(-cd+af+cex)}{a\left(c^2d^2+a^2f^2+ac\left(e^2-2df\right)\right)\sqrt{a+cx^2}} + \frac{2\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+cx^2}}{\sqrt{a}}\right)}{a^{3/2}d} + \frac{\text{RootSum}\left[a^2f+2a\sqrt{c}\,e\#1+4cd\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{ace^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)-acdf^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+a^2f^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)+2c^{3/2}e^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-4c^{3/2}def\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1+2a\sqrt{c}\,ef^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1-ce^2f\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2+cdf^2\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2-af^3\log\left(-\sqrt{c}\,x+\sqrt{a+cx^2}-\#1\right)\#1^2}{a\sqrt{c}\,e+4cd\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{d\left(c^2d^2+a^2f^2+ac\left(e^2-2df\right)\right)}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -((c*(-(c*d) + a*f + c*e*x))/(a*(c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f))*Sqr
t[a + c*x^2])) + (2*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/Sqrt[a]])/(a^(3/2
)*d) + RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*Sqrt[
c]*e*#1^3 + f*#1^4 & , (a*c*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]
- a*c*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + a^2*f^3*Log[-(Sqrt[c
]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(3/2)*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*
x^2] - #1]*#1 - 4*c^(3/2)*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1
 + 2*a*Sqrt[c]*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - c*e^2*f*
Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + c*d*f^2*Log[-(Sqrt[c]*x) +
```

```
Sqrt[a + c*x^2] - #1]*#1^2 - a*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]
*#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) &
]/(d*(c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)))
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 1564 vs. $2(480) = 960$.

time = 0.14, size = 1565, normalized size = 2.98

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1565 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(2/((-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-4*d*f
+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e+(-4*d*f+e^2)^(1/2))*f/((-4*d*f+e^
2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)-c
*(e+(-4*d*f+e^2)^(1/2))/f)/(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e
^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*
d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^
2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*(
((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*
d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f)))+2*f/(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/
2)*(2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*
(e-(-4*d*f+e^2)^(1/2))*f/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2
*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/(2*c*(-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)^(1/2))^2
/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-
1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f
+c*e^2)/f^2)^(1/2)-2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^
(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(1/2))/f*(
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-4*c*
(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/
```

```
2)))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(1/a/(c*x^2+a)^(1
/2)-1/a^(3/2)*ln((2*a+2*a^(1/2)*(c*x^2+a)^(1/2))/x))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(1/((c*x^2 + a)^(3/2)*(f*x^2 + x*e + d)*x), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x\left(a + cx^{2}\right)^{\frac{3}{2}}\left(d + ex + fx^{2}\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
[Out] Integral(1/(x*(a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\,{\left(c\,x^2+a\right)}^{3/2}\,\left(f\,x^2+e\,x+d\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
[Out] int(1/(x*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.76 $\int \frac{1}{x^2(a+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=618

$$-\frac{e}{ad^2\sqrt{a+cx^2}}-\frac{1}{adx\sqrt{a+cx^2}}-\frac{2cx}{a^2d\sqrt{a+cx^2}}+\frac{ae(af^2+c(e^2-2df))+cd(af^2+c(e^2-df))\,x}{ad^2\,(ace^2+(cd-af)^2)\,\sqrt{a+cx^2}}+\frac{f\Big(e\Big(e}{}$$

```
[Out] e*arctanh((c*x^2+a)^(1/2)/a^(1/2))/a^(3/2)/d^2-e/a/d^2/(c*x^2+a)^(1/2)-1/a/
d/x/(c*x^2+a)^(1/2)-2*c*x/a^2/d/(c*x^2+a)^(1/2)+(a*e*(a*f^2+c*(-2*d*f+e^2))
+c*d*(a*f^2+c*(-d*f+e^2))*x)/a/d^2/(a*c*e^2+(-a*f+c*d)^2)/(c*x^2+a)^(1/2)+1
/2*f*arctanh(1/2*(2*a*f-c*x*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)
/(2*a*f^2+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2))*(-2*a*f^2*(-d*f+e^2)-2
*c*(d^2*f^2-3*d*e^2*f+e^4)+e*(a*f^2+c*(-2*d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))
/d^2/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*
f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*f*arctanh(1/2*(2*a*f-c*x*(e+(-4*d*f+e^2)
^(1/2)))*2^(1/2)/(c*x^2+a)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)
))^(1/2))*(-2*a*f^2*(-d*f+e^2)-2*c*(d^2*f^2-3*d*e^2*f+e^4)+e*(a*f^2+c*(-2*d
*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/d^2/(a*c*e^2+(-a*f+c*d)^2)*2^(1/2)/(-4*d*f
+e^2)^(1/2)/(2*a*f^2+c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 1.43, antiderivative size = 618, normalized size of antiderivative = 1.00, number of steps used = 14, number of rules used = 11, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.407$, Rules used = {6860, 277, 197, 272, 53, 65, 214, 1031, 1048, 739, 212}

$$\frac{e\tanh^{-1}\left(\frac{\sqrt{a+cx^2}}{\sqrt{a}}\right)}{a^{3/2}d^2}-\frac{2cx}{a^2d\sqrt{a+cx^2}}+\frac{cdx(af^2+c(e^2-df))+ae(af^2+c(e^2-2df))}{ad^2\sqrt{a+cx^2}\,((cd-af)^2+ace^2)}+\frac{f\left(e\left(e-\sqrt{e^2-4df}\right)(af^2+c(e^2-2df))-2(af^2(e^2-df)+c(d^2f^2-3de^2f+e^4))\right)\tanh^{-1}\left(\frac{2af-cx\left(e-\sqrt{e^2-4df}\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^2\sqrt{e^2-4df}\,((cd-af)^2+ace^2)\sqrt{2af^2+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{f\left(e\left(\sqrt{e^2-4df}+e\right)(af^2+c(e^2-2df))-2(af^2(e^2-df)+c(d^2f^2-3de^2f+e^4))\right)\tanh^{-1}\left(\frac{2af-cx\left(\sqrt{e^2-4df}+e\right)}{\sqrt{2}\sqrt{a+cx^2}\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}\right)}{\sqrt{2}d^2\sqrt{e^2-4df}\,((cd-af)^2+ace^2)\sqrt{2af^2+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{e}{ad^2\sqrt{a+cx^2}}-\frac{1}{adx\sqrt{a+cx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -(e/(a*d^2*Sqrt[a + c*x^2])) - 1/(a*d*x*Sqrt[a + c*x^2]) - (2*c*x)/(a^2*d*S
qrt[a + c*x^2]) + (a*e*(a*f^2 + c*(e^2 - 2*d*f)) + c*d*(a*f^2 + c*(e^2 - d*
f))*x)/(a*d^2*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[a + c*x^2]) + (f*(e*(e - Sqrt[
e^2 - 4*d*f])*(a*f^2 + c*(e^2 - 2*d*f)) - 2*(a*f^2*(e^2 - d*f) + c*(e^4 - 3
*d*e^2*f + d^2*f^2)))*ArcTanh[(2*a*f - c*(e - Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2
]*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/
(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f^2 + c*(
e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])]) - (f*(e*(e + Sqrt[e^2 - 4*d*f])*(a*f^2
 + c*(e^2 - 2*d*f)) - 2*(a*f^2*(e^2 - d*f) + c*(e^4 - 3*d*e^2*f + d^2*f^2))
)*ArcTanh[(2*a*f - c*(e + Sqrt[e^2 - 4*d*f])*x)/(Sqrt[2]*Sqrt[2*a*f^2 + c*(
```

```
e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])]*Sqrt[a + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^
2 - 4*d*f]*(a*c*e^2 + (c*d - a*f)^2)*Sqrt[2*a*f^2 + c*(e^2 - 2*d*f + e*Sqrt
[e^2 - 4*d*f])]) + (e*ArcTanh[Sqrt[a + c*x^2]/Sqrt[a]])/(a^(3/2)*d^2)
```

Rule 53

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> Simp[
(a + b*x)^(m + 1)*((c + d*x)^(n + 1)/((b*c - a*d)*(m + 1))), x] - Dist[d*((
m + n + 2)/((b*c - a*d)*(m + 1))), Int[(a + b*x)^(m + 1)*(c + d*x)^n, x], x
] /; FreeQ[{a, b, c, d, n}, x] && NeQ[b*c - a*d, 0] && LtQ[m, -1] &&  !(LtQ
[n, -1] && (EqQ[a, 0] || (NeQ[c, 0] && LtQ[m - n, 0] && IntegerQ[n]))) && I
ntLinearQ[a, b, c, d, m, n, x]
```

Rule 65

```
Int[((a_.) + (b_.)*(x_))^(m_)*((c_.) + (d_.)*(x_))^(n_), x_Symbol] :> With[
{p = Denominator[m]}, Dist[p/b, Subst[Int[x^(p*(m + 1) - 1)*(c - a*(d/b) +
d*(x^p/b))^n, x], x, (a + b*x)^(1/p)], x]] /; FreeQ[{a, b, c, d}, x] && NeQ
[b*c - a*d, 0] && LtQ[-1, m, 0] && LeQ[-1, n, 0] && LeQ[Denominator[n], Den
ominator[m]] && IntLinearQ[a, b, c, d, m, n, x]
```

Rule 197

```
Int[((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x*((a + b*x^n)^(p + 1)
/a), x] /; FreeQ[{a, b, n, p}, x] && EqQ[1/n + p + 1, 0]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 214

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(Rt[-a/b, 2]/a)*ArcTanh[x
/Rt[-a/b, 2]], x] /; FreeQ[{a, b}, x] && NegQ[a/b]
```

Rule 272

```
Int[(x_)^(m_.)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Dist[1/n, Subst[
Int[x^(Simplify[(m + 1)/n] - 1)*(a + b*x)^p, x], x, x^n], x] /; FreeQ[{a, b
, m, n, p}, x] && IntegerQ[Simplify[(m + 1)/n]]
```

Rule 277

```
Int[(x_)^(m_)*((a_) + (b_.)*(x_)^(n_))^(p_), x_Symbol] :> Simp[x^(m + 1)*((
a + b*x^n)^(p + 1)/(a*(m + 1))), x] - Dist[b*((m + n*(p + 1) + 1)/(a*(m + 1
```

```
))), Int[x^(m + n)*(a + b*x^n)^p, x], x] /; FreeQ[{a, b, m, n, p}, x] && IL
tQ[Simplify[(m + 1)/n + p + 1], 0] && NeQ[m, -1]
```

Rule 739

```
Int[1/(((d_) + (e_.)*(x_))*Sqrt[(a_) + (c_.)*(x_)^2]), x_Symbol] :> -Subst[
Int[1/(c*d^2 + a*e^2 - x^2), x], x, (a*e - c*d*x)/Sqrt[a + c*x^2]] /; FreeQ
[{a, c, d, e}, x]
```

Rule 1031

```
Int[((g_.) + (h_.)*(x_))*((a_) + (c_.)*(x_)^2)^(p_)*((d_) + (e_.)*(x_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + c*x^2)^(p + 1)*((d + e*x + f*x^2)^
(q + 1)/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)))*(g*c*(2*a*c*e) + ((-a
)*h)*(2*c^2*d - c*(2*a*f)) + c*(g*(2*c^2*d - c*(2*a*f)) - h*(-2*a*c*e))*x),
 x] + Dist[1/((-4*a*c)*(a*c*e^2 + (c*d - a*f)^2)*(p + 1)), Int[(a + c*x^2)^
(p + 1)*(d + e*x + f*x^2)^q*Simp[(-2*g*c)*((c*d - a*f)^2 - ((-a)*e)*(c*e))*
(p + 1) + (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(a*f*(p + 1) - c*d*(p + 2))
- e*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d - c*((Plus[2])*a*f)))*(p + q + 2)
- (2*f*((g*c)*(2*a*c*e) + ((-a)*h)*(2*c^2*d + (-c)*((Plus[2])*a*f)))*(p + q
 + 2) - (2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*((-c)*e*(2*p + q + 4)))*x - c*
f*(2*(g*c*(c*d - a*f) - a*((-h)*c*e)))*(2*p + 2*q + 5)*x^2, x], x], x] /; F
reeQ[{a, c, d, e, f, g, h, q}, x] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && N
eQ[a*c*e^2 + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1048

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (f
_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[(2*c*g - h*(
b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + f*x^2]), x], x] - Dist[(2*c*g -
h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + f*x^2]), x], x]] /; FreeQ[{a,
 b, c, d, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{1}{dx^2\left(a+cx^2\right)^{3/2}} - \frac{e}{d^2x\left(a+cx^2\right)^{3/2}} + \frac{e^2-df+efx}{d^2\left(a+cx^2\right)^{3/2}\left(d+ex+\right.}\right.$$

$$= \frac{\int \frac{e^2-df+efx}{\left(a+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx}{d^2} + \frac{\int \frac{1}{x^2\left(a+cx^2\right)^{3/2}}\,dx}{d} - \frac{e\int \frac{1}{x\left(a+cx^2\right)^{3/2}}\,dx}{d^2}$$

$$= -\frac{1}{adx\sqrt{a+cx^2}} + \frac{ae(af^2+c(e^2-2df))+cd(af^2+c(e^2-df))\,x}{ad^2\left(ace^2+(cd-af)^2\right)\sqrt{a+cx^2}} -$$

$$= -\frac{e}{ad^2\sqrt{a+cx^2}} - \frac{1}{adx\sqrt{a+cx^2}} - \frac{2cx}{a^2d\sqrt{a+cx^2}} + \frac{ae(af^2+c(e^2}{ad^2\left(ace^2\right.}$$

$$= -\frac{e}{ad^2\sqrt{a+cx^2}} - \frac{1}{adx\sqrt{a+cx^2}} - \frac{2cx}{a^2d\sqrt{a+cx^2}} + \frac{ae(af^2+c(e^2}{ad^2\left(ace^2\right.}$$

$$= -\frac{e}{ad^2\sqrt{a+cx^2}} - \frac{1}{adx\sqrt{a+cx^2}} - \frac{2cx}{a^2d\sqrt{a+cx^2}} + \frac{ae(af^2+c(e^2}{ad^2\left(ace^2\right.}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.22, size = 684, normalized size = 1.11

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] -(((d*(a^3*f^2 + 2*c^3*d^2*x^2 + a*c^2*(d^2 + e^2*x^2 + d*x*(e - 3*f*x)) +
a^2*c*(e^2 + f*(-2*d + f*x^2))))/(a^2*(c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f
))*x*Sqrt[a + c*x^2]) + (2*e*ArcTanh[(Sqrt[c]*x - Sqrt[a + c*x^2])/Sqrt[a]]
)/a^(3/2) + RootSum[a^2*f + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 - 2*a*f*#1^2 - 2*
Sqrt[c]*e*#1^3 + f*#1^4 & , (a*c*e^3*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] -
 #1] - 2*a*c*d*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + a^2*e*f^3*L
og[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1] + 2*c^(3/2)*e^4*Log[-(Sqrt[c]*x) +
Sqrt[a + c*x^2] - #1]*#1 - 6*c^(3/2)*d*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + c*
x^2] - #1]*#1 + 2*c^(3/2)*d^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*
```

```
#1 + 2*a*Sqrt[c]*e^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - 2*a*
Sqrt[c]*d*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1 - c*e^3*f*Log[-(S
qrt[c]*x) + Sqrt[a + c*x^2] - #1]*#1^2 + 2*c*d*e*f^2*Log[-(Sqrt[c]*x) + Sqr
t[a + c*x^2] - #1]*#1^2 - a*e*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + c*x^2] - #1]*
#1^2)/(a*Sqrt[c]*e + 4*c*d*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ]
/(c^2*d^2 + a^2*f^2 + a*c*(e^2 - 2*d*f)))/d^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1638 vs. $2(565) = 1130$.
time = 0.18, size = 1639, normalized size = 2.65

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1639 |
| risch | Expression too large to display | 1788 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*(2/((-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c-c*(e+(-
4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*((-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)+2*c*(e+(-4*d*f+e^2)^(1/2))*f/((-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))
/f)-c*(e+(-4*d*f+e^2)^(1/2))/f)/(2*c*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*
f+c*e^2)/f^2-c^2*(e+(-4*d*f+e^2)^(1/2))^2/f^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)^2*c-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(
(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/((-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2*2^(1/2)/(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)/f^2-c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1
/2)*(((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e
+(-4*d*f+e^2)^(1/2))/f)^2*c-4*c*(e+(-4*d*f+e^2)^(1/2))/f*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)+2*((-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))
/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+4*f^2/(-e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f
+e^2)^(1/2)*(2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)*f^2/((x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(
1/2)+2*c*(e-(-4*d*f+e^2)^(1/2))*f/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+
c*e^2)*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))-c*(e-(-4*d*f+e^2)^(1/2))/f)/(
2*c*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c^2*(e-(-4*d*f+e^2)
^(1/2))^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c-c*(e-(-4*d*f+e^2)^(1/
2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^
2)*f^2*2^(1/2)/((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*
ln(((-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2-c*(e-(-4*d*f+e^2)^(
```

```
1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((-(-4*d*f+e^2)^(1/2)
*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))
^2*c-4*c*(e-(-4*d*f+e^2)^(1/2))/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(-1/a/x/
(c*x^2+a)^(1/2)-2*c/a^2*x/(c*x^2+a)^(1/2))-16*f^2*e/(-e+(-4*d*f+e^2)^(1/2))
^2/(e+(-4*d*f+e^2)^(1/2))^2*(1/a/(c*x^2+a)^(1/2)-1/a^(3/2)*ln((2*a+2*a^(1/2
)*(c*x^2+a)^(1/2))/x))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(1/((c*x^2 + a)^(3/2)*(f*x^2 + x*e + d)*x^2), x)
```

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^2 (a + cx^2)^{\frac{3}{2}} (d + ex + fx^2)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(c*x**2+a)**(3/2)/(f*x**2+e*x+d),x)
[Out] Integral(1/(x**2*(a + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^2\,{\left(c\,x^2+a\right)}^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x^2*(a + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

## 3.77 $\int \frac{x^3\sqrt{a+bx+cx^2}}{d-fx^2}\,dx$

**Optimal**. Leaf size=392

$$-\frac{d\sqrt{a+bx+cx^2}}{f^2}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{(a+bx+cx^2)^{3/2}}{3cf}-\frac{bd\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2}-\frac{b(b$$

```
[Out] -1/3*(c*x^2+b*x+a)^(3/2)/c/f-1/16*b*(-4*a*c+b^2)*arctanh(1/2*(2*c*x+b)/c^(1
/2)/(c*x^2+b*x+a)^(1/2))/c^(5/2)/f-1/2*b*d*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c
*x^2+b*x+a)^(1/2))/f^2/c^(1/2)-d*(c*x^2+b*x+a)^(1/2)/f^2+1/8*b*(2*c*x+b)*(c
*x^2+b*x+a)^(1/2)/c^2/f-1/2*d*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(
1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*
d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/f^(5/2)+1/2*d*arctanh(1/2*(b*d^(1/2)+2*a*f^(
1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1
/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)/f^(5/2)
```

**Rubi** [A]

time = 0.60, antiderivative size = 392, normalized size of antiderivative = 1.00, number of steps used = 15, number of rules used = 9, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.321$, Rules used = {6857, 654, 626, 635, 212, 1035, 1092, 1047, 738}

$$-\frac{b(b^2-4ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{5/2}f}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{d\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^{5/2}}+\frac{d\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^{5/2}}-\frac{d\sqrt{a+bx+cx^2}}{f^2}-\frac{bd\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2}-\frac{(a+bx+cx^2)^{3/2}}{3cf}$$

Antiderivative was successfully verified.

```
[In] Int[(x^3*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -((d*Sqrt[a + b*x + c*x^2])/f^2) + (b*(b + 2*c*x)*Sqrt[a + b*x + c*x^2])/(8
*c^2*f) - (a + b*x + c*x^2)^(3/2)/(3*c*f) - (b*d*ArcTanh[(b + 2*c*x)/(2*Sqr
t[c]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[c]*f^2) - (b*(b^2 - 4*a*c)*ArcTanh[(b
 + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(16*c^(5/2)*f) - (d*Sqrt[c*d
- b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d]
- b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2
])])/(2*f^(5/2)) + (d*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d
] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt
[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^(5/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$
\begin{aligned}
\int \frac{x^3\sqrt{a+bx+cx^2}}{d-fx^2}\,dx &= \int \left(-\frac{x\sqrt{a+bx+cx^2}}{f}+\frac{dx\sqrt{a+bx+cx^2}}{f\left(d-fx^2\right)}\right) dx\\
&= -\frac{\int x\sqrt{a+bx+cx^2}\,dx}{f}+\frac{d\int \frac{x\sqrt{a+bx+cx^2}}{d-fx^2}\,dx}{f}\\
&= -\frac{d\sqrt{a+bx+cx^2}}{f^2}-\frac{\left(a+bx+cx^2\right)^{3/2}}{3cf}+\frac{d\int \frac{\frac{bd}{2}+(cd+af)x+\frac{1}{2}bfx^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f^2}+\frac{b\int \sqrt{c}
\\
&= -\frac{d\sqrt{a+bx+cx^2}}{f^2}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{\left(a+bx+cx^2\right)^{3/2}}{3cf}-\frac{d\int \frac{}{\sqrt{a+}}
\\
&= -\frac{d\sqrt{a+bx+cx^2}}{f^2}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{\left(a+bx+cx^2\right)^{3/2}}{3cf}-\frac{(bd)\text{Subs}}{}
\\
&= -\frac{d\sqrt{a+bx+cx^2}}{f^2}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{\left(a+bx+cx^2\right)^{3/2}}{3cf}-\frac{bd\tanh^{-}}{}
\\
&= -\frac{d\sqrt{a+bx+cx^2}}{f^2}+\frac{b(b+2cx)\sqrt{a+bx+cx^2}}{8c^2f}-\frac{\left(a+bx+cx^2\right)^{3/2}}{3cf}-\frac{bd\tanh^{-}}{}
\end{aligned}
$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.75, size = 409, normalized size = 1.04

$$\frac{-2\sqrt{c}\,\sqrt{a+x(b+cx)}\left(-3b^2f+2cf(4a+bx)+8c^2(3d+fx^2)\right)+3b(8c^2d+b^2f-4acf)\log\left(b+2cx-2\sqrt{c}\,\sqrt{a+x(b+cx)}\right)-24c^{5/2}d\,\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-acd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-a^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+af\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{48c^{5/2}f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x^3*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*Sqrt[a + x*(b + c*x)]*(-3*b^2*f + 2*c*f*(4*a + b*x) + 8*c^2*(3*
d + f*x^2)) + 3*b*(8*c^2*d + b^2*f - 4*a*c*f)*Log[b + 2*c*x - 2*Sqrt[c]*Sqr
t[a + x*(b + c*x)]] - 24*c^(5/2)*d*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1
 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a +
b*x + c*x^2] - #1] - a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] -
 a^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-
(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1]*#1^2 + a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(48*c^(5/2)*f^2
)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 862 vs. $2(306) = 612$.
time = 0.18, size = 863, normalized size = 2.20 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/3*(c*x^2+b*x+a)^(3/2)/c-1/2*b/c*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2
)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))))-1/2
*d/f^2*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f
*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)
^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/
2))/(x+(d*f)^(1/2)/f)))-1/2*d/f^2*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+
b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1
/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x
-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2
)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+
c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d
*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*
c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x
-(d*f)^(1/2)/f)))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^3\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3*(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**3*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3\,\sqrt{c\,x^2+b\,x+a}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^3*(a + b*x + c*x^2)^(1/2))/(d - f*x^2),x)
```

```
[Out] int((x^3*(a + b*x + c*x^2)^(1/2))/(d - f*x^2), x)
```

## 3.78 $\int \frac{x^2\sqrt{a+bx+cx^2}}{d-fx^2}\,dx$

**Optimal**. Leaf size=316

$$-\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf}-\frac{(8c^2d-b^2f+4acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8c^{3/2}f^2}+\frac{\sqrt{d}\sqrt{cd-b\sqrt{d}\sqrt{f}}\ \cdots}{\cdots}$$

```
[Out] -1/8*(4*a*c*f-b^2*f+8*c^2*d)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1
/2))/c^(3/2)/f^2-1/4*(2*c*x+b)*(c*x^2+b*x+a)^(1/2)/c/f+1/2*arctanh(1/2*(b*d
^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-
b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)*(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/f^2+1/2
*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a
)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)*(c*d+a*f+b*d^(1/2)*f^(1/
2))^(1/2)/f^2
```

**Rubi** [A]
time = 0.30, antiderivative size = 316, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.214$, Rules used = {1085, 1092, 635, 212, 1047, 738}

$$-\frac{(4acf+b^2(-f)+8c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8c^{3/2}f^2}+\frac{\sqrt{d}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^2}+\frac{\sqrt{d}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^2}-\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf}$$

Antiderivative was successfully verified.

```
[In] Int[(x^2*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -1/4*((b + 2*c*x)*Sqrt[a + b*x + c*x^2])/(c*f) - ((8*c^2*d - b^2*f + 4*a*c*
f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(8*c^(3/2)*f^2)
+ (Sqrt[d]*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqr
t[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*
Sqrt[a + b*x + c*x^2])])/(2*f^2) + (Sqrt[d]*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] +
a*f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqr
t[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^2)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1085

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(
a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3
))), x] - Dist[1/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + b*x + c*x^
2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*(C*((-b)*f)*(q + 1)) + (p + q + 1)*(b
^2*C*d*f*p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*
f)*(C*((-b)*f)*(q + 1)) + (p + q + 1)*((-b)*c*(C*(-4*d*f)*(2*p + q + 2) + f
*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*((-b)*f)*(q + 1)) +
(p + q + 1)*(C*f^2*p*(b^2 - 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C
*d + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A,
C, q}, x] && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2
*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{x^2\sqrt{a+bx+cx^2}}{d-fx^2}\,dx = -\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf} - \frac{\int \frac{-\frac{1}{4}(b^2+4ac)df-2bcdfx-\frac{1}{4}f(8c^2d-b^2f+4acf)x^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{2cf^2}$$

$$= -\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf} + \frac{\int \frac{\frac{1}{4}(b^2+4ac)df^2+\frac{1}{4}df(8c^2d-b^2f+4acf)+2bcdf^2x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{2cf^3} - \frac{(8c^2d}{}$$

$$= -\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf} - \frac{\left(\sqrt{d}\left(cd-b\sqrt{d}\sqrt{f}+af\right)\right)\int \frac{1}{\left(-\sqrt{d}\sqrt{f}-fx\right)}}{2f^{3/2}}$$

$$= -\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf} - \frac{(8c^2d-b^2f+4acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right.}{8c^{3/2}f^2}$$

$$= -\frac{(b+2cx)\sqrt{a+bx+cx^2}}{4cf} - \frac{(8c^2d-b^2f+4acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right.}{8c^{3/2}f^2}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.57, size = 320, normalized size = 1.01

$$\frac{-2\sqrt{c}f(b+2cx)\sqrt{a+x(b+cx)}+(8c^2d-b^2f+4acf)\log\left(c\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)-4c^{3/2}d\,\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+bf\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}d-2cd\#1-af\#1+f\#1^3}\&\right]}{8c^{3/2}f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x^2*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*f*(b + 2*c*x)*Sqrt[a + x*(b + c*x)] + (8*c^2*d - b^2*f + 4*a*c*
f)*Log[c*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])] - 4*c^(3/2)*d*RootS
um[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & ,
(b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*c^(3/2)*d*Log[-(S
qrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*f*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1 + b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(8*c^(3/2
)*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 848 vs. $2(246) = 492$.
time = 0.15, size = 849, normalized size = 2.69 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b
+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+1/2*d/(d*f)^(1/2)/f*(((x+(d*f)^(1/2)/f)
^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c
*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*
(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f
)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d
*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(
1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)
^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x
+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/
2*d/(d*f)^(1/2)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^
(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1
/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^
2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1
/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln
((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*(
(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*
f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
)
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^2\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**2*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2\,\sqrt{c\,x^2+b\,x+a}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2*(a + b*x + c*x^2)^(1/2))/(d - f*x^2),x)
```

```
[Out] int((x^2*(a + b*x + c*x^2)^(1/2))/(d - f*x^2), x)
```

# 3.79 $\int \frac{x\sqrt{a+bx+cx^2}}{d-fx^2}\,dx$

**Optimal**. Leaf size=282

$$-\frac{\sqrt{a+bx+cx^2}}{f}-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}-\frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\,\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\cdots}{2\sqrt{cd-b\sqrt{d}\sqrt{f}\cdots}}\right.}{2f^{3/2}}$$

```
[Out] -1/2*b*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f/c^(1/2)-(c*x^2+
b*x+a)^(1/2)/f-1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1
/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(
1/2)*f^(1/2))^(1/2)/f^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d
^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(
c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)/f^(3/2)
```

**Rubi** [A]

time = 0.19, antiderivative size = 282, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.231, Rules used = {1035, 1092, 635, 212, 1047, 738}

$$-\frac{\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^{3/2}}+\frac{\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^{3/2}}-\frac{\sqrt{a+bx+cx^2}}{f}-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}$$

Antiderivative was successfully verified.

```
[In] Int[(x*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/f) - (b*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x
 + c*x^2])])/(2*Sqrt[c]*f) - (Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(
b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt
[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^(3/2)) + (Sqrt[c*d + b*Sqr
t[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqr
t[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2
*f^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{x\sqrt{a+bx+cx^2}}{d-fx^2}\,dx = -\frac{\sqrt{a+bx+cx^2}}{f}+\frac{\int \frac{\frac{bd}{2}+(cd+af)x+\frac{1}{2}bfx^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{f}-\frac{\int \frac{-bdf-f(cd+af)x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f^2}-\frac{b\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{2f}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{f}-\frac{b\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f}+\frac{\left(cd-b\sqrt{d}\,\sqrt{f}\right.}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{f}-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}-\frac{\left(cd-b\sqrt{d}\,\sqrt{f}+af\right)}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{f}-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f}-\frac{\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.41, size = 349, normalized size = 1.24

$$-\frac{2\sqrt{a+x(b+cx)}-\frac{b\log\left(f\left(b+2cx-2\sqrt{c}\,\sqrt{a+x(b+cx)}\right)\right)}{\sqrt{c}}+\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-acd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-a^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+af\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2f}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*Sqrt[a + b*x + c*x^2])/(d - f*x^2),x]
```

```
[Out] -1/2*(2*Sqrt[a + x*(b + c*x)] - (b*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*
(b + c*x)])])/Sqrt[c] + RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1
^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1] - a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*f*Log[
-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1]*#1^2 + a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)
/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 767 vs. 2(214) = 428.
time = 0.14, size = 768, normalized size = 2.72

| method | result |
|---|---|

| | |
|---|---|
| default | $-\frac{\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2} c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}+\frac{\left(-2c\sqrt{df}+bf\right)\ln\left(\frac{\frac{-2c\sqrt{df}+bf}{2f}+c\left(\cdots\right.}{\sqrt{c}\cdots}\right.}{\cdots}}{\cdots}$ |
| risch | $-\frac{\sqrt{c\,x^{2}+bx+a}}{f}-\frac{b\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^{2}+bx+a}\right)}{2f\sqrt{c}}+\frac{\ln\left(\frac{\frac{2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(2c\sqrt{df}+bf\right)\left(x-\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\cdots}}{\cdots}\right.}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/f*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f
*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)
^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/
2))/(x+(d*f)^(1/2)/f)))-1/2/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)
/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+
b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*
f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*
a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)
/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^
(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d
*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*
f)^(1/2)/f)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 595 vs. $2(214) = 428$.
time = 157.33, size = 1192, normalized size = 4.23

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(c*f*sqrt((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3)*log((2*sqrt(c*x^2 + b
*x + a)*f^4*sqrt(b^2*d/f^5)*sqrt((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3) + 2
*b*c*d*x + b^2*d + (b*f^3*x + 2*a*f^3)*sqrt(b^2*d/f^5))/x) - c*f*sqrt((f^3*
sqrt(b^2*d/f^5) + c*d + a*f)/f^3)*log(-(2*sqrt(c*x^2 + b*x + a)*f^4*sqrt(b^
2*d/f^5)*sqrt((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3) - 2*b*c*d*x - b^2*d -
(b*f^3*x + 2*a*f^3)*sqrt(b^2*d/f^5))/x) - c*f*sqrt(-(f^3*sqrt(b^2*d/f^5) -
c*d - a*f)/f^3)*log((2*sqrt(c*x^2 + b*x + a)*f^4*sqrt(b^2*d/f^5)*sqrt(-(f^3
*sqrt(b^2*d/f^5) - c*d - a*f)/f^3) + 2*b*c*d*x + b^2*d - (b*f^3*x + 2*a*f^3
)*sqrt(b^2*d/f^5))/x) + c*f*sqrt(-(f^3*sqrt(b^2*d/f^5) - c*d - a*f)/f^3)*lo
g(-(2*sqrt(c*x^2 + b*x + a)*f^4*sqrt(b^2*d/f^5)*sqrt(-(f^3*sqrt(b^2*d/f^5)
- c*d - a*f)/f^3) - 2*b*c*d*x - b^2*d + (b*f^3*x + 2*a*f^3)*sqrt(b^2*d/f^5)
)/x) + b*sqrt(c)*log(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*sqrt(c*x^2 + b*x + a)*(
2*c*x + b)*sqrt(c) - 4*a*c) - 4*sqrt(c*x^2 + b*x + a)*c)/(c*f), 1/4*(c*f*sq
rt((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3)*log((2*sqrt(c*x^2 + b*x + a)*f^4*
sqrt(b^2*d/f^5)*sqrt((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3) + 2*b*c*d*x + b
^2*d + (b*f^3*x + 2*a*f^3)*sqrt(b^2*d/f^5))/x) - c*f*sqrt((f^3*sqrt(b^2*d/f
^5) + c*d + a*f)/f^3)*log(-(2*sqrt(c*x^2 + b*x + a)*f^4*sqrt(b^2*d/f^5)*sqr
t((f^3*sqrt(b^2*d/f^5) + c*d + a*f)/f^3) - 2*b*c*d*x - b^2*d - (b*f^3*x + 2
*a*f^3)*sqrt(b^2*d/f^5))/x) - c*f*sqrt(-(f^3*sqrt(b^2*d/f^5) - c*d - a*f)/f
^3)*log((2*sqrt(c*x^2 + b*x + a)*f^4*sqrt(b^2*d/f^5)*sqrt(-(f^3*sqrt(b^2*d/
f^5) - c*d - a*f)/f^3) + 2*b*c*d*x + b^2*d - (b*f^3*x + 2*a*f^3)*sqrt(b^2*d
/f^5))/x) + c*f*sqrt(-(f^3*sqrt(b^2*d/f^5) - c*d - a*f)/f^3)*log(-(2*sqrt(c
*x^2 + b*x + a)*f^4*sqrt(b^2*d/f^5)*sqrt(-(f^3*sqrt(b^2*d/f^5) - c*d - a*f)
/f^3) - 2*b*c*d*x - b^2*d + (b*f^3*x + 2*a*f^3)*sqrt(b^2*d/f^5))/x) + 2*b*s
qrt(-c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(-c)/(c^2*x^2 + b*
c*x + a*c)) - 4*sqrt(c*x^2 + b*x + a)*c)/(c*f)]
```

**Sympy** [F]

time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x\,\sqrt{c\,x^2+b\,x+a}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(a + b*x + c*x^2)^(1/2))/(d - f*x^2),x)
```

```
[Out] int((x*(a + b*x + c*x^2)^(1/2))/(d - f*x^2), x)
```

# 3.80 $\int \frac{\sqrt{a+bx+cx^2}}{d-fx^2}\,dx$

**Optimal**. Leaf size=266

$$-\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f}+\frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx}}\right.}{2\sqrt{d}f}$$

```
[Out] -arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))*c^(1/2)/f+1/2*arctanh(1
/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c
*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/f/d^(1/2
)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+
b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))
^(1/2)/f/d^(1/2)
```

**Rubi** [A]
time = 0.14, antiderivative size = 266, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {1004, 635, 212, 1047, 738}

$$\frac{\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}f}+\frac{\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{d}f}-\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(d - f*x^2),x]
```

```
[Out] -((Sqrt[c]*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/f) + (Sq
rt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*S
qrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x
+ c*x^2])])/(2*Sqrt[d]*f) + (Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b
*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[
d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*f)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1004

```
Int[Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2]/((d_) + (f_.)*(x_)^2), x_Symbol]
 :> Dist[c/f, Int[1/Sqrt[a + b*x + c*x^2], x], x] - Dist[1/f, Int[(c*d - a*
f - b*f*x)/(Sqrt[a + b*x + c*x^2]*(d + f*x^2)), x], x] /; FreeQ[{a, b, c, d
, f}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{d-fx^2}\,dx = \frac{\int \frac{cd+af+bfx}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f} - \frac{c\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f}$$

$$= -\frac{(2c)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f} + \frac{1}{2}\left(b-\frac{cd+af}{\sqrt{d}\sqrt{f}}\right)\int \frac{\cdots}{\left(-\sqrt{d}\sqrt{f}\cdots\right.}$$

$$= -\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f} + \left(-b-\frac{cd+af}{\sqrt{d}\sqrt{f}}\right)\text{Subst}\left(\int \frac{\cdots}{4cdf+4b\cdots}\right.$$

$$= -\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f} + \frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\tanh^{-1}\left(\frac{b\cdots}{2\sqrt{cd\cdots}}\right.}{2\sqrt{d}\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 0.26, size = 268, normalized size = 1.01

$$-\frac{-2\sqrt{c}\,\log\left(f\left(b+2cx-2\sqrt{c}\,\sqrt{a+x(b+cx)}\right)\right)+\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}\,f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+bf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2f}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(d - f*x^2),x]
```

```
[Out] -1/2*(-2*Sqrt[c]*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])] + Roo
tSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 &
, (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*c^(3/2)*d*Log[-
(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*f*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 771 vs. $2(202) = 404$.
time = 0.13, size = 772, normalized size = 2.90

| method | result |
|---|---|
| default | $\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}+\frac{\left(-2c\sqrt{df}+bf\right)\ln\left(\frac{\frac{-2c\sqrt{df}+bf}{2f}+c\left(x+\right.}{\sqrt{c}}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2/(d*f)^(1/2)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)
^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*
ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/
2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f
)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+
c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2/(d*f)^(1/2)*(((x-(d*f)^(1/2)/f)^2*c+(2
*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/
2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/
f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f
```

```
)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(
d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/
2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(
1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*
d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

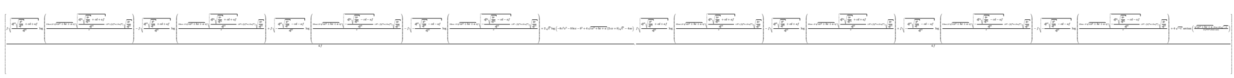
Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 568 vs. $2(202) = 404$.
time = 56.21, size = 1139, normalized size = 4.28



Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f^2))*log((2*b*c*x +
2*sqrt(c*x^2 + b*x + a)*b*f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f
^2)) + b^2 + (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f^3)))/x) - f*sqrt((d*f^2*sqrt
(b^2/(d*f^3)) + c*d + a*f)/(d*f^2))*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*
b*f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f^2)) + b^2 + (b*f^2*x +
2*a*f^2)*sqrt(b^2/(d*f^3)))/x) + f*sqrt(-(d*f^2*sqrt(b^2/(d*f^3)) - c*d - a
*f)/(d*f^2))*log((2*b*c*x + 2*sqrt(c*x^2 + b*x + a)*b*f*sqrt(-(d*f^2*sqrt(b
^2/(d*f^3)) - c*d - a*f)/(d*f^2)) + b^2 - (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f
^3)))/x) - f*sqrt(-(d*f^2*sqrt(b^2/(d*f^3)) - c*d - a*f)/(d*f^2))*log((2*b*
c*x - 2*sqrt(c*x^2 + b*x + a)*b*f*sqrt(-(d*f^2*sqrt(b^2/(d*f^3)) - c*d - a*
f)/(d*f^2)) + b^2 - (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f^3)))/x) + 2*sqrt(c)*l
og(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(c)
 - 4*a*c))/f, 1/4*(f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f^2))*lo
g((2*b*c*x + 2*sqrt(c*x^2 + b*x + a)*b*f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*
d + a*f)/(d*f^2)) + b^2 + (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f^3)))/x) - f*sqr
t((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f^2))*log((2*b*c*x - 2*sqrt(c*x^
2 + b*x + a)*b*f*sqrt((d*f^2*sqrt(b^2/(d*f^3)) + c*d + a*f)/(d*f^2)) + b^2
```

```
+ (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f^3)))/x) + f*sqrt(-(d*f^2*sqrt(b^2/(d*f^
3)) - c*d - a*f)/(d*f^2))*log((2*b*c*x + 2*sqrt(c*x^2 + b*x + a)*b*f*sqrt(-
(d*f^2*sqrt(b^2/(d*f^3)) - c*d - a*f)/(d*f^2)) + b^2 - (b*f^2*x + 2*a*f^2)*
sqrt(b^2/(d*f^3)))/x) - f*sqrt(-(d*f^2*sqrt(b^2/(d*f^3)) - c*d - a*f)/(d*f^
2))*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*b*f*sqrt(-(d*f^2*sqrt(b^2/(d*f^3
)) - c*d - a*f)/(d*f^2)) + b^2 - (b*f^2*x + 2*a*f^2)*sqrt(b^2/(d*f^3)))/x)
+ 4*sqrt(-c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(-c)/(c^2*x^2
 + b*c*x + a*c)))/f]
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{\sqrt{a + bx + cx^2}}{-d + fx^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(sqrt(a + b*x + c*x**2)/(-d + f*x**2), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2 + b\,x + a}}{d - f\,x^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(d - f*x^2),x)
```

```
[Out] int((a + b*x + c*x^2)^(1/2)/(d - f*x^2), x)
```

# 3.81 $\int \frac{\sqrt{a+bx+cx^2}}{x(d-fx^2)}\,dx$

**Optimal**. Leaf size=267

$$-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}-\frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right.}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+b}}\right.}{2d\sqrt{f}}$$

```
[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))*a^(1/2)/d-1/2*arctanh(1
/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c
*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/d/f^(1/2
)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+
b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))
^(1/2)/d/f^(1/2)
```

**Rubi** [A]
time = 0.50, antiderivative size = 267, normalized size of antiderivative = 1.00, number of steps used = 17, number of rules used = 9, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.321$, Rules used = {6857, 748, 857, 635, 212, 738, 1035, 1092, 1047}

$$-\frac{\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d\sqrt{f}}+\frac{\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d\sqrt{f}}-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(x*(d - f*x^2)),x]
```

```
[Out] -((Sqrt[a]*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/d) - (Sq
rt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*S
qrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x
+ c*x^2])])/(2*d*Sqrt[f]) + (Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b
*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[
d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d*Sqrt[f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\begin{aligned}
\int \frac{\sqrt{a+bx+cx^2}}{x\left(d-fx^2\right)}\,dx &= \int \left(\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{fx\sqrt{a+bx+cx^2}}{d\left(-d+fx^2\right)}\right)dx\\
&= \frac{\int \frac{\sqrt{a+bx+cx^2}}{x}\,dx}{d}-\frac{f\int \frac{x\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx}{d}\\
&= -\frac{\int \frac{-2a-bx}{x\sqrt{a+bx+cx^2}}\,dx}{2d}+\frac{\int \frac{-\frac{bd}{2}-(cd+af)x-\frac{1}{2}bfx^2}{\sqrt{a+bx+cx^2}\left(-d+fx^2\right)}\,dx}{d}\\
&= \frac{a\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{d}+\frac{\int \frac{-bdf+f(-cd-af)x}{\sqrt{a+bx+cx^2}\left(-d+fx^2\right)}\,dx}{df}\\
&= -\frac{(2a)\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx,x,\frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{d}-\frac{\left(cd-b\sqrt{d}\sqrt{f}+af\right)\int \frac{\cdots}{\left(\sqrt{d}\cdots\right.}}{2d\cdots}\\
&= -\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}+\frac{\left(cd-b\sqrt{d}\sqrt{f}+af\right)\text{Subst}\left(\int \frac{\cdots}{4cdf-4b\cdots}\right.}{\cdots}\\
&= -\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}-\frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\tanh^{-1}\left(\frac{\cdots}{2\sqrt{cd\cdots}}\right.}{2d\sqrt{\cdots}}
\end{aligned}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.25, size = 334, normalized size = 1.25

$$-\frac{-4\sqrt{a}\tanh^{-1}\left(\frac{\sqrt{c}x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)+\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-acd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-a^2f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+af\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}d-2cd\#1-af\#1+f\#1^3}\&\right]}{2d}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(x*(d - f*x^2)),x]
```

```
[Out] -1/2*(-4*Sqrt[a]*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]] + Roo
tSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 &
, (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*c*d*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
 + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
#1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a*f*Log[
-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 -
a*f*#1 + f*#1^3) & ])/d
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 849 vs. $2(203) = 406$.
time = 0.14, size = 850, normalized size = 3.18

| method | result |
|---|---|
| default | $-\frac{\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2} c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}+\frac{\left(-2c\sqrt{df}+bf\right)\ln\left(\frac{\frac{-2c\sqrt{df}+bf}{2f}+c\left(x\right.}{\sqrt{c}}\right.}{}}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/x/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/d*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f
*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)
^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/
2))/(x+(d*f)^(1/2)/f)))+1/d*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/
2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)
^(1/2))/x))-1/2/d*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^
(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1
/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^
2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1
/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln
```

```
((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*(
(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*
f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
)
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(sqrt(c*x^2 + b*x + a)/((f*x^2 - d)*x), x)
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 622 vs. 2(203) = 406.
time = 10.12, size = 1253, normalized size = 4.69

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(d*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d + a*f)/(d^2*f))*log((2*sqrt(c*x
^2 + b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d +
 a*f)/(d^2*f)) + 2*b*c*x + b^2 + (b*d*f*x + 2*a*d*f)*sqrt(b^2/(d^3*f)))/x)
- d*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d + a*f)/(d^2*f))*log(-(2*sqrt(c*x^2
+ b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d + a*
f)/(d^2*f)) - 2*b*c*x - b^2 - (b*d*f*x + 2*a*d*f)*sqrt(b^2/(d^3*f)))/x) - d
*sqrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)/(d^2*f))*log((2*sqrt(c*x^2 + b
*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)
/(d^2*f)) + 2*b*c*x + b^2 - (b*d*f*x + 2*a*d*f)*sqrt(b^2/(d^3*f)))/x) + d*s
qrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)/(d^2*f))*log(-(2*sqrt(c*x^2 + b*
x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)/
(d^2*f)) - 2*b*c*x - b^2 + (b*d*f*x + 2*a*d*f)*sqrt(b^2/(d^3*f)))/x) + 2*sq
rt(a)*log(-(8*a*b*x + (b^2 + 4*a*c)*x^2 - 4*sqrt(c*x^2 + b*x + a)*(b*x + 2*
a)*sqrt(a) + 8*a^2)/x^2))/d, 1/4*(d*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d + a
*f)/(d^2*f))*log((2*sqrt(c*x^2 + b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt((d^2
*f*sqrt(b^2/(d^3*f)) + c*d + a*f)/(d^2*f)) + 2*b*c*x + b^2 + (b*d*f*x + 2*a
*d*f)*sqrt(b^2/(d^3*f)))/x) - d*sqrt((d^2*f*sqrt(b^2/(d^3*f)) + c*d + a*f)/
(d^2*f))*log(-(2*sqrt(c*x^2 + b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt((d^2*f*
sqrt(b^2/(d^3*f)) + c*d + a*f)/(d^2*f)) - 2*b*c*x - b^2 - (b*d*f*x + 2*a*d*
f)*sqrt(b^2/(d^3*f)))/x) - d*sqrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)/(d
^2*f))*log((2*sqrt(c*x^2 + b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt(-(d^2*f*sq
```

```
rt(b^2/(d^3*f)) - c*d - a*f)/(d^2*f)) + 2*b*c*x + b^2 - (b*d*f*x + 2*a*d*f)
*sqrt(b^2/(d^3*f)))/x) + d*sqrt(-(d^2*f*sqrt(b^2/(d^3*f)) - c*d - a*f)/(d^2
*f))*log(-(2*sqrt(c*x^2 + b*x + a)*d^2*f*sqrt(b^2/(d^3*f))*sqrt(-(d^2*f*sqr
t(b^2/(d^3*f)) - c*d - a*f)/(d^2*f)) - 2*b*c*x - b^2 + (b*d*f*x + 2*a*d*f)*
sqrt(b^2/(d^3*f)))/x) + 4*sqrt(-a)*arctan(1/2*sqrt(c*x^2 + b*x + a)*(b*x +
2*a)*sqrt(-a)/(a*c*x^2 + a*b*x + a^2)))/d]
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{\sqrt{a+bx+cx^2}}{-dx+fx^3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/x/(-f*x**2+d),x)
```

```
[Out] -Integral(sqrt(a + b*x + c*x**2)/(-d*x + f*x**3), x)
```

**Giac** [**F**]
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] sage2
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+b\,x+a}}{x\,\left(d-f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(x*(d - f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(1/2)/(x*(d - f*x^2)), x)
```

## 3.82 $\int \frac{\sqrt{a+bx+cx^2}}{x^2(d-fx^2)}\,dx$

**Optimal**. Leaf size=286

$$-\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d}+\frac{\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\,\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}\ \cdots}{2\sqrt{cd-b\sqrt{d}\sqrt{f}\ \cdots}}\right)}{2d^{3/2}}$$

```
[Out] -1/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d/a^(1/2)-(c*x^2+
b*x+a)^(1/2)/d/x+1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^
(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d
^(1/2)*f^(1/2))^(1/2)/d^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c
*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))
*(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)/d^(3/2)
```

**Rubi** [A]
time = 0.46, antiderivative size = 286, normalized size of antiderivative = 1.00, number of steps used = 16, number of rules used = 8, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.286$, Rules used = {6857, 746, 857, 635, 212, 738, 1004, 1047}

$$\frac{\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^{3/2}}+\frac{\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^{3/2}}-\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(x^2*(d - f*x^2)),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/(d*x)) - (b*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a +
 b*x + c*x^2])])/(2*Sqrt[a]*d) + (Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTa
nh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*
Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]])/(2*d^(3/2)) + (Sqrt[c*d + b
*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b
*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]
)/(2*d^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 746

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 1))), x] - Di
st[p/(e*(m + 1)), Int[(d + e*x)^(m + 1)*(b + 2*c*x)*(a + b*x + c*x^2)^(p -
1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*
d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && GtQ[p, 0] && (IntegerQ[p]
 || LtQ[m, -1]) && NeQ[m, -1] &&  !ILtQ[m + 2*p + 1, 0] && IntQuadraticQ[a,
 b, c, d, e, m, p, x]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1004

```
Int[Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2]/((d_) + (f_.)*(x_)^2), x_Symbol]
 :> Dist[c/f, Int[1/Sqrt[a + b*x + c*x^2], x], x] - Dist[1/f, Int[(c*d - a*
f - b*f*x)/(Sqrt[a + b*x + c*x^2]*(d + f*x^2)), x], x] /; FreeQ[{a, b, c, d
, f}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
```

```
[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{x^2\left(d-fx^2\right)}\,dx = \int \left(\frac{\sqrt{a+bx+cx^2}}{dx^2}+\frac{f\sqrt{a+bx+cx^2}}{d\left(d-fx^2\right)}\right)dx$$

$$= \frac{\int \frac{\sqrt{a+bx+cx^2}}{x^2}\,dx}{d}+\frac{f\int \frac{\sqrt{a+bx+cx^2}}{d-fx^2}\,dx}{d}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx}+\frac{\int \frac{b+2cx}{x\sqrt{a+bx+cx^2}}\,dx}{2d}+\frac{\int \frac{cd+af+bfx}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{d}-\frac{c\int -}{\sqrt{}}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx}+\frac{b\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{2d}+\frac{c\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{d}-\frac{(2c)\text{Subs}}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{\sqrt{c}\,\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{d}-\frac{b\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx,x\right.}{d}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d}+\frac{\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.41, size = 288, normalized size = 1.01

$$-\frac{\frac{2\sqrt{a+x(b+cx)}}{x}-\frac{2b\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{\sqrt{a}}+\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}\,f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+bf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2d}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(x^2*(d - f*x^2)),x]
```

```
[Out] -1/2*((2*Sqrt[a + x*(b + c*x)])/x - (2*b*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b
 + c*x)])/Sqrt[a]])/Sqrt[a] + RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*
c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1] - 2*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*
#1 - 2*a*Sqrt[c]*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + b*f*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#
1 - a*f*#1 + f*#1^3) & ])/d
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 958 vs. $2(218) = 436$.
time = 0.15, size = 959, normalized size = 3.35 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/x^2/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/a/x*(c*x^2+b*x+a)^(3/2)+1/2*b/a*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*
b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(
c*x^2+b*x+a)^(1/2))/x))+2*c/a*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a
*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))))+1/2*f/d/(d*f)
^(1/2)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f
*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)
^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/
2))/(x+(d*f)^(1/2)/f)))-1/2*f/d/(d*f)^(1/2)*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d
*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*
c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c
^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*
(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^
(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*
f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/
f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)
^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

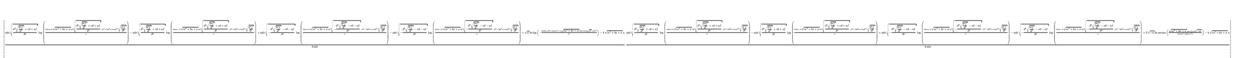
Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(sqrt(c*x^2 + b*x + a)/((f*x^2 - d)*x^2), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 543 vs. $2(218) = 436$.
time = 15.01, size = 1094, normalized size = 3.83



Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(a*d*x*sqrt((d^3*sqrt(b^2*f/d^5) + c*d + a*f)/d^3)*log((2*b*c*x + 2*sq
rt(c*x^2 + b*x + a)*b*d*sqrt((d^3*sqrt(b^2*f/d^5) + c*d + a*f)/d^3) + b^2 +
 (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))/x) - a*d*x*sqrt((d^3*sqrt(b^2*f/d^5)
+ c*d + a*f)/d^3)*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*b*d*sqrt((d^3*sqrt
(b^2*f/d^5) + c*d + a*f)/d^3) + b^2 + (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))/
x) + a*d*x*sqrt(-(d^3*sqrt(b^2*f/d^5) - c*d - a*f)/d^3)*log((2*b*c*x + 2*sq
rt(c*x^2 + b*x + a)*b*d*sqrt(-(d^3*sqrt(b^2*f/d^5) - c*d - a*f)/d^3) + b^2
- (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))/x) - a*d*x*sqrt(-(d^3*sqrt(b^2*f/d^5
) - c*d - a*f)/d^3)*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*b*d*sqrt(-(d^3*s
qrt(b^2*f/d^5) - c*d - a*f)/d^3) + b^2 - (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5
))/x) + sqrt(a)*b*x*log(-(8*a*b*x + (b^2 + 4*a*c)*x^2 - 4*sqrt(c*x^2 + b*x
+ a)*(b*x + 2*a)*sqrt(a) + 8*a^2)/x^2) - 4*sqrt(c*x^2 + b*x + a)*a)/(a*d*x)
, 1/4*(a*d*x*sqrt((d^3*sqrt(b^2*f/d^5) + c*d + a*f)/d^3)*log((2*b*c*x + 2*s
qrt(c*x^2 + b*x + a)*b*d*sqrt((d^3*sqrt(b^2*f/d^5) + c*d + a*f)/d^3) + b^2
+ (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))/x) - a*d*x*sqrt((d^3*sqrt(b^2*f/d^5)
 + c*d + a*f)/d^3)*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*b*d*sqrt((d^3*sqr
t(b^2*f/d^5) + c*d + a*f)/d^3) + b^2 + (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))
/x) + a*d*x*sqrt(-(d^3*sqrt(b^2*f/d^5) - c*d - a*f)/d^3)*log((2*b*c*x + 2*s
qrt(c*x^2 + b*x + a)*b*d*sqrt(-(d^3*sqrt(b^2*f/d^5) - c*d - a*f)/d^3) + b^2
 - (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^5))/x) - a*d*x*sqrt(-(d^3*sqrt(b^2*f/d^
5) - c*d - a*f)/d^3)*log((2*b*c*x - 2*sqrt(c*x^2 + b*x + a)*b*d*sqrt(-(d^3*
sqrt(b^2*f/d^5) - c*d - a*f)/d^3) + b^2 - (b*d^2*x + 2*a*d^2)*sqrt(b^2*f/d^
5))/x) + 2*sqrt(-a)*b*x*arctan(1/2*sqrt(c*x^2 + b*x + a)*(b*x + 2*a)*sqrt(-
a)/(a*c*x^2 + a*b*x + a^2)) - 4*sqrt(c*x^2 + b*x + a)*a)/(a*d*x)]
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{\sqrt{a+bx+cx^2}}{-dx^2+fx^4}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/x**2/(-f*x**2+d),x)
```

```
[Out] -Integral(sqrt(a + b*x + c*x**2)/(-d*x**2 + f*x**4), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**

time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+b\,x+a}}{x^2\,\left(d-f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(x^2*(d - f*x^2)),x)
[Out] int((a + b*x + c*x^2)^(1/2)/(x^2*(d - f*x^2)), x)
```

## 3.83 $\int \frac{\sqrt{a+bx+cx^2}}{x^3(d-fx^2)}\,dx$

**Optimal**. Leaf size=353

$$-\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2}+\frac{(b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{3/2}d}-\frac{\sqrt{a}\,f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+c}\right.}{d^2}$$

```
[Out] 1/8*(-4*a*c+b^2)*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)
/d-f*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))*a^(1/2)/d^2-1/4*(b*
x+2*a)*(c*x^2+b*x+a)^(1/2)/a/d/x^2-1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x
*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(
1/2))*f^(1/2)*(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)/d^2+1/2*arctanh(1/2*(b*d^(1
/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d
^(1/2)*f^(1/2))^(1/2))*f^(1/2)*(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)/d^2
```

**Rubi** [A]
time = 0.56, antiderivative size = 353, normalized size of antiderivative = 1.00, number of steps used = 20, number of rules used = 10, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.357$, Rules used = {6857, 734, 738, 212, 748, 857, 635, 1035, 1092, 1047}

$$\frac{(b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{3/2}d}-\frac{\sqrt{a}\,f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d^2}-\frac{\sqrt{f}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^2}+\frac{\sqrt{f}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^2}-\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(x^3*(d - f*x^2)),x]
```

```
[Out] -1/4*((2*a + b*x)*Sqrt[a + b*x + c*x^2])/(a*d*x^2) + ((b^2 - 4*a*c)*ArcTanh
[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(8*a^(3/2)*d) - (Sqrt[a]*f
*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/d^2 - (Sqrt[f]*Sqr
t[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sq
rt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x +
 c*x^2])])/(2*d^2) + (Sqrt[f]*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*ArcTanh[(
b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt
[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^2)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 734

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(-(d + e*x)^(m + 1))*(d*b - 2*a*e + (2*c*d - b*e)*x)*((a + b
*x + c*x^2)^p/(2*(m + 1)*(c*d^2 - b*d*e + a*e^2))), x] + Dist[p*((b^2 - 4*a
*c)/(2*(m + 1)*(c*d^2 - b*d*e + a*e^2))), Int[(d + e*x)^(m + 2)*(a + b*x +
c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && EqQ[m + 2*p + 2,
 0] && GtQ[p, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
```

```
1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{x^3 (d-fx^2)} \, dx = \int \left( \frac{\sqrt{a+bx+cx^2}}{dx^3} + \frac{f\sqrt{a+bx+cx^2}}{d^2x} + \frac{f^2x\sqrt{a+bx+cx^2}}{d^2(d-fx^2)} \right) dx$$

$$= \frac{\int \frac{\sqrt{a+bx+cx^2}}{x^3} dx}{d} + \frac{f \int \frac{\sqrt{a+bx+cx^2}}{x} dx}{d^2} + \frac{f^2 \int \frac{x\sqrt{a+bx+cx^2}}{d-fx^2} dx}{d^2}$$

$$= -\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2} - \frac{(b^2-4ac)\int \frac{1}{x\sqrt{a+bx+cx^2}} dx}{8ad} - \frac{f \int \frac{-2a-bx}{x\sqrt{a+bx+c}}}{2d^2}$$

$$= -\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2} - \frac{\int \frac{-bdf-f(cd+af)x}{\sqrt{a+bx+cx^2}(d-fx^2)} dx}{d^2} + \frac{(b^2-4ac)\,\mathrm{Subst}\left(\int \frac{}{4a}\right.}{}$$

$$= -\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2} + \frac{(b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{3/2}d} - \frac{(2af)\mathrm{S}}{}$$

$$= -\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2} + \frac{(b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{3/2}d} - \frac{\sqrt{a}\,f\,\mathrm{t}}{}$$

$$= -\frac{(2a+bx)\sqrt{a+bx+cx^2}}{4adx^2} + \frac{(b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{3/2}d} - \frac{\sqrt{a}\,f\,\mathrm{t}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.65, size = 381, normalized size = 1.08

$$\frac{-\frac{d(2a+bx)\sqrt{a+x(b+cx)}}{ax^2}+\frac{(b^2d-4a(cd+2af))\tanh^{-1}\left(\frac{-\sqrt{c}x+\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{3/2}}-2f\,\mathrm{RootSum}\left[b^2d-a^2f-4b\sqrt{c}d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-acd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-a^2f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+af\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}d-2cd\#1-af\#1+f\#1^3}\&\right]}{4d^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(x^3*(d - f*x^2)),x]
```

```
[Out] (-((d*(2*a + b*x)*Sqrt[a + x*(b + c*x)])/(a*x^2)) + ((b^2*d - 4*a*(c*d + 2*
a*f))*ArcTanh[(-(Sqrt[c]*x) + Sqrt[a + x*(b + c*x)])/Sqrt[a]])/a^(3/2) - 2*
f*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1
^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*c*d*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*f*Log[-(Sqrt[c]*x) + Sqrt[a
+ b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^
2] - #1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a*f
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*
#1 - a*f*#1 + f*#1^3) & ])/(4*d^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1142 vs. $2(275) = 550$.
time = 0.15, size = 1143, normalized size = 3.24 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/x^3/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/2/a/x^2*(c*x^2+b*x+a)^(3/2)-1/4*b/a*(-1/a/x*(c*x^2+b*x+a)^(3/2)+1/2
*b/a*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))
/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))+2*c/a*(1/4*
(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1
/2)+(c*x^2+b*x+a)^(1/2))))+1/2*c/a*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x
)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2
+b*x+a)^(1/2))/x)))-1/2/d^2*f*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)
+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*
f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)
+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-
b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-
b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d
*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x
+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d
*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))+f/d^2*((c*x^2+b*x+a)^(1/2)+1
/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*
x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))-1/2/d^2*f*(((x-(d*f)^(1/2)/f)^2*c+(2*c
*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*
(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f)
)/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+
(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*
f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)
+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/
2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)
/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^3/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(sqrt(c*x^2 + b*x + a)/((f*x^2 - d)*x^3), x)
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 738 vs. $2(275) = 550$.
time = 78.19, size = 1485, normalized size = 4.21

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^3/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/16*(4*a^2*d^2*x^2*sqrt((d^4*sqrt(b^2*f^3/d^7) + c*d*f + a*f^2)/d^4)*log(
(2*sqrt(c*x^2 + b*x + a)*d^5*sqrt(b^2*f^3/d^7)*sqrt((d^4*sqrt(b^2*f^3/d^7)
+ c*d*f + a*f^2)/d^4) + 2*b*c*f^2*x + b^2*f^2 + (b*d^3*f*x + 2*a*d^3*f)*sqr
t(b^2*f^3/d^7))/x) - 4*a^2*d^2*x^2*sqrt((d^4*sqrt(b^2*f^3/d^7) + c*d*f + a*
f^2)/d^4)*log(-(2*sqrt(c*x^2 + b*x + a)*d^5*sqrt(b^2*f^3/d^7)*sqrt((d^4*sqr
t(b^2*f^3/d^7) + c*d*f + a*f^2)/d^4) - 2*b*c*f^2*x - b^2*f^2 - (b*d^3*f*x +
 2*a*d^3*f)*sqrt(b^2*f^3/d^7))/x) - 4*a^2*d^2*x^2*sqrt(-(d^4*sqrt(b^2*f^3/d
^7) - c*d*f - a*f^2)/d^4)*log((2*sqrt(c*x^2 + b*x + a)*d^5*sqrt(b^2*f^3/d^7
)*sqrt(-(d^4*sqrt(b^2*f^3/d^7) - c*d*f - a*f^2)/d^4) + 2*b*c*f^2*x + b^2*f^
2 - (b*d^3*f*x + 2*a*d^3*f)*sqrt(b^2*f^3/d^7))/x) + 4*a^2*d^2*x^2*sqrt(-(d^
4*sqrt(b^2*f^3/d^7) - c*d*f - a*f^2)/d^4)*log(-(2*sqrt(c*x^2 + b*x + a)*d^5
*sqrt(b^2*f^3/d^7)*sqrt(-(d^4*sqrt(b^2*f^3/d^7) - c*d*f - a*f^2)/d^4) - 2*b
*c*f^2*x - b^2*f^2 + (b*d^3*f*x + 2*a*d^3*f)*sqrt(b^2*f^3/d^7))/x) + (8*a^2
*f - (b^2 - 4*a*c)*d)*sqrt(a)*x^2*log(-(8*a*b*x + (b^2 + 4*a*c)*x^2 - 4*sqr
t(c*x^2 + b*x + a)*(b*x + 2*a)*sqrt(a) + 8*a^2)/x^2) - 4*(a*b*d*x + 2*a^2*d
)*sqrt(c*x^2 + b*x + a))/(a^2*d^2*x^2), 1/8*(2*a^2*d^2*x^2*sqrt((d^4*sqrt(b
^2*f^3/d^7) + c*d*f + a*f^2)/d^4)*log((2*sqrt(c*x^2 + b*x + a)*d^5*sqrt(b^2
*f^3/d^7)*sqrt((d^4*sqrt(b^2*f^3/d^7) + c*d*f + a*f^2)/d^4) + 2*b*c*f^2*x +
 b^2*f^2 + (b*d^3*f*x + 2*a*d^3*f)*sqrt(b^2*f^3/d^7))/x) - 2*a^2*d^2*x^2*sq
rt((d^4*sqrt(b^2*f^3/d^7) + c*d*f + a*f^2)/d^4)*log(-(2*sqrt(c*x^2 + b*x +
a)*d^5*sqrt(b^2*f^3/d^7)*sqrt((d^4*sqrt(b^2*f^3/d^7) + c*d*f + a*f^2)/d^4)
- 2*b*c*f^2*x - b^2*f^2 - (b*d^3*f*x + 2*a*d^3*f)*sqrt(b^2*f^3/d^7))/x) - 2
*a^2*d^2*x^2*sqrt(-(d^4*sqrt(b^2*f^3/d^7) - c*d*f - a*f^2)/d^4)*log((2*sqrt
(c*x^2 + b*x + a)*d^5*sqrt(b^2*f^3/d^7)*sqrt(-(d^4*sqrt(b^2*f^3/d^7) - c*d*
f - a*f^2)/d^4) + 2*b*c*f^2*x + b^2*f^2 - (b*d^3*f*x + 2*a*d^3*f)*sqrt(b^2*
f^3/d^7))/x) + 2*a^2*d^2*x^2*sqrt(-(d^4*sqrt(b^2*f^3/d^7) - c*d*f - a*f^2)/
d^4)*log(-(2*sqrt(c*x^2 + b*x + a)*d^5*sqrt(b^2*f^3/d^7)*sqrt(-(d^4*sqrt(b^
2*f^3/d^7) - c*d*f - a*f^2)/d^4) - 2*b*c*f^2*x - b^2*f^2 + (b*d^3*f*x + 2*a
*d^3*f)*sqrt(b^2*f^3/d^7))/x) + (8*a^2*f - (b^2 - 4*a*c)*d)*sqrt(-a)*x^2*ar
ctan(1/2*sqrt(c*x^2 + b*x + a)*(b*x + 2*a)*sqrt(-a)/(a*c*x^2 + a*b*x + a^2)
) - 2*(a*b*d*x + 2*a^2*d)*sqrt(c*x^2 + b*x + a))/(a^2*d^2*x^2)]
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{\sqrt{a + bx + cx^2}}{-dx^3 + fx^5}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/x**3/(-f*x**2+d),x)
```

```
[Out] -Integral(sqrt(a + b*x + c*x**2)/(-d*x**3 + f*x**5), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

$$\text{Timed out}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^3/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+b\,x+a}}{x^3\,\left(d-f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(x^3*(d - f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(1/2)/(x^3*(d - f*x^2)), x)
```

# 3.84 $\int \frac{x^3 (a+bx+cx^2)^{3/2}}{d-fx^2} \, dx$

**Optimal**. Leaf size=501

$$-\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f}-\frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^3}-\frac{d(a+bx+cx^2)^{3/2}}{3f^2}$$

```
[Out] -1/3*d*(c*x^2+b*x+a)^(3/2)/f^2+1/16*b*(2*c*x+b)*(c*x^2+b*x+a)^(3/2)/c^2/f-1
/5*(c*x^2+b*x+a)^(5/2)/c/f+3/256*b*(-4*a*c+b^2)^2*arctanh(1/2*(2*c*x+b)/c^(
1/2)/(c*x^2+b*x+a)^(1/2))/c^(7/2)/f-1/16*b*d*(12*a*c*f-b^2*f+24*c^2*d)*arct
anh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f^3-1/2*d*arctanh(1/
2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*
d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)/f^(7/2)+1
/2*d*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b
*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^
(3/2)/f^(7/2)-3/128*b*(-4*a*c+b^2)*(2*c*x+b)*(c*x^2+b*x+a)^(1/2)/c^3/f-1/8*
d*(2*b*c*f*x+8*a*c*f+b^2*f+8*c^2*d)*(c*x^2+b*x+a)^(1/2)/c/f^3
```

**Rubi** [A]
time = 0.91, antiderivative size = 501, normalized size of antiderivative = 1.00, number of steps used = 17, number of rules used = 10, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.357$, Rules used = {6857, 654, 626, 635, 212, 1035, 1084, 1092, 1047, 738}

$$\frac{3b(b^2-4ac)^2\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{256c^{7/2}f}-\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f}-\frac{d\sqrt{a+bx+cx^2}(8acf+b^2f+2bcfx+8c^2d)}{8cf^3}-\frac{bd(12acf+b^2(-f)+24c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}f^3}+\frac{b(b+2cx)(a+bx+cx^2)^{3/2}}{16c^2f}-\frac{d\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^{7/2}}+\frac{d\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^{7/2}}-\frac{d(a+bx+cx^2)^{3/2}}{3f^2}-\frac{(a+bx+cx^2)^{5/2}}{5cf}$$

Antiderivative was successfully verified.

```
[In] Int[(x^3*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] (-3*b*(b^2 - 4*a*c)*(b + 2*c*x)*Sqrt[a + b*x + c*x^2])/(128*c^3*f) - (d*(8*
c^2*d + b^2*f + 8*a*c*f + 2*b*c*f*x)*Sqrt[a + b*x + c*x^2])/(8*c*f^3) - (d*
(a + b*x + c*x^2)^(3/2))/(3*f^2) + (b*(b + 2*c*x)*(a + b*x + c*x^2)^(3/2))/
(16*c^2*f) - (a + b*x + c*x^2)^(5/2)/(5*c*f) + (3*b*(b^2 - 4*a*c)^2*ArcTanh
[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(256*c^(7/2)*f) - (b*d*(24
*c^2*d - b^2*f + 12*a*c*f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*
x^2])])/(16*c^(3/2)*f^3) - (d*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh
[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sq
rt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^(7/2)) + (d*(c*d + b*Sqr
t[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] +
 b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2]
)])/(2*f^(7/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
```

```
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1084

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1
)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*
(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*
(C*((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*
p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*((-
b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*((-b)*c*(C*(-4*d*
f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*
((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2
- 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3
))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 -
4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !I
GtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{x^3(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx = \int \left(-\frac{x(a+bx+cx^2)^{3/2}}{f} + \frac{dx(a+bx+cx^2)^{3/2}}{f(d-fx^2)}\right) dx$$

$$= -\frac{\int x(a+bx+cx^2)^{3/2}\,dx}{f} + \frac{d\int \frac{x(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx}{f}$$

$$= -\frac{d(a+bx+cx^2)^{3/2}}{3f^2} - \frac{(a+bx+cx^2)^{5/2}}{5cf} + \frac{d\int \frac{\sqrt{a+bx+cx^2}\left(\frac{3bd}{2}+3(cd+af)x+\right.}{d-fx^2}}{3f^2}$$

$$= -\frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^3} - \frac{d(a+bx+cx^2)^{3/2}}{3f^2} + \frac{b(b+}{}$$

$$= -\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f} - \frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a-}}{8cf^3}$$

$$= -\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f} - \frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a-}}{8cf^3}$$

$$= -\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f} - \frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a-}}{8cf^3}$$

$$= -\frac{3b(b^2-4ac)(b+2cx)\sqrt{a+bx+cx^2}}{128c^3f} - \frac{d(8c^2d+b^2f+8acf+2bcfx)\sqrt{a-}}{8cf^3}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 2.27, size = 734, normalized size = 1.47

Antiderivative was successfully verified.

```
[In] Integrate[(x^3*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*Sqrt[a + x*(b + c*x)]*(45*b^4*f^2 - 30*b^2*c*f^2*(10*a + b*x) +
 16*c^3*f*(160*a*d + 70*b*d*x + 48*a*f*x^2 + 33*b*f*x^3) + 128*c^4*(15*d^2
+ 5*d*f*x^2 + 3*f^2*x^4) + 24*c^2*f*(16*a^2*f + 7*a*b*f*x + b^2*(10*d + f*x
^2))) - 15*b*(-384*c^4*d^2 - 192*a*c^3*d*f + 3*b^4*f^2 - 24*a*b^2*c*f^2 + 1
6*c^2*f*(b^2*d + 3*a^2*f))*Log[b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)]]
```

```
 - 1920*c^(7/2)*d*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2
*a*f*#1^2 - f*#1^4 & , (2*b^2*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1] - a*c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*b^2*
d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a^2*c*d*f*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^3*f^2*Log[-(Sqrt[c]*x) + Sqrt[a +
b*x + c*x^2] - #1] - 4*b*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^
2] - #1]*#1 - 4*a*b*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
#1]*#1 + c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + b^2*
d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*c*d*f*Log[-(S
qrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a^2*f^2*Log[-(Sqrt[c]*x) + S
qrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3)
 & ])/(3840*c^(7/2)*f^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1606 vs. $2(405) = 810$.
time = 0.17, size = 1607, normalized size = 3.21

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1607 |
| risch | Expression too large to display | 2577 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/5*(c*x^2+b*x+a)^(5/2)/c-1/2*b/c*(1/8*(2*c*x+b)/c*(c*x^2+b*x+a)^(3/2
)+3/16*(4*a*c-b^2)/c*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c
^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))))-1/2*d/f^2*(1/3*((x+(d
*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)
^(1/2)+f*a+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*(2*c*(x+(d*f)^(1/2
)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(
1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/8*(4*c/f*
(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/c^(3/2)*ln((1/2/f*
(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+
1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^
(1/2)))+1/f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f
)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-
2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/
c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/
(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(
-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1
/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f
*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))-1/2*d/f^2*(1/3*((x-(
d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+
f*a+c*d)/f)^(3/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(
```

```
2*c*(d*f)^(1/2)+b*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x
-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*
a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)
/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f
)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a
+c*d)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(
b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d
*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(
d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/
2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*
f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(
1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d
*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3*(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] Timed out
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3\,{\left(c\,x^2+b\,x+a\right)}^{3/2}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^3*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x)
```

```
[Out] int((x^3*(a + b*x + c*x^2)^(3/2))/(d - f*x^2), x)
```

## 3.85 $\int \frac{x^2 (a+bx+cx^2)^{3/2}}{d-fx^2} \, dx$

**Optimal**. Leaf size=417

$$-\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2}-\frac{(b+2cx)\,(a+bx+cx^2)^{3/2}}{8cf}-$$

```
[Out] -1/8*(2*c*x+b)*(c*x^2+b*x+a)^(3/2)/c/f-1/128*(128*c^4*d^2+192*a*c^3*d*f+3*b
^4*f^2-24*a*b^2*c*f^2+48*c^2*f*(a^2*f+b^2*d))*arctanh(1/2*(2*c*x+b)/c^(1/2)
/(c*x^2+b*x+a)^(1/2))/c^(5/2)/f^3+1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*
(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1
/2))*d^(1/2)*(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)/f^3+1/2*arctanh(1/2*(b*d^(1/
2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^
(1/2)*f^(1/2))^(1/2))*d^(1/2)*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/f^3-1/64*(b
*(12*a*c*f-3*b^2*f+80*c^2*d)+2*c*(12*a*c*f-3*b^2*f+16*c^2*d)*x)*(c*x^2+b*x+
a)^(1/2)/c^2/f^2
```

**Rubi** [A]

time = 0.63, antiderivative size = 417, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {1085, 1084, 1092, 635, 212, 1047, 738}

$$-\frac{(48c^2f(a^2f+b^2d)-24ab^2cf^2+192ac^3df+3b^4f^2+128c^4d^2)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{128c^{5/2}f^3}-\frac{\sqrt{a+bx+cx^2}\,(2cx(12acf-3b^2f+16c^2d)+b(12acf-3b^2f+80c^2d))}{64c^2f^2}+\frac{\sqrt{d}\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^3}+\frac{\sqrt{d}\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^3}-\frac{(b+2cx)\,(a+bx+cx^2)^{3/2}}{8cf}$$

Antiderivative was successfully verified.

```
[In] Int[(x^2*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] -1/64*((b*(80*c^2*d - 3*b^2*f + 12*a*c*f) + 2*c*(16*c^2*d - 3*b^2*f + 12*a*
c*f)*x)*Sqrt[a + b*x + c*x^2])/(c^2*f^2) - ((b + 2*c*x)*(a + b*x + c*x^2)^(
3/2))/(8*c*f) - ((128*c^4*d^2 + 192*a*c^3*d*f + 3*b^4*f^2 - 24*a*b^2*c*f^2
+ 48*c^2*f*(b^2*d + a^2*f))*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c
*x^2])])/(128*c^(5/2)*f^3) + (Sqrt[d]*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)
*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*
d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^3) + (Sqrt[d]*(c
*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c
*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*
x + c*x^2])])/(2*f^3)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1084

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1
)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*
(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*
(C*((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*
p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*((-
b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*((-b)*c*(C*(-4*d*
f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*
((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2
- 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3
))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 -
4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !I
GtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1085

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(
a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3
))), x] - Dist[1/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3)), Int[(a + b*x + c*x^
```

```
2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*(C*((-b)*f)*(q + 1)) + (p + q + 1)*(b
^2*C*d*f*p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*
f)*(C*((-b)*f)*(q + 1)) + (p + q + 1)*((-b)*c*(C*(-4*d*f)*(2*p + q + 2) + f
*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*((-b)*f)*(q + 1)) +
(p + q + 1)*(C*f^2*p*(b^2 - 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C
*d + 2*A*f)*(2*p + 2*q + 3))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A,
C, q}, x] && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2
*p + 2*q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{x^2(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx = -\frac{(b+2cx)\,(a+bx+cx^2)^{3/2}}{8cf} - \frac{\int \frac{\sqrt{a+bx+cx^2}\,\left(-\frac{3}{4}(3b^2+4ac)df-12bcdfx-\frac{3}{4}f(16c^2d}{d-fx^2}}{12cf^2}$$

$$= -\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2} -$$

$$= -\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2} -$$

$$= -\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2} -$$

$$= -\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2} -$$

$$= -\frac{(b(80c^2d-3b^2f+12acf)+2c(16c^2d-3b^2f+12acf)\,x)\sqrt{a+bx+cx^2}}{64c^2f^2} -$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 1.88, size = 608, normalized size = 1.46

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[(x^2*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*f*Sqrt[a + x*(b + c*x)]*(-3*b^3*f + 2*b^2*c*f*x + 8*c^2*x*(4*c*
d + 5*a*f + 2*c*f*x^2) + 4*b*c*(20*c*d + 5*a*f + 6*c*f*x^2)) + (128*c^4*d^2
 + 192*a*c^3*d*f + 3*b^4*f^2 - 24*a*b^2*c*f^2 + 48*c^2*f*(b^2*d + a^2*f))*L
og[b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)]] - 64*c^(5/2)*d*RootSum[b^2*
d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c^2*
d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^3*d*f*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*b*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1] - 2*c^(5/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1]*#1 - 2*b^2*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*
#1 - 4*a*c^(3/2)*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*
a^2*Sqrt[c]*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*b*c*d
*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*b*f^2*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*
#1 + f*#1^3) & ])/(128*c^(5/2)*f^3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1592 vs. $2(343) = 686$.
time = 0.15, size = 1593, normalized size = 3.82

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1593 |
| risch | Expression too large to display | 2506 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/8*(2*c*x+b)/c*(c*x^2+b*x+a)^(3/2)+3/16*(4*a*c-b^2)/c*(1/4*(2*c*x+b)
/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^
2+b*x+a)^(1/2))))+1/2*d/(d*f)^(1/2)/f*(1/3*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c
*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(3/2)+1/2
/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)
+b*f))/c*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f
)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/8*(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1
/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/c^(3/2)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(
x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)))+1/f*(-b*(d*f)^(1/2
)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2
)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1
```

```
/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)
^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c
*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c
*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(
d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2))/(x+(d*f)^(1/2)/f))))-1/2*d/(d*f)^(1/2)/f*(1/3*((x-(d*f)^(1/2)/f)^2*
c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(3/2
)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+
b*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+
(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d
*f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1
/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/
2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a+c*d)/f*(((x-(d*
f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*
a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f
+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/
f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2
)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d
)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)
^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*
(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{ax^2\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx-\int \frac{bx^3\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx-\int \frac{cx^4\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(a*x**2*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(b*x**3
*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(c*x**4*sqrt(a + b*x +
c*x**2)/(-d + f*x**2), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2\,{\left(c\,x^2+b\,x+a\right)}^{3/2}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x)
```

```
[Out] int((x^2*(a + b*x + c*x^2)^(3/2))/(d - f*x^2), x)
```

# 3.86 $\int \frac{x(a+bx+cx^2)^{3/2}}{d-fx^2} dx$

**Optimal**. Leaf size=349

$$-\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2}-\frac{(a+bx+cx^2)^{3/2}}{3f}-\frac{b(24c^2d-b^2f+12acf)\tanh^{-1}\left(\frac{\phantom{x}}{2\sqrt{c}\ldots}\right.}{16c^{3/2}f^2}$$

```
[Out] -1/3*(c*x^2+b*x+a)^(3/2)/f-1/16*b*(12*a*c*f-b^2*f+24*c^2*d)*arctanh(1/2*(2*
c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f^2-1/2*arctanh(1/2*(b*d^(1/2)-
2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/
2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)/f^(5/2)+1/2*arctanh(1/
2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*
d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/f^(5/2)-1
/8*(2*b*c*f*x+8*a*c*f+b^2*f+8*c^2*d)*(c*x^2+b*x+a)^(1/2)/c/f^2
```

**Rubi** [A]
time = 0.32, antiderivative size = 349, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.269$, Rules used = {1035, 1084, 1092, 635, 212, 1047, 738}

$$-\frac{\sqrt{a+bx+cx^2}\,(8acf+b^2f+2bcfx+8c^2d)}{8cf^2}-\frac{b(12acf+b^2(-f)+24c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}f^2}-\frac{\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^{5/2}}+\frac{\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^{5/2}}-\frac{(a+bx+cx^2)^{3/2}}{3f}$$

Antiderivative was successfully verified.

```
[In] Int[(x*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] -1/8*((8*c^2*d + b^2*f + 8*a*c*f + 2*b*c*f*x)*Sqrt[a + b*x + c*x^2])/(c*f^2
) - (a + b*x + c*x^2)^(3/2)/(3*f) - (b*(24*c^2*d - b^2*f + 12*a*c*f)*ArcTan
h[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(16*c^(3/2)*f^2) - ((c*d
- b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sq
rt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x +
 c*x^2])])/(2*f^(5/2)) + ((c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*
Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d
]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^(5/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1084

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1
)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*
(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*
(C*((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*
p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*((-
b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*((-b)*c*(C*(-4*d*
f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*
((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2
- 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3
))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 -
4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !I
GtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{x(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx = -\frac{(a+bx+cx^2)^{3/2}}{3f} + \frac{\int \frac{\sqrt{a+bx+cx^2}\left(\frac{3bd}{2}+3(cd+af)x+\frac{3}{2}bfx^2\right)}{d-fx^2}\,dx}{3f}$$

$$= -\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2} - \frac{(a+bx+cx^2)^{3/2}}{3f} - \frac{\int \frac{-\frac{3}{8}bdf(}{}}{}$$

$$= -\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2} - \frac{(a+bx+cx^2)^{3/2}}{3f} + \frac{\int \frac{\frac{3}{8}bdf^2(2}{}}{}$$

$$= -\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2} - \frac{(a+bx+cx^2)^{3/2}}{3f} + \frac{(cd-b}{}$$

$$= -\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2} - \frac{(a+bx+cx^2)^{3/2}}{3f} - \frac{b(24c^2d}{}$$

$$= -\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cf^2} - \frac{(a+bx+cx^2)^{3/2}}{3f} - \frac{b(24c^2d}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.02, size = 619, normalized size = 1.77

$$\frac{-2\sqrt{c}\sqrt{a+x(b+cx)}\left(3b^2f+2cf(16a+7bx)+8c^2(3d+fx^2)\right)+\left(-3b^3f+36bc(2cd+af)\right)\log\left(c\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)-24c^{3/2}\mathrm{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\ \ldots\right]}{48c^{3/2}f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*Sqrt[a + x*(b + c*x)]*(3*b^2*f + 2*c*f*(16*a + 7*b*x) + 8*c^2*(
3*d + f*x^2)) + (-3*b^3*f + 36*b*c*(2*c*d + a*f))*Log[c*(b + 2*c*x - 2*Sqrt
[c]*Sqrt[a + x*(b + c*x)])] - 24*c^(3/2)*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c
```

```
]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (2*b^2*c*d^2*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1] - a*c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
+ c*x^2] - #1] + a*b^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] -
 2*a^2*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^3*f^2*Log[-
(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 4*b*c^(3/2)*d^2*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 4*a*b*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + S
qrt[a + b*x + c*x^2] - #1]*#1 + c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1]*#1^2 + b^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#
1^2 + 2*a*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a^2*f
^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*
d*#1 - a*f*#1 + f*#1^3) & ])/(48*c^(3/2)*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1472 vs. $2(275) = 550$.
time = 0.15, size = 1473, normalized size = 4.22

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1473 |
| risch | Expression too large to display | 2337 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/f*(1/3*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2
)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*
(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)+1/8*(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/
c^(3/2)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(
d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f
)^(1/2)+f*a+c*d))^(1/2)))+1/f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^
2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*
d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(
x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*
f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1
/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^
(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+
(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))-1/
2/f*(1/3*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(
b*(d*f)^(1/2)+f*a+c*d)/f)^(3/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d
*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(
1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*
(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d
*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(
```

```
d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(
d*f)^(1/2)+f*a+c*d)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d
*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*l
n((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)
/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f
)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2
)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)
+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2
)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)
/f))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{ax\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx - \int \frac{bx^2\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx - \int \frac{cx^3\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(a*x*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(b*x**2*sq
rt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(c*x**3*sqrt(a + b*x + c*x
**2)/(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x\,(c\,x^2+b\,x+a)^{3/2}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(a + b*x + c*x^2)^(3/2))/(d - f*x^2),x)
```

```
[Out] int((x*(a + b*x + c*x^2)^(3/2))/(d - f*x^2), x)
```

## 3.87 $\int \frac{(a+bx+cx^2)^{3/2}}{d-fx^2} dx$

**Optimal**. Leaf size=315

$$-\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f}-\frac{(8c^2d+3b^2f+12acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8\sqrt{c}f^2}+\frac{\left(cd-b\sqrt{d}\sqrt{f}+\right.}{}$$

```
[Out] -1/8*(12*a*c*f+3*b^2*f+8*c^2*d)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)
^(1/2))/f^2/c^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b
*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-
b*d^(1/2)*f^(1/2))^(3/2)/f^2/d^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)
+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))
^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/f^2/d^(1/2)-1/4*(2*c*x+5*b)*(c*x^
2+b*x+a)^(1/2)/f
```

**Rubi** [A]
time = 0.32, antiderivative size = 315, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.240$, Rules used = {992, 1092, 635, 212, 1047, 738}

$$-\frac{(12acf+3b^2f+8c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8\sqrt{c}f^2}+\frac{\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}f^2}+\frac{\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{d}f^2}-\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*x + c*x^2)^(3/2)/(d - f*x^2),x]
```

```
[Out] -1/4*((5*b + 2*c*x)*Sqrt[a + b*x + c*x^2])/f - ((8*c^2*d + 3*b^2*f + 12*a*c
*f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(8*Sqrt[c]*f^2)
 + ((c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f]
+ (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[
a + b*x + c*x^2])])/(2*Sqrt[d]*f^2) + ((c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2
)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c
*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*f^2)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 992

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b*(3*p + 2*q) + 2*c*(p + q)*x)*(a + b*x + c*x^2)^(p - 1)*
((d + f*x^2)^(q + 1)/(2*f*(p + q)*(2*p + 2*q + 1))), x] - Dist[1/(2*f*(p +
q)*(2*p + 2*q + 1)), Int[(a + b*x + c*x^2)^(p - 2)*(d + f*x^2)^q*Simp[b^2*d
*(p - 1)*(2*p + q) - (p + q)*(b^2*d*(1 - p) - 2*a*(c*d - a*f*(2*p + 2*q + 1
))) - (2*b*(c*d - a*f)*(1 - p)*(2*p + q) - 2*(p + q)*b*(2*c*d*(2*p + q) - (
c*d + a*f)*(2*p + 2*q + 1)))*x + (b^2*f*p*(1 - p) + 2*c*(p + q)*(c*d*(2*p -
 1) - a*f*(4*p + 2*q - 1)))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x]
 && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 1] && NeQ[p + q, 0] && NeQ[2*p + 2*q + 1,
 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx = -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f} + \frac{\int \frac{\frac{1}{4}(5b^2d+4a(cd+2af))+4b(cd+af)x+\frac{1}{4}(8c^2d+3b^2f+12acf)x^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)} \cdots}{2f}$$

$$= -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f} - \frac{\int \frac{-\frac{1}{4}d(8c^2d+3b^2f+12acf)-\frac{1}{4}f(5b^2d+4a(cd+2af))-4bf(cd+af \cdots}{\sqrt{a+bx+cx^2}\,(d-fx^2)}}{2f^2}$$

$$= -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f} - \frac{\left(cd-b\sqrt{d}\sqrt{f}+af\right)^2 \int \frac{1}{\left(-\sqrt{d}\sqrt{f}-fx\right)\sqrt{a+\cdots}}}{2\sqrt{d}\,f^{3/2}}$$

$$= -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f} - \frac{(8c^2d+3b^2f+12acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+\cdots}}\right.}{8\sqrt{c}\,f^2}$$

$$= -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4f} - \frac{(8c^2d+3b^2f+12acf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+\cdots}}\right.}{8\sqrt{c}\,f^2}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.75, size = 524, normalized size = 1.66

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*x + c*x^2)^(3/2)/(d - f*x^2),x]
```

```
[Out] (-2*Sqrt[c]*f*(5*b + 2*c*x)*Sqrt[a + x*(b + c*x)] + (8*c^2*d + 3*b^2*f + 12
*a*c*f)*Log[b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)]] - 4*Sqrt[c]*RootSu
m[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (
b*c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^3*d*f*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*b*f^2*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] - 2*c^(5/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1]*#1 - 2*b^2*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1]*#1 - 4*a*c^(3/2)*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#
1 - 2*a^2*Sqrt[c]*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2
*b*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*b*f^2*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1
- a*f*#1 + f*#1^3) & ])/(8*Sqrt[c]*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1476 vs. $2(245) = 490$.

time = 0.15, size = 1477, normalized size = 4.69

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1477 |
| risch | Expression too large to display | 2314 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2/(d*f)^(1/2)*(1/3*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(
d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b
*f)*(1/4*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/
2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2)+1/8*(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)
+b*f)^2)/c^(3/2)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1
/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f
*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)))+1/f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^
(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/
2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)
+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(
1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/
f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b
*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(
-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)
+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)
/f))))-1/2/(d*f)^(1/2)*(1/3*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*
(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(3/2)+1/2*(2*c*(d*f)^(1/2)+b*f
)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/c*((x-(d*f)^(1/2)/
f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)
^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/c^(3
/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^
(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c
*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a+c*d)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)
^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(
d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1
/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*
f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/
2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/
f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^
2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1
/2))/(x-(d*f)^(1/2)/f))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{a\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx-\int \frac{bx\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx-\int \frac{cx^2\sqrt{a+bx+cx^2}}{-d+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(a*sqrt(a + b*x + c*x**2)/(-d + f*x**2), x) - Integral(b*x*sqrt(a
+ b*x + c*x**2)/(-d + f*x**2), x) - Integral(c*x**2*sqrt(a + b*x + c*x**2)/
(-d + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(c\,x^2+b\,x+a\right)^{3/2}}{d-f\,x^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(3/2)/(d - f*x^2),x)
[Out] int((a + b*x + c*x^2)^(3/2)/(d - f*x^2), x)
```

# 3.88 $\int \frac{(a+bx+cx^2)^{3/2}}{x(d-fx^2)}\,dx$

**Optimal**. Leaf size=469

$$\frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd}-\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}-\frac{a^{3/2}\tanh^{-1}\left(\frac{2a}{2\sqrt{a}\sqrt{a+}\right.}{d}$$

```
[Out] -a^(3/2)*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d-1/16*b*(-12*a
*c+b^2)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/d-1/16*b
*(12*a*c*f-b^2*f+24*c^2*d)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2
))/c^(3/2)/d/f-1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1
/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(
1/2)*f^(1/2))^(3/2)/d/f^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c
*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))
*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/d/f^(3/2)+1/8*(2*b*c*x+8*a*c+b^2)*(c*x^2
+b*x+a)^(1/2)/c/d-1/8*(2*b*c*f*x+8*a*c*f+b^2*f+8*c^2*d)*(c*x^2+b*x+a)^(1/2)
/c/d/f
```

**Rubi** [A]
time = 0.84, antiderivative size = 469, normalized size of antiderivative = 1.00, number of steps used = 19, number of rules used = 11, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.393$, Rules used = {6857, 748, 828, 857, 635, 212, 738, 1035, 1084, 1092, 1047}

$$-\frac{a^{3/2}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}-\frac{b(b^2-12ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}d}-\frac{\sqrt{a+bx+cx^2}\,(8acf+b^2f+2bcfx+8c^2d)}{8cdf}-\frac{b(12acf+b^2(-f)+24c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}df}+\frac{(8ac+b^2+2bcx)\sqrt{a+bx+cx^2}}{8cd}-\frac{\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2df^{3/2}}+\frac{\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2df^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*x + c*x^2)^(3/2)/(x*(d - f*x^2)),x]
```

```
[Out] ((b^2 + 8*a*c + 2*b*c*x)*Sqrt[a + b*x + c*x^2])/(8*c*d) - ((8*c^2*d + b^2*f
 + 8*a*c*f + 2*b*c*f*x)*Sqrt[a + b*x + c*x^2])/(8*c*d*f) - (a^(3/2)*ArcTanh
[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/d - (b*(b^2 - 12*a*c)*ArcT
anh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(16*c^(3/2)*d) - (b*(24
*c^2*d - b^2*f + 12*a*c*f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*
x^2])])/(16*c^(3/2)*d*f) - ((c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(
b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt
[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d*f^(3/2)) + ((c*d + b*Sqrt[
d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b
*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]
)/(2*d*f^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 828

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(c*e*f*(m + 2*p + 2)
 - g*(c*d + 2*c*d*p - b*e*p) + g*c*e*(m + 2*p + 1)*x)*((a + b*x + c*x^2)^p/
(c*e^2*(m + 2*p + 1)*(m + 2*p + 2))), x] - Dist[p/(c*e^2*(m + 2*p + 1)*(m +
 2*p + 2)), Int[(d + e*x)^m*(a + b*x + c*x^2)^(p - 1)*Simp[c*e*f*(b*d - 2*a
*e)*(m + 2*p + 2) + g*(a*e*(b*e - 2*c*d*m + b*e*m) + b*d*(b*e*p - c*d - 2*c
*d*p)) + (c*e*f*(2*c*d - b*e)*(m + 2*p + 2) + g*(b^2*e^2*(p + m + 1) - 2*c^
2*d^2*(1 + 2*p) - c*e*(b*d*(m - 2*p) + 2*a*e*(m + 2*p + 1))))*x, x], x], x]
 /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2
- b*d*e + a*e^2, 0] && GtQ[p, 0] && (IntegerQ[p] ||  !RationalQ[m] || (GeQ[
m, -1] && LtQ[m, 0])) &&  !ILtQ[m + 2*p, 0] && (IntegerQ[m] || IntegerQ[p]
|| IntegersQ[2*m, 2*p])
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
```

```
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1084

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1
)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*
(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*
(C*((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*
p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*((-
b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*((-b)*c*(C*(-4*d*
f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*
((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2
- 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3
))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 -
4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !I
GtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{(a+bx+cx^2)^{3/2}}{x(d-fx^2)}\,dx = \int \left(\frac{(a+bx+cx^2)^{3/2}}{dx} - \frac{fx(a+bx+cx^2)^{3/2}}{d(-d+fx^2)}\right)dx$$

$$= \frac{\int \frac{(a+bx+cx^2)^{3/2}}{x}\,dx}{d} - \frac{f\int \frac{x(a+bx+cx^2)^{3/2}}{-d+fx^2}\,dx}{d}$$

$$= \frac{\int \frac{\sqrt{a+bx+cx^2}\left(-\frac{3bd}{2}-3(cd+af)x-\frac{3}{2}bfx^2\right)}{-d+fx^2}\,dx}{3d} - \frac{\int \frac{(-2a-bx)\sqrt{a+bx+cx^2}}{x}\,dx}{2d}$$

$$= \frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd} - \frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}$$

$$= \frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd} - \frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}$$

$$= \frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd} - \frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}$$

$$= \frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd} - \frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}$$

$$= \frac{(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd} - \frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx+cx^2}}{8cdf}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.80, size = 601, normalized size = 1.28

$$-\frac{2cd\sqrt{a+x(b+cx)} - 4a^{3/2}f\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right) - 3b\sqrt{c}\,d\log\left(f\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right) + \text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{2b^2cd^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-ac^2d^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+ab^2df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2a^2cdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-a^3f^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-4bc^{3/2}d^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-4ab\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+c^2d^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+b^2df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+2acdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+a^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2df}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*x + c*x^2)^(3/2)/(x*(d - f*x^2)),x]
```

```
[Out] -1/2*(2*c*d*Sqrt[a + x*(b + c*x)] - 4*a^(3/2)*f*ArcTanh[(Sqrt[c]*x - Sqrt[a
 + x*(b + c*x)])/Sqrt[a]] - 3*b*Sqrt[c]*d*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt
[a + x*(b + c*x)])] + RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2
 + 2*a*f*#1^2 - f*#1^4 & , (2*b^2*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1] - a*c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*
b^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a^2*c*d*f*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^3*f^2*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] - 4*b*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1]*#1 - 4*a*b*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1]*#1 + c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 +
b^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*c*d*f*Log
[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a^2*f^2*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#
1^3) & ])/(d*f)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1638 vs. $2(377) = 754$.
time = 0.13, size = 1639, normalized size = 3.49

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1639 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(3/2)/x/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/d*(1/3*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2
)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*
(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/2)/f)^2*c
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))
^(1/2)+1/8*(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/
c^(3/2)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(
d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f
)^(1/2)+f*a+c*d))^(1/2)))+1/f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^
2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*
d))^(1/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(
x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*
f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1
/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^
(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+
(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))+1/
d*(1/3*(c*x^2+b*x+a)^(3/2)+1/2*b*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(
4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+a*((c*x^2+b
*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/
2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x)))-1/2/d*(1/3*((x-(d*f)^(1/
```

```
2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)
/f)^(3/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f
)^(1/2)+b*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(
1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f
-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-
(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(
d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a+c*d)/f*
(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^
(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2
)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/
2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d
*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)
+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a
+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2
)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate((c*x^2 + b*x + a)^(3/2)/((f*x^2 - d)*x), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{a\sqrt{a+bx+cx^2}}{-dx+fx^3}\,dx-\int \frac{bx\sqrt{a+bx+cx^2}}{-dx+fx^3}\,dx-\int \frac{cx^2\sqrt{a+bx+cx^2}}{-dx+fx^3}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(3/2)/x/(-f*x**2+d),x)
```

```
[Out] -Integral(a*sqrt(a + b*x + c*x**2)/(-d*x + f*x**3), x) - Integral(b*x*sqrt(
a + b*x + c*x**2)/(-d*x + f*x**3), x) - Integral(c*x**2*sqrt(a + b*x + c*x*
*2)/(-d*x + f*x**3), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:index.cc index_m operator + Error: Ba
d Argument Value
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(c\,x^2+b\,x+a\right)^{3/2}}{x\,\left(d-f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(3/2)/(x*(d - f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(3/2)/(x*(d - f*x^2)), x)
```

# 3.89 $\int \frac{\left(a+bx+cx^2\right)^{3/2}}{x^2\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=463

$$\frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d}-\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d}-\frac{\left(a+bx+cx^2\right)^{3/2}}{dx}-\frac{3\sqrt{a}\,b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+}\right.}{2d}$$

```
[Out] -(c*x^2+b*x+a)^(3/2)/d/x-3/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^
(1/2))*a^(1/2)/d+3/8*(4*a*c+b^2)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a
)^(1/2))/d/c^(1/2)-1/8*(12*a*c*f+3*b^2*f+8*c^2*d)*arctanh(1/2*(2*c*x+b)/c^(
1/2)/(c*x^2+b*x+a)^(1/2))/d/f/c^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2
)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2)
)^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)/d^(3/2)/f+1/2*arctanh(1/2*(b*d^(
1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*
d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/d^(3/2)/f+3/4*(2*
c*x+3*b)*(c*x^2+b*x+a)^(1/2)/d-1/4*(2*c*x+5*b)*(c*x^2+b*x+a)^(1/2)/d
```

**Rubi** [A]
time = 0.74, antiderivative size = 463, normalized size of antiderivative = 1.00, number of steps used = 18, number of rules used = 10, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.357$, Rules used = {6857, 746, 828, 857, 635, 212, 738, 992, 1092, 1047}

$$-\frac{\left(12acf+3b^2f+8c^2d\right)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8\sqrt{c}\,df}+\frac{3\left(4ac+b^2\right)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8\sqrt{c}\,d}+\frac{\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^{3/2}f}+\frac{\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^{3/2}f}-\frac{\left(a+bx+cx^2\right)^{3/2}}{dx}+\frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d}-\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d}-\frac{3\sqrt{a}\,b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*x + c*x^2)^(3/2)/(x^2*(d - f*x^2)),x]
```

```
[Out] (3*(3*b + 2*c*x)*Sqrt[a + b*x + c*x^2])/(4*d) - ((5*b + 2*c*x)*Sqrt[a + b*x
 + c*x^2])/(4*d) - (a + b*x + c*x^2)^(3/2)/(d*x) - (3*Sqrt[a]*b*ArcTanh[(2*
a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(2*d) + (3*(b^2 + 4*a*c)*ArcTa
nh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(8*Sqrt[c]*d) - ((8*c^2*
d + 3*b^2*f + 12*a*c*f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2
])])/(8*Sqrt[c]*d*f) + ((c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sq
rt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*
Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^(3/2)*f) + ((c*d + b*Sqrt[d]*S
qrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqr
t[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2
*d^(3/2)*f)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 746

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 1))), x] - Di
st[p/(e*(m + 1)), Int[(d + e*x)^(m + 1)*(b + 2*c*x)*(a + b*x + c*x^2)^(p -
1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*
d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && GtQ[p, 0] && (IntegerQ[p]
 || LtQ[m, -1]) && NeQ[m, -1] &&  !ILtQ[m + 2*p + 1, 0] && IntQuadraticQ[a,
 b, c, d, e, m, p, x]
```

Rule 828

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(c*e*f*(m + 2*p + 2)
 - g*(c*d + 2*c*d*p - b*e*p) + g*c*e*(m + 2*p + 1)*x)*((a + b*x + c*x^2)^p/
(c*e^2*(m + 2*p + 1)*(m + 2*p + 2))), x] - Dist[p/(c*e^2*(m + 2*p + 1)*(m +
 2*p + 2)), Int[(d + e*x)^m*(a + b*x + c*x^2)^(p - 1)*Simp[c*e*f*(b*d - 2*a
*e)*(m + 2*p + 2) + g*(a*e*(b*e - 2*c*d*m + b*e*m) + b*d*(b*e*p - c*d - 2*c
*d*p)) + (c*e*f*(2*c*d - b*e)*(m + 2*p + 2) + g*(b^2*e^2*(p + m + 1) - 2*c^
2*d^2*(1 + 2*p) - c*e*(b*d*(m - 2*p) + 2*a*e*(m + 2*p + 1))))*x, x], x], x]
 /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2
- b*d*e + a*e^2, 0] && GtQ[p, 0] && (IntegerQ[p] ||  !RationalQ[m] || (GeQ[
m, -1] && LtQ[m, 0])) &&  !ILtQ[m + 2*p, 0] && (IntegerQ[m] || IntegerQ[p]
|| IntegersQ[2*m, 2*p])
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
```

```
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 992

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b*(3*p + 2*q) + 2*c*(p + q)*x)*(a + b*x + c*x^2)^(p - 1)*
((d + f*x^2)^(q + 1)/(2*f*(p + q)*(2*p + 2*q + 1))), x] - Dist[1/(2*f*(p +
q)*(2*p + 2*q + 1)), Int[(a + b*x + c*x^2)^(p - 2)*(d + f*x^2)^q*Simp[b^2*d
*(p - 1)*(2*p + q) - (p + q)*(b^2*d*(1 - p) - 2*a*(c*d - a*f*(2*p + 2*q + 1
))) - (2*b*(c*d - a*f)*(1 - p)*(2*p + q) - 2*(p + q)*b*(2*c*d*(2*p + q) - (
c*d + a*f)*(2*p + 2*q + 1)))*x + (b^2*f*p*(1 - p) + 2*c*(p + q)*(c*d*(2*p -
 1) - a*f*(4*p + 2*q - 1)))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x]
 && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 1] && NeQ[p + q, 0] && NeQ[2*p + 2*q + 1,
 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{(a+bx+cx^2)^{3/2}}{x^2(d-fx^2)}\,dx = \int \left(\frac{(a+bx+cx^2)^{3/2}}{dx^2} + \frac{f(a+bx+cx^2)^{3/2}}{d(d-fx^2)}\right)dx$$

$$= \frac{\int \frac{(a+bx+cx^2)^{3/2}}{x^2}\,dx}{d} + \frac{f\int \frac{(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx}{d}$$

$$= -\frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(a+bx+cx^2)^{3/2}}{dx} + \frac{\int \frac{\frac{1}{4}(5b^2d+4a(cd+2af))+4b(cd+af)x}{\sqrt{a+bx+cx^2}}}{2d}$$

$$= \frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(a+bx+cx^2)^{3/2}}{dx} -$$

$$= \frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(a+bx+cx^2)^{3/2}}{dx} +$$

$$= \frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(a+bx+cx^2)^{3/2}}{dx} -$$

$$= \frac{3(3b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(5b+2cx)\sqrt{a+bx+cx^2}}{4d} - \frac{(a+bx+cx^2)^{3/2}}{dx} -$$

**<u>Mathematica</u> [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.69, size = 541, normalized size = 1.17

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*x + c*x^2)^(3/2)/(x^2*(d - f*x^2)),x]
```

```
[Out] (-2*a*f*Sqrt[a + x*(b + c*x)] + 6*Sqrt[a]*b*f*x*ArcTanh[(Sqrt[c]*x - Sqrt[a
 + x*(b + c*x)])/Sqrt[a]] + 2*c^(3/2)*d*x*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt
[a + x*(b + c*x)])] - x*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1
^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1] + b^3*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*
b*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*c^(5/2)*d^2*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*b^2*Sqrt[c]*d*f*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 4*a*c^(3/2)*d*f*Log[-(Sqrt[c]*x) +
```

```
 Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a^2*Sqrt[c]*f^2*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1]*#1 + 2*b*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1]*#1^2 + 2*a*b*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*
#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(2*d*f*x)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1786 vs. $2(367) = 734$.
time = 0.15, size = 1787, normalized size = 3.86

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1787 |
| risch | Expression too large to display | 2283 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(3/2)/x^2/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/a/x*(c*x^2+b*x+a)^(5/2)+3/2*b/a*(1/3*(c*x^2+b*x+a)^(3/2)+1/2*b*(1/4
*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(
1/2)+(c*x^2+b*x+a)^(1/2)))+a*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1
/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a
)^(1/2))/x)))+4*c/a*(1/8*(2*c*x+b)/c*(c*x^2+b*x+a)^(3/2)+3/16*(4*a*c-b^2)/c
*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x
)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))))+1/2*f/d/(d*f)^(1/2)*(1/3*((x+(d*f)^(1/2)/
f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a
+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*(2*c*(x+(d*f)^(1/2)/f)+1/f*(
-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*
(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/8*(4*c/f*(-b*(d*f)^
(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/c^(3/2)*ln((1/2/f*(-2*c*(d*f
)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*
(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)))+1/
f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*
f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f*(-2*c*(d*f)^
(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((
x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(
d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(
d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)
^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d
*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)
^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))-1/2*f/d/(d*f)^(1/2)*(1/3*((x-(
d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+
f*a+c*d)/f)^(3/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(
2*c*(d*f)^(1/2)+b*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x
-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*
a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)
/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f
```

```
)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a
+c*d)/f*(((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(
b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d
*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(
d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/
2)-(b*(d*f)^(1/2)+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*
f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(
1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d
*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^2/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate((c*x^2 + b*x + a)^(3/2)/((f*x^2 - d)*x^2), x)
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^2/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{a\sqrt{a+bx+cx^2}}{-dx^2+fx^4}\,dx-\int \frac{bx\sqrt{a+bx+cx^2}}{-dx^2+fx^4}\,dx-\int \frac{cx^2\sqrt{a+bx+cx^2}}{-dx^2+fx^4}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(3/2)/x**2/(-f*x**2+d),x)
```

```
[Out] -Integral(a*sqrt(a + b*x + c*x**2)/(-d*x**2 + f*x**4), x) - Integral(b*x*sq
rt(a + b*x + c*x**2)/(-d*x**2 + f*x**4), x) - Integral(c*x**2*sqrt(a + b*x
+ c*x**2)/(-d*x**2 + f*x**4), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^2/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\left(c\,x^2+b\,x+a\right)^{3/2}}{x^2\,\left(d-f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(3/2)/(x^2*(d - f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(3/2)/(x^2*(d - f*x^2)), x)
```

## 3.90 $\int \frac{\left(a+bx+cx^2\right)^{3/2}}{x^3\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=614

$$-\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx}+\frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2}-\frac{(8c^2d+b^2f+8acf+2bcfx)\sqrt{a+bx-}}{8cd^2}$$

```
[Out] -1/2*(c*x^2+b*x+a)^(3/2)/d/x^2-a^(3/2)*f*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x
^2+b*x+a)^(1/2))/d^2-1/16*b*(-12*a*c+b^2)*f*arctanh(1/2*(2*c*x+b)/c^(1/2)/(
c*x^2+b*x+a)^(1/2))/c^(3/2)/d^2-1/16*b*(12*a*c*f-b^2*f+24*c^2*d)*arctanh(1/
2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/d^2-3/8*(4*a*c+b^2)*arctan
h(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d/a^(1/2)+3/2*b*arctanh(1/2*(2
*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))*c^(1/2)/d-1/2*arctanh(1/2*(b*d^(1/2)-2
*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2
)*f^(1/2))^(1/2))*(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2)/d^2/f^(1/2)+1/2*arctanh
(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/
(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*(c*d+a*f+b*d^(1/2)*f^(1/2))^(3/2)/d^2/f^
(1/2)-3/4*(-2*c*x+b)*(c*x^2+b*x+a)^(1/2)/d/x+1/8*f*(2*b*c*x+8*a*c+b^2)*(c*x
^2+b*x+a)^(1/2)/c/d^2-1/8*(2*b*c*f*x+8*a*c*f+b^2*f+8*c^2*d)*(c*x^2+b*x+a)^(
1/2)/c/d^2
```

**Rubi** [A]

time = 0.91, antiderivative size = 614, normalized size of antiderivative = 1.00, number of steps used = 26, number of rules used = 13, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.464$, Rules used = {6857, 746, 826, 857, 635, 212, 738, 748, 828, 1035, 1084, 1092, 1047}

$$-\frac{a^{3/2}f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d^2}-\frac{bf(b^2-12ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}d^2}-\frac{\sqrt{a+bx+cx^2}\,(8acf+b^2f+2bcfx+8c^2d)}{8cd^2}-\frac{b(12acf+b^2(-f)+24c^2d)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{16c^{3/2}d^2}+\frac{f(8ac+b^2+2bcx)\sqrt{a+bx+cx^2}}{8cd^2}-\frac{3(4ac+b^2)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8\sqrt{a}\,d}-\frac{\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^2\sqrt{f}}+\frac{\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^2\sqrt{f}}-\frac{(a+bx+cx^2)^{3/2}}{2dx^2}-\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx}+\frac{3b\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2d}$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*x + c*x^2)^(3/2)/(x^3*(d - f*x^2)),x]
```

```
[Out] (-3*(b - 2*c*x)*Sqrt[a + b*x + c*x^2])/(4*d*x) + (f*(b^2 + 8*a*c + 2*b*c*x)
*Sqrt[a + b*x + c*x^2])/(8*c*d^2) - ((8*c^2*d + b^2*f + 8*a*c*f + 2*b*c*f*x
)*Sqrt[a + b*x + c*x^2])/(8*c*d^2) - (a + b*x + c*x^2)^(3/2)/(2*d*x^2) - (3
*(b^2 + 4*a*c)*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(8*S
qrt[a]*d) - (a^(3/2)*f*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2]
)])/d^2 + (3*b*Sqrt[c]*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2]
)])/(2*d) - (b*(b^2 - 12*a*c)*f*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x
 + c*x^2])])/(16*c^(3/2)*d^2) - (b*(24*c^2*d - b^2*f + 12*a*c*f)*ArcTanh[(b
 + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(16*c^(3/2)*d^2) - ((c*d - b*
Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d
] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x
```

```
^2])])/(2*d^2*Sqrt[f]) + ((c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)*ArcTanh[(b*
Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d
]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^2*Sqrt[f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 746

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 1))), x] - Di
st[p/(e*(m + 1)), Int[(d + e*x)^(m + 1)*(b + 2*c*x)*(a + b*x + c*x^2)^(p -
1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*
d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && GtQ[p, 0] && (IntegerQ[p]
 || LtQ[m, -1]) && NeQ[m, -1] &&  !ILtQ[m + 2*p + 1, 0] && IntQuadraticQ[a,
 b, c, d, e, m, p, x]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 826

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(e*f*(m + 2*p + 2) -
```

```
 d*g*(2*p + 1) + e*g*(m + 1)*x)*((a + b*x + c*x^2)^p/(e^2*(m + 1)*(m + 2*p
+ 2))), x] + Dist[p/(e^2*(m + 1)*(m + 2*p + 2)), Int[(d + e*x)^(m + 1)*(a +
 b*x + c*x^2)^(p - 1)*Simp[g*(b*d + 2*a*e + 2*a*e*m + 2*b*d*p) - f*b*e*(m +
 2*p + 2) + (g*(2*c*d + b*e + b*e*m + 4*c*d*p) - 2*c*e*f*(m + 2*p + 2))*x,
x], x], x] /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] && RationalQ[p] && p > 0 && (LtQ[m, -1] || Eq
Q[p, 1] || (IntegerQ[p] &&  !RationalQ[m])) && NeQ[m, -1] &&  !ILtQ[m + 2*p
 + 1, 0] && (IntegerQ[m] || IntegerQ[p] || IntegersQ[2*m, 2*p])
```

Rule 828

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(c*e*f*(m + 2*p + 2)
 - g*(c*d + 2*c*d*p - b*e*p) + g*c*e*(m + 2*p + 1)*x)*((a + b*x + c*x^2)^p/
(c*e^2*(m + 2*p + 1)*(m + 2*p + 2))), x] - Dist[p/(c*e^2*(m + 2*p + 1)*(m +
 2*p + 2)), Int[(d + e*x)^m*(a + b*x + c*x^2)^(p - 1)*Simp[c*e*f*(b*d - 2*a
*e)*(m + 2*p + 2) + g*(a*e*(b*e - 2*c*d*m + b*e*m) + b*d*(b*e*p - c*d - 2*c
*d*p)) + (c*e*f*(2*c*d - b*e)*(m + 2*p + 2) + g*(b^2*e^2*(p + m + 1) - 2*c^
2*d^2*(1 + 2*p) - c*e*(b*d*(m - 2*p) + 2*a*e*(m + 2*p + 1))))*x, x], x], x]
 /; FreeQ[{a, b, c, d, e, f, g, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2
- b*d*e + a*e^2, 0] && GtQ[p, 0] && (IntegerQ[p] ||  !RationalQ[m] || (GeQ[
m, -1] && LtQ[m, 0])) &&  !ILtQ[m + 2*p, 0] && (IntegerQ[m] || IntegerQ[p]
|| IntegersQ[2*m, 2*p])
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1035

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d + f*x^2)^(q +
 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), Int[(a + b*x + c*x^2
)^(p - 1)*(d + f*x^2)^q*Simp[h*p*(b*d) + a*(-2*g*f)*(p + q + 1) + (2*h*p*(c
*d - a*f) + b*(-2*g*f)*(p + q + 1))*x + (h*p*((-b)*f) + c*(-2*g*f)*(p + q +
 1))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*a
*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
```

```
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1084

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (B_.)*(x_) + (C_.)*(x_
)^2)*((d_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(B*c*f*(2*p + 2*q + 3) +
 C*(b*f*p) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + f*x^2)^(q + 1
)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1)*
(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + f*x^2)^q*Simp[p*(b*d)*
(C*((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(b^2*C*d*f*
p + a*c*(C*(2*d*f) + f*(-2*A*f)*(2*p + 2*q + 3))) + (2*p*(c*d - a*f)*(C*((-
b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*((-b)*c*(C*(-4*d*
f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*((-b)*f)*(C*
((-b)*f)*(q + 1) - c*((-B)*f)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2
- 4*a*c) - c^2*(C*(-4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3
))))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, A, B, C, q}, x] && NeQ[b^2 -
4*a*c, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*q + 3, 0] &&  !I
GtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{(a+bx+cx^2)^{3/2}}{x^3(d-fx^2)}\,dx = \int \left(\frac{(a+bx+cx^2)^{3/2}}{dx^3} + \frac{f(a+bx+cx^2)^{3/2}}{d^2x} + \frac{f^2x(a+bx+cx^2)^{3/2}}{d^2(d-fx^2)}\right) dx$$

$$= \frac{\int \frac{(a+bx+cx^2)^{3/2}}{x^3}\,dx}{d} + \frac{f\int \frac{(a+bx+cx^2)^{3/2}}{x}\,dx}{d^2} + \frac{f^2\int \frac{x(a+bx+cx^2)^{3/2}}{d-fx^2}\,dx}{d^2}$$

$$= -\frac{(a+bx+cx^2)^{3/2}}{2dx^2} + \frac{3\int \frac{(b+2cx)\sqrt{a+bx+cx^2}}{x^2}\,dx}{4d} + \frac{f\int \frac{\sqrt{a+bx+cx^2}\left(\frac{3bd}{2}+3(\right.}{d-fx^2}}{3d^2}$$

$$= -\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx} + \frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2} - \frac{(8c^2d+b^2}{}$$

$$= -\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx} + \frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2} - \frac{(8c^2d+b^2}{}$$

$$= -\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx} + \frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2} - \frac{(8c^2d+b^2}{}$$

$$= -\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx} + \frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2} - \frac{(8c^2d+b^2}{}$$

$$= -\frac{3(b-2cx)\sqrt{a+bx+cx^2}}{4dx} + \frac{f(b^2+8ac+2bcx)\sqrt{a+bx+cx^2}}{8cd^2} - \frac{(8c^2d+b^2}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.05, size = 587, normalized size = 0.96

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*x + c*x^2)^(3/2)/(x^3*(d - f*x^2)),x]
```

```
[Out] -1/4*((d*(2*a + 5*b*x)*Sqrt[a + x*(b + c*x)])/x^2 + ((3*b^2*d + 4*a*(3*c*d
+ 2*a*f))*ArcTanh[(-(Sqrt[c]*x) + Sqrt[a + x*(b + c*x)])/Sqrt[a]])/Sqrt[a]
+ 2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*
#1^4 & , (2*b^2*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*c^
2*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*b^2*d*f*Log[-(Sqrt
```

```
[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a^2*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] - a^3*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1] - 4*b*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 -
4*a*b*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + c^2*d
^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + b^2*d*f*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + 2*a*c*d*f*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1]*#1^2 + a^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/d^2
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 2138 vs. $2(494) = 988$.
time = 0.14, size = 2139, normalized size = 3.48

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2139 |
| risch | Expression too large to display | 2312 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(3/2)/x^3/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/2/a/x^2*(c*x^2+b*x+a)^(5/2)+1/4*b/a*(-1/a/x*(c*x^2+b*x+a)^(5/2)+3/2
*b/a*(1/3*(c*x^2+b*x+a)^(3/2)+1/2*b*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/
8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+a*((c*x^
2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^
(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x)))+4*c/a*(1/8*(2*c*x+b)/
c*(c*x^2+b*x+a)^(3/2)+3/16*(4*a*c-b^2)/c*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/
2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))))+3
/2*c/a*(1/3*(c*x^2+b*x+a)^(3/2)+1/2*b*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+
1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+a*((c*
x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-
a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))))-1/2/d^2*f*(1/3*((x
+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d
*f)^(1/2)+f*a+c*d))^(3/2)+1/2/f*(-2*c*(d*f)^(1/2)+b*f)*(1/4*(2*c*(x+(d*f)^(
1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/c*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f
)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/8*(4*c
/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/c^(3/2)*ln((1/2
/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f))/c^(1/2)+((x+(d*f)^(1/2)/f)^2
*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d
))^(1/2)))+1/f*(-b*(d*f)^(1/2)+f*a+c*d)*(((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(
d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)+1/2/f
*(-2*c*(d*f)^(1/2)+b*f)*ln((1/2/f*(-2*c*(d*f)^(1/2)+b*f)+c*(x+(d*f)^(1/2)/f
))/c^(1/2)+((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)
/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/c^(1/2)-1/f*(-b*(d*f)^(1/2)+f*a+c*
d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/
f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))
```

```
^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+
1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))))+f/d^2*(1/3*(c*x^2
+b*x+a)^(3/2)+1/2*b*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^
(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+a*((c*x^2+b*x+a)^(1/2)+1
/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*
x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x)))-1/2/d^2*f*(1/3*((x-(d*f)^(1/2)/f)^2*c
+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(3/2)
+1/2*(2*c*(d*f)^(1/2)+b*f)/f*(1/4*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b
*f)/f)/c*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(
b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)+1/8*(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*
f)^(1/2)+b*f)^2/f^2)/c^(3/2)*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+c*(x-(d*f)^(1/
2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2
)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)))+(b*(d*f)^(1/2)+f*a+c*d)/f*(((x-(d*f
)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a
+c*d)/f)^(1/2)+1/2*(2*c*(d*f)^(1/2)+b*f)/f*ln((1/2*(2*c*(d*f)^(1/2)+b*f)/f+
c*(x-(d*f)^(1/2)/f))/c^(1/2)+((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f
*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/c^(1/2)-(b*(d*f)^(1/2)
+f*a+c*d)/f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)
/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^
(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(
d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^3/(-f*x^2+d),x, algorithm="maxima")
[Out] -integrate((c*x^2 + b*x + a)^(3/2)/((f*x^2 - d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^3/(-f*x^2+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{a\sqrt{a+bx+cx^2}}{-dx^3+fx^5}\,dx-\int \frac{bx\sqrt{a+bx+cx^2}}{-dx^3+fx^5}\,dx-\int \frac{cx^2\sqrt{a+bx+cx^2}}{-dx^3+fx^5}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(3/2)/x**3/(-f*x**2+d),x)
```

```
[Out] -Integral(a*sqrt(a + b*x + c*x**2)/(-d*x**3 + f*x**5), x) - Integral(b*x*sq
rt(a + b*x + c*x**2)/(-d*x**3 + f*x**5), x) - Integral(c*x**2*sqrt(a + b*x
+ c*x**2)/(-d*x**3 + f*x**5), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/x^3/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{{\left(c\,x^2+b\,x+a\right)}^{3/2}}{x^3\,\left(d-f\,x^2\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(3/2)/(x^3*(d - f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(3/2)/(x^3*(d - f*x^2)), x)
```

# 3.91 $\int \frac{\left(a+bx+cx^2\right)^{3/2}}{1-x^2}\,dx$

**Optimal**. Leaf size=189

$$-\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2}-\frac{1}{2}(a-b+c)^{3/2}\tanh^{-1}\left(\frac{2a-b+(b-2c)x}{2\sqrt{a-b+c}\,\sqrt{a+bx+cx^2}}\right)-\frac{(3b^2+12ac+8c^2)\,\text{ta}}{}$$

```
[Out] -1/2*(a-b+c)^(3/2)*arctanh(1/2*(2*a-b+(b-2*c)*x)/(a-b+c)^(1/2)/(c*x^2+b*x+a
)^(1/2))+1/2*(a+b+c)^(3/2)*arctanh(1/2*(2*a+b+(b+2*c)*x)/(a+b+c)^(1/2)/(c*x
^2+b*x+a)^(1/2))-1/8*(12*a*c+3*b^2+8*c^2)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*
x^2+b*x+a)^(1/2))/c^(1/2)-1/4*(2*c*x+5*b)*(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]
time = 0.18, antiderivative size = 189, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 24, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {992, 1092, 635, 212, 1047, 738}

$$-\frac{(12ac+3b^2+8c^2)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{8\sqrt{c}}-\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2}-\frac{1}{2}(a-b+c)^{3/2}\tanh^{-1}\left(\frac{2a+x(b-2c)-b}{2\sqrt{a-b+c}\,\sqrt{a+bx+cx^2}}\right)+\frac{1}{2}(a+b+c)^{3/2}\tanh^{-1}\left(\frac{2a+x(b+2c)+b}{2\sqrt{a+b+c}\,\sqrt{a+bx+cx^2}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(a + b*x + c*x^2)^(3/2)/(1 - x^2),x]
```

```
[Out] -1/4*((5*b + 2*c*x)*Sqrt[a + b*x + c*x^2]) - ((a - b + c)^(3/2)*ArcTanh[(2*
a - b + (b - 2*c)*x)/(2*Sqrt[a - b + c]*Sqrt[a + b*x + c*x^2])])/2 - ((3*b^
2 + 12*a*c + 8*c^2)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])
/(8*Sqrt[c]) + ((a + b + c)^(3/2)*ArcTanh[(2*a + b + (b + 2*c)*x)/(2*Sqrt[a
 + b + c]*Sqrt[a + b*x + c*x^2])])/2
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 992

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b*(3*p + 2*q) + 2*c*(p + q)*x)*(a + b*x + c*x^2)^(p - 1)*
((d + f*x^2)^(q + 1)/(2*f*(p + q)*(2*p + 2*q + 1))), x] - Dist[1/(2*f*(p +
q)*(2*p + 2*q + 1)), Int[(a + b*x + c*x^2)^(p - 2)*(d + f*x^2)^q*Simp[b^2*d
*(p - 1)*(2*p + q) - (p + q)*(b^2*d*(1 - p) - 2*a*(c*d - a*f*(2*p + 2*q + 1
))) - (2*b*(c*d - a*f)*(1 - p)*(2*p + q) - 2*(p + q)*b*(2*c*d*(2*p + q) - (
c*d + a*f)*(2*p + 2*q + 1)))*x + (b^2*f*p*(1 - p) + 2*c*(p + q)*(c*d*(2*p -
 1) - a*f*(4*p + 2*q - 1)))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x]
 && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 1] && NeQ[p + q, 0] && NeQ[2*p + 2*q + 1,
 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{(a+bx+cx^2)^{3/2}}{1-x^2}\,dx = -\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2} + \frac{1}{2}\int \frac{\frac{1}{4}(8a^2+5b^2+4ac)+4b(a+c)x+\frac{1}{4}(3b^2+}{(1-x^2)\sqrt{a+bx+cx^2}}$$

$$= -\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2} - \frac{1}{2}\int \frac{\frac{1}{4}(-8a^2-5b^2-4ac)+\frac{1}{4}(-3b^2-12ac-8}{(1-x^2)\sqrt{a+bx+cx^2}}$$

$$= -\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2} - \frac{1}{2}(a-b+c)^2\int \frac{1}{(-1-x)\sqrt{a+bx+cx^2}}\,dx +$$

$$= -\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2} - \frac{(3b^2+12ac+8c^2)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right.}{8\sqrt{c}}$$

$$= -\frac{1}{4}(5b+2cx)\sqrt{a+bx+cx^2} - \frac{1}{2}(a-b+c)^{3/2}\tanh^{-1}\left(\frac{2a-b+(b-2c)x}{2\sqrt{a-b+c}\sqrt{a+bx}}\right.$$

**Mathematica [A]**
time = 0.65, size = 187, normalized size = 0.99

$$-\frac{1}{4}(5b+2cx)\sqrt{a+x(b+cx)} - (-a+b-c)^{3/2}\tan^{-1}\left(\frac{\sqrt{c}(1+x)-\sqrt{a+x(b+cx)}}{\sqrt{-a+b-c}}\right) - (-a-b-c)^{3/2}\tan^{-1}\left(\frac{-\sqrt{c}(-1+x)+\sqrt{a+x(b+cx)}}{\sqrt{-a-b-c}}\right) + \frac{(3b^2+4c(3a+2c))\log\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)}{8\sqrt{c}}$$

Antiderivative was successfully verified.

```
[In] Integrate[(a + b*x + c*x^2)^(3/2)/(1 - x^2),x]
```

```
[Out] -1/4*((5*b + 2*c*x)*Sqrt[a + x*(b + c*x)]) - (-a + b - c)^(3/2)*ArcTan[(Sqr
t[c]*(1 + x) - Sqrt[a + x*(b + c*x)])/Sqrt[-a + b - c]] - (-a - b - c)^(3/2
)*ArcTan[(-(Sqrt[c]*(-1 + x)) + Sqrt[a + x*(b + c*x)])/Sqrt[-a - b - c]] +
((3*b^2 + 4*c*(3*a + 2*c))*Log[b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)]]
)/(8*Sqrt[c])
```

**Maple [B]** Leaf count of result is larger than twice the leaf count of optimal. 567 vs. $2(155) = 310$.
time = 0.16, size = 568, normalized size = 3.01 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(3/2)/(-x^2+1),x,method=_RETURNVERBOSE)
```

```
[Out] -1/6*((-1+x)^2*c+(b+2*c)*(-1+x)+a+b+c)^(3/2)-1/4*(b+2*c)*(1/4*(2*c*(-1+x)+b
+2*c)/c*((-1+x)^2*c+(b+2*c)*(-1+x)+a+b+c)^(1/2)+1/8*(4*c*(a+b+c)-(b+2*c)^2)
/c^(3/2)*ln((1/2*b+c+c*(-1+x))/c^(1/2)+((-1+x)^2*c+(b+2*c)*(-1+x)+a+b+c)^(1
/2)))-1/2*(a+b+c)*(((-1+x)^2*c+(b+2*c)*(-1+x)+a+b+c)^(1/2)+1/2*(b+2*c)*ln((
1/2*b+c+c*(-1+x))/c^(1/2)+((-1+x)^2*c+(b+2*c)*(-1+x)+a+b+c)^(1/2))/c^(1/2)-
(a+b+c)^(1/2)*ln((2*a+2*b+2*c+(b+2*c)*(-1+x)+2*(a+b+c)^(1/2)*((-1+x)^2*c+(b
+2*c)*(-1+x)+a+b+c)^(1/2))/(-1+x)))+1/6*((1+x)^2*c+(b-2*c)*(1+x)+a-b+c)^(3/
```

```
2)+1/4*(b-2*c)*(1/4*(2*c*(1+x)+b-2*c)/c*((1+x)^2*c+(b-2*c)*(1+x)+a-b+c)^(1/
2)+1/8*(4*c*(a-b+c)-(b-2*c)^2)/c^(3/2)*ln((1/2*b-c+c*(1+x))/c^(1/2)+((1+x)^
2*c+(b-2*c)*(1+x)+a-b+c)^(1/2)))+1/2*(a-b+c)*(((1+x)^2*c+(b-2*c)*(1+x)+a-b+
c)^(1/2)+1/2*(b-2*c)*ln((1/2*b-c+c*(1+x))/c^(1/2)+((1+x)^2*c+(b-2*c)*(1+x)+
a-b+c)^(1/2))/c^(1/2)-(a-b+c)^(1/2)*ln((2*a-2*b+2*c+(b-2*c)*(1+x)+2*(a-b+c)
^(1/2)*((1+x)^2*c+(b-2*c)*(1+x)+a-b+c)^(1/2))/(1+x)))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-x^2+1),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*a*c-b^2>0)', see `assume?` for mo
re deta
```

**Fricas** [**A**]
time = 94.79, size = 2579, normalized size = 13.65

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-x^2+1),x, algorithm="fricas")
```

```
[Out] [1/16*((3*b^2 + 12*a*c + 8*c^2)*sqrt(c)*log(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*
sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(c) - 4*a*c) + 4*((a - b)*c + c^2)*sq
rt(a - b + c)*log(-((b^2 + 4*(a - 2*b)*c + 8*c^2)*x^2 - 4*sqrt(c*x^2 + b*x
+ a)*((b - 2*c)*x + 2*a - b)*sqrt(a - b + c) + 8*a^2 - 8*a*b + b^2 + 4*a*c
+ 2*(4*a*b - 3*b^2 - 4*(a - b)*c)*x)/(x^2 + 2*x + 1)) + 4*((a + b)*c + c^2)
*sqrt(a + b + c)*log(-((b^2 + 4*(a + 2*b)*c + 8*c^2)*x^2 + 4*sqrt(c*x^2 + b
*x + a)*((b + 2*c)*x + 2*a + b)*sqrt(a + b + c) + 8*a^2 + 8*a*b + b^2 + 4*a
*c + 2*(4*a*b + 3*b^2 + 4*(a + b)*c)*x)/(x^2 - 2*x + 1)) - 4*(2*c^2*x + 5*b
*c)*sqrt(c*x^2 + b*x + a))/c, -1/16*(8*((a - b)*c + c^2)*sqrt(-a + b - c)*a
rctan(-1/2*sqrt(c*x^2 + b*x + a)*((b - 2*c)*x + 2*a - b)*sqrt(-a + b - c)/(
((a - b)*c + c^2)*x^2 + a^2 - a*b + a*c + (a*b - b^2 + b*c)*x)) - (3*b^2 +
12*a*c + 8*c^2)*sqrt(c)*log(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*sqrt(c*x^2 + b*x
 + a)*(2*c*x + b)*sqrt(c) - 4*a*c) - 4*((a + b)*c + c^2)*sqrt(a + b + c)*lo
g(-((b^2 + 4*(a + 2*b)*c + 8*c^2)*x^2 + 4*sqrt(c*x^2 + b*x + a)*((b + 2*c)*
x + 2*a + b)*sqrt(a + b + c) + 8*a^2 + 8*a*b + b^2 + 4*a*c + 2*(4*a*b + 3*b
^2 + 4*(a + b)*c)*x)/(x^2 - 2*x + 1)) + 4*(2*c^2*x + 5*b*c)*sqrt(c*x^2 + b*
x + a))/c, -1/16*(8*((a + b)*c + c^2)*sqrt(-a - b - c)*arctan(1/2*sqrt(c*x^
2 + b*x + a)*((b + 2*c)*x + 2*a + b)*sqrt(-a - b - c)/(((a + b)*c + c^2)*x^
```

2 + a^2 + a*b + a*c + (a*b + b^2 + b*c)*x)) - (3*b^2 + 12*a*c + 8*c^2)*sqrt
(c)*log(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sq
rt(c) - 4*a*c) - 4*((a - b)*c + c^2)*sqrt(a - b + c)*log(-((b^2 + 4*(a - 2*
b)*c + 8*c^2)*x^2 - 4*sqrt(c*x^2 + b*x + a)*((b - 2*c)*x + 2*a - b)*sqrt(a
- b + c) + 8*a^2 - 8*a*b + b^2 + 4*a*c + 2*(4*a*b - 3*b^2 - 4*(a - b)*c)*x)
/(x^2 + 2*x + 1)) + 4*(2*c^2*x + 5*b*c)*sqrt(c*x^2 + b*x + a))/c, -1/16*(8*
((a - b)*c + c^2)*sqrt(-a + b - c)*arctan(-1/2*sqrt(c*x^2 + b*x + a)*((b -
2*c)*x + 2*a - b)*sqrt(-a + b - c)/(((a - b)*c + c^2)*x^2 + a^2 - a*b + a*c
 + (a*b - b^2 + b*c)*x)) + 8*((a + b)*c + c^2)*sqrt(-a - b - c)*arctan(1/2*
sqrt(c*x^2 + b*x + a)*((b + 2*c)*x + 2*a + b)*sqrt(-a - b - c)/(((a + b)*c
+ c^2)*x^2 + a^2 + a*b + a*c + (a*b + b^2 + b*c)*x)) - (3*b^2 + 12*a*c + 8*
c^2)*sqrt(c)*log(-8*c^2*x^2 - 8*b*c*x - b^2 + 4*sqrt(c*x^2 + b*x + a)*(2*c*
x + b)*sqrt(c) - 4*a*c) + 4*(2*c^2*x + 5*b*c)*sqrt(c*x^2 + b*x + a))/c, 1/8
*((3*b^2 + 12*a*c + 8*c^2)*sqrt(-c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*(2*c*x
 + b)*sqrt(-c)/(c^2*x^2 + b*c*x + a*c)) + 2*((a - b)*c + c^2)*sqrt(a - b +
c)*log(-((b^2 + 4*(a - 2*b)*c + 8*c^2)*x^2 - 4*sqrt(c*x^2 + b*x + a)*((b -
2*c)*x + 2*a - b)*sqrt(a - b + c) + 8*a^2 - 8*a*b + b^2 + 4*a*c + 2*(4*a*b
- 3*b^2 - 4*(a - b)*c)*x)/(x^2 + 2*x + 1)) + 2*((a + b)*c + c^2)*sqrt(a + b
 + c)*log(-((b^2 + 4*(a + 2*b)*c + 8*c^2)*x^2 + 4*sqrt(c*x^2 + b*x + a)*((b
 + 2*c)*x + 2*a + b)*sqrt(a + b + c) + 8*a^2 + 8*a*b + b^2 + 4*a*c + 2*(4*a
*b + 3*b^2 + 4*(a + b)*c)*x)/(x^2 - 2*x + 1)) - 2*(2*c^2*x + 5*b*c)*sqrt(c*
x^2 + b*x + a))/c, -1/8*(4*((a - b)*c + c^2)*sqrt(-a + b - c)*arctan(-1/2*s
qrt(c*x^2 + b*x + a)*((b - 2*c)*x + 2*a - b)*sqrt(-a + b - c)/(((a - b)*c +
 c^2)*x^2 + a^2 - a*b + a*c + (a*b - b^2 + b*c)*x)) - (3*b^2 + 12*a*c + 8*c
^2)*sqrt(-c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(-c)/(c^2*x^2
 + b*c*x + a*c)) - 2*((a + b)*c + c^2)*sqrt(a + b + c)*log(-((b^2 + 4*(a +
2*b)*c + 8*c^2)*x^2 + 4*sqrt(c*x^2 + b*x + a)*((b + 2*c)*x + 2*a + b)*sqrt(
a + b + c) + 8*a^2 + 8*a*b + b^2 + 4*a*c + 2*(4*a*b + 3*b^2 + 4*(a + b)*c)*
x)/(x^2 - 2*x + 1)) + 2*(2*c^2*x + 5*b*c)*sqrt(c*x^2 + b*x + a))/c, -1/8*(4
*((a + b)*c + c^2)*sqrt(-a - b - c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*((b +
2*c)*x + 2*a + b)*sqrt(-a - b - c)/(((a + b)*c + c^2)*x^2 + a^2 + a*b + a*c
 + (a*b + b^2 + b*c)*x)) - (3*b^2 + 12*a*c + 8*c^2)*sqrt(-c)*arctan(1/2*sqr
t(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(-c)/(c^2*x^2 + b*c*x + a*c)) - 2*((a -
b)*c + c^2)*sqrt(a - b + c)*log(-((b^2 + 4*(a - 2*b)*c + 8*c^2)*x^2 - 4*sqr
t(c*x^2 + b*x + a)*((b - 2*c)*x + 2*a - b)*sqrt(a - b + c) + 8*a^2 - 8*a*b
+ b^2 + 4*a*c + 2*(4*a*b - 3*b^2 - 4*(a - b)*c)*x)/(x^2 + 2*x + 1)) + 2*(2*
c^2*x + 5*b*c)*sqrt(c*x^2 + b*x + a))/c, -1/8*(4*((a - b)*c + c^2)*sqrt(-a
+ b - c)*arctan(-1/2*sqrt(c*x^2 + b*x + a)*((b - 2*c)*x + 2*a - b)*sqrt(-a
+ b - c)/(((a - b)*c + c^2)*x^2 + a^2 - a*b + a*c + (a*b - b^2 + b*c)*x)) +
 4*((a + b)*c + c^2)*sqrt(-a - b - c)*arctan(1/2*sqrt(c*x^2 + b*x + a)*((b
+ 2*c)*x + 2*a + b)*sqrt(-a - b - c)/(((a + b)*c + c^2)*x^2 + a^2 + a*b + a
*c + (a*b + b^2 + b*c)*x)) - (3*b^2 + 12*a*c + 8*c^2)*sqrt(-c)*arctan(1/2*s
qrt(c*x^2 + b*x + a)*(2*c*x + b)*sqrt(-c)/(c^2*x^2 + b*c*x + a*c)) + 2*(2*c
^2*x + 5*b*c)*sqrt(c*x^2 + b*x + a))/c]

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{a\sqrt{a+bx+cx^2}}{x^2-1}\,dx - \int \frac{bx\sqrt{a+bx+cx^2}}{x^2-1}\,dx - \int \frac{cx^2\sqrt{a+bx+cx^2}}{x^2-1}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(3/2)/(-x**2+1),x)
```

```
[Out] -Integral(a*sqrt(a + b*x + c*x**2)/(x**2 - 1), x) - Integral(b*x*sqrt(a + b
*x + c*x**2)/(x**2 - 1), x) - Integral(c*x**2*sqrt(a + b*x + c*x**2)/(x**2
- 1), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(3/2)/(-x^2+1),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$-\int \frac{{\left(c\,x^2+b\,x+a\right)}^{3/2}}{x^2-1}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(-(a + b*x + c*x^2)^(3/2)/(x^2 - 1),x)
```

```
[Out] -int((a + b*x + c*x^2)^(3/2)/(x^2 - 1), x)
```

## 3.92 $\int \frac{\sqrt{-1-x+x^2}}{1-x^2} \, dx$

**Optimal**. Leaf size=75

$$-\frac{1}{2}\tan^{-1}\left(\frac{3-x}{2\sqrt{-1-x+x^2}}\right)+\tanh^{-1}\left(\frac{1-2x}{2\sqrt{-1-x+x^2}}\right)+\frac{1}{2}\tanh^{-1}\left(\frac{1+3x}{2\sqrt{-1-x+x^2}}\right)$$

```
[Out] -1/2*arctan(1/2*(3-x)/(x^2-x-1)^(1/2))+arctanh(1/2*(1-2*x)/(x^2-x-1)^(1/2))
+1/2*arctanh(1/2*(1+3*x)/(x^2-x-1)^(1/2))
```

**Rubi** [A]
time = 0.03, antiderivative size = 75, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 6, integrand size = 22, $\frac{\text{number of rules}}{\text{integrand size}} = 0.273$, Rules used = {1004, 635, 212, 1047, 738, 210}

$$-\frac{1}{2}\text{ArcTan}\left(\frac{3-x}{2\sqrt{x^2-x-1}}\right)+\tanh^{-1}\left(\frac{1-2x}{2\sqrt{x^2-x-1}}\right)+\frac{1}{2}\tanh^{-1}\left(\frac{3x+1}{2\sqrt{x^2-x-1}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[-1 - x + x^2]/(1 - x^2),x]
```

```
[Out] -1/2*ArcTan[(3 - x)/(2*Sqrt[-1 - x + x^2])] + ArcTanh[(1 - 2*x)/(2*Sqrt[-1
- x + x^2])] + ArcTanh[(1 + 3*x)/(2*Sqrt[-1 - x + x^2])]/2
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1004

```
Int[Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2]/((d_) + (f_.)*(x_)^2), x_Symbol]
 :> Dist[c/f, Int[1/Sqrt[a + b*x + c*x^2], x], x] - Dist[1/f, Int[(c*d - a*
f - b*f*x)/(Sqrt[a + b*x + c*x^2]*(d + f*x^2)), x], x] /; FreeQ[{a, b, c, d
, f}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\begin{aligned}\int \frac{\sqrt{-1-x+x^2}}{1-x^2}\,dx &= -\int \frac{1}{\sqrt{-1-x+x^2}}\,dx-\int \frac{x}{(1-x^2)\sqrt{-1-x+x^2}}\,dx\\ &= -\left(\frac{1}{2}\int \frac{1}{(-1-x)\sqrt{-1-x+x^2}}\,dx\right)-\frac{1}{2}\int \frac{1}{(1-x)\sqrt{-1-x+x^2}}\,dx-2\text{Subst}\Big(\\ &= \tanh^{-1}\left(\frac{1-2x}{2\sqrt{-1-x+x^2}}\right)+\text{Subst}\left(\int \frac{1}{-4-x^2}\,dx,x,\frac{3-x}{\sqrt{-1-x+x^2}}\right)+\text{Subst}\Big(\\ &= -\frac{1}{2}\tan^{-1}\left(\frac{3-x}{2\sqrt{-1-x+x^2}}\right)+\tanh^{-1}\left(\frac{1-2x}{2\sqrt{-1-x+x^2}}\right)+\frac{1}{2}\tanh^{-1}\Big(\frac{1-}{2\sqrt{-1-}}\end{aligned}$$

**Mathematica [A]**
time = 0.09, size = 57, normalized size = 0.76

$$\tan^{-1}\left(1-x+\sqrt{-1-x+x^2}\right)+\tanh^{-1}\left(1+x-\sqrt{-1-x+x^2}\right)+\log\left(1-2x+2\sqrt{-1-x+x^2}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[-1 - x + x^2]/(1 - x^2),x]
```

```
[Out] ArcTan[1 - x + Sqrt[-1 - x + x^2]] + ArcTanh[1 + x - Sqrt[-1 - x + x^2]] +
Log[1 - 2*x + 2*Sqrt[-1 - x + x^2]]
```

**Maple [A]**
time = 0.18, size = 102, normalized size = 1.36

| method | result |
|---|---|
| default | $-\frac{\sqrt{(-1+x)^2-2+x}}{2}-\frac{\ln\left(-\frac{1}{2}+x+\sqrt{(-1+x)^2-2+x}\right)}{4}+\frac{\arctan\left(\frac{x-3}{2\sqrt{(-1+x)^2-2+x}}\right)}{2}+$ |
| trager | $\frac{\operatorname{RootOf}\left(\_Z^2+1\right)\ln\left(-\frac{-x\operatorname{RootOf}\left(\_Z^2+1\right)+2\sqrt{x^2-x-1}+3\operatorname{RootOf}\left(\_Z^2+1\right)}{-1+x}\right)}{2}-\frac{\ln\left(\frac{8\sqrt{x^2-x-1}\,x^2+8x^3+12}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2-x-1)^(1/2)/(-x^2+1),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*((-1+x)^2-2+x)^(1/2)-1/4*ln(-1/2+x+((-1+x)^2-2+x)^(1/2))+1/2*arctan(1/
2*(x-3)/((-1+x)^2-2+x)^(1/2))+1/2*((1+x)^2-2-3*x)^(1/2)-3/4*ln(-1/2+x+((1+x
)^2-2-3*x)^(1/2))-1/2*arctanh(1/2*(-1-3*x)/((1+x)^2-2-3*x)^(1/2))
```

**Maxima [A]**
time = 0.49, size = 83, normalized size = 1.11

$$\frac{1}{2}\arcsin\left(\frac{2\sqrt{5}\,x}{5\,|2\,x-2|}-\frac{6\sqrt{5}}{5\,|2\,x-2|}\right)-\log\left(x+\sqrt{x^2-x-1}-\frac{1}{2}\right)-\frac{1}{2}\log\left(\frac{2\sqrt{x^2-x-1}}{|2\,x+2|}+\frac{2}{|2\,x+2|}-\frac{3}{2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2-x-1)^(1/2)/(-x^2+1),x, algorithm="maxima")
```

```
[Out] 1/2*arcsin(2/5*sqrt(5)*x/abs(2*x - 2) - 6/5*sqrt(5)/abs(2*x - 2)) - log(x +
 sqrt(x^2 - x - 1) - 1/2) - 1/2*log(2*sqrt(x^2 - x - 1)/abs(2*x + 2) + 2/ab
s(2*x + 2) - 3/2)
```

**Fricas [A]**
time = 0.35, size = 70, normalized size = 0.93

$$\arctan\left(-x+\sqrt{x^2-x-1}+1\right)-\frac{1}{2}\log\left(-x+\sqrt{x^2-x-1}\right)+\frac{1}{2}\log\left(-x+\sqrt{x^2-x-1}-2\right)+\log\left(-2\,x+2\sqrt{x^2-x-1}+1\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2-x-1)^(1/2)/(-x^2+1),x, algorithm="fricas")
```

```
[Out] arctan(-x + sqrt(x^2 - x - 1) + 1) - 1/2*log(-x + sqrt(x^2 - x - 1)) + 1/2*
log(-x + sqrt(x^2 - x - 1) - 2) + log(-2*x + 2*sqrt(x^2 - x - 1) + 1)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int\frac{\sqrt{x^2-x-1}}{x^2-1}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x**2-x-1)**(1/2)/(-x**2+1),x)
```

```
[Out] -Integral(sqrt(x**2 - x - 1)/(x**2 - 1), x)
```

**Giac [A]**
time = 5.64, size = 73, normalized size = 0.97

$$\arctan\left(-x+\sqrt{x^2-x-1}+1\right)-\frac{1}{2}\log\left(\left|-x+\sqrt{x^2-x-1}\right|\right)+\frac{1}{2}\log\left(\left|-x+\sqrt{x^2-x-1}-2\right|\right)+\log\left(\left|-2\,x+2\,\sqrt{x^2-x-1}+1\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2-x-1)^(1/2)/(-x^2+1),x, algorithm="giac")
```

```
[Out] arctan(-x + sqrt(x^2 - x - 1) + 1) - 1/2*log(abs(-x + sqrt(x^2 - x - 1))) +
 1/2*log(abs(-x + sqrt(x^2 - x - 1) - 2)) + log(abs(-2*x + 2*sqrt(x^2 - x -
 1) + 1))
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$-\int \frac{\sqrt{x^2-x-1}}{x^2-1}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(-(x^2 - x - 1)^(1/2)/(x^2 - 1),x)
```

```
[Out] -int((x^2 - x - 1)^(1/2)/(x^2 - 1), x)
```

## 3.93 $\int \frac{(x+x^2)^{3/2}}{1+x^2}\,dx$

**Optimal**. Leaf size=130

$$\frac{1}{4}(5+2x)\sqrt{x+x^2}+\sqrt{1+\sqrt{2}}\,\tan^{-1}\left(\frac{1+\sqrt{2}-x}{\sqrt{2\left(1+\sqrt{2}\right)}\,\sqrt{x+x^2}}\right)-\sqrt{-1+\sqrt{2}}\,\tanh^{-1}\left(\frac{1-\sqrt{}}{\sqrt{2\left(-1+\sqrt{}\right.}}\right.$$

```
[Out] -5/4*arctanh(x/(x^2+x)^(1/2))+1/4*(5+2*x)*(x^2+x)^(1/2)-arctanh((1-x-2^(1/2
))/(x^2+x)^(1/2)/(-2+2*2^(1/2))^(1/2))*(2^(1/2)-1)^(1/2)+arctan((1-x+2^(1/2
))/(x^2+x)^(1/2)/(2+2*2^(1/2))^(1/2))*(1+2^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.10, antiderivative size = 130, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 9, integrand size = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 0.529$, Rules used = {992, 1092, 634, 212, 12, 1050, 1044, 213, 209}

$$\sqrt{1+\sqrt{2}}\,\text{ArcTan}\left(\frac{-x+\sqrt{2}+1}{\sqrt{2\left(1+\sqrt{2}\right)}\,\sqrt{x^2+x}}\right)+\frac{1}{4}\sqrt{x^2+x}\,(2x+5)-\sqrt{\sqrt{2}-1}\,\tanh^{-1}\left(\frac{-x-\sqrt{2}+1}{\sqrt{2\left(\sqrt{2}-1\right)}\,\sqrt{x^2+x}}\right)-\frac{5}{4}\tanh^{-1}\left(\frac{x}{\sqrt{x^2+x}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[(x + x^2)^(3/2)/(1 + x^2),x]
```

```
[Out] ((5 + 2*x)*Sqrt[x + x^2])/4 + Sqrt[1 + Sqrt[2]]*ArcTan[(1 + Sqrt[2] - x)/(S
qrt[2*(1 + Sqrt[2])]*Sqrt[x + x^2])] - Sqrt[-1 + Sqrt[2]]*ArcTanh[(1 - Sqrt
[2] - x)/(Sqrt[2*(-1 + Sqrt[2])]*Sqrt[x + x^2])] - (5*ArcTanh[x/Sqrt[x + x^
2]])/4
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 209

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[b, 2]))*A
rcTan[Rt[b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] && (GtQ[a
, 0] || GtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 213

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[b, 2])^(-
1))*ArcTanh[Rt[b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] &&
 (LtQ[a, 0] || GtQ[b, 0])
```

Rule 634

```
Int[1/Sqrt[(b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[Int[1/(1
- c*x^2), x], x, x/Sqrt[b*x + c*x^2]], x] /; FreeQ[{b, c}, x]
```

Rule 992

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b*(3*p + 2*q) + 2*c*(p + q)*x)*(a + b*x + c*x^2)^(p - 1)*
((d + f*x^2)^(q + 1)/(2*f*(p + q)*(2*p + 2*q + 1))), x] - Dist[1/(2*f*(p +
q)*(2*p + 2*q + 1)), Int[(a + b*x + c*x^2)^(p - 2)*(d + f*x^2)^q*Simp[b^2*d
*(p - 1)*(2*p + q) - (p + q)*(b^2*d*(1 - p) - 2*a*(c*d - a*f*(2*p + 2*q + 1
))) - (2*b*(c*d - a*f)*(1 - p)*(2*p + q) - 2*(p + q)*b*(2*c*d*(2*p + q) - (
c*d + a*f)*(2*p + 2*q + 1)))*x + (b^2*f*p*(1 - p) + 2*c*(p + q)*(c*d*(2*p -
 1) - a*f*(4*p + 2*q - 1)))*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x]
 && NeQ[b^2 - 4*a*c, 0] && GtQ[p, 1] && NeQ[p + q, 0] && NeQ[2*p + 2*q + 1,
 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1044

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f
_.)*(x_)^2]), x_Symbol] :> Dist[-2*a*g*h, Subst[Int[1/Simp[2*a^2*g*h*c + a*
e*x^2, x], x], x, Simp[a*h - g*c*x, x]/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[
{a, c, d, e, f, g, h}, x] && EqQ[a*h^2*e + 2*g*h*(c*d - a*f) - g^2*c*e, 0]
```

Rule 1050

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 + a*c*e^2, 2]}, Dist
[1/(2*q), Int[Simp[(-a)*h*e - g*(c*d - a*f - q) + (h*(c*d - a*f + q) - g*c*
e)*x, x]/((a + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[Si
mp[(-a)*h*e - g*(c*d - a*f + q) + (h*(c*d - a*f - q) - g*c*e)*x, x]/((a + c
*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f, g, h}, x] &&
 NeQ[e^2 - 4*d*f, 0] && NegQ[(-a)*c]
```

Rule 1092

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + B*c*x)/((a + c*x^2)*Sqrt[d + e*x
```

```
 + f*x^2]), x], x] /; FreeQ[{a, c, d, e, f, A, B, C}, x] && NeQ[e^2 - 4*d*f
, 0]
```

Rubi steps

$$\int \frac{(x+x^2)^{3/2}}{1+x^2}\,dx = \frac{1}{4}(5+2x)\sqrt{x+x^2} - \frac{1}{2}\int \frac{\frac{5}{4}+4x+\frac{5x^2}{4}}{(1+x^2)\sqrt{x+x^2}}\,dx$$

$$= \frac{1}{4}(5+2x)\sqrt{x+x^2} - \frac{1}{2}\int \frac{4x}{(1+x^2)\sqrt{x+x^2}}\,dx - \frac{5}{8}\int \frac{1}{\sqrt{x+x^2}}\,dx$$

$$= \frac{1}{4}(5+2x)\sqrt{x+x^2} - \frac{5}{4}\text{Subst}\left(\int \frac{1}{1-x^2}\,dx, x, \frac{x}{\sqrt{x+x^2}}\right) - 2\int \frac{x}{(1+x^2)\sqrt{x+x^2}}\,d$$

$$= \frac{1}{4}(5+2x)\sqrt{x+x^2} - \frac{5}{4}\tanh^{-1}\left(\frac{x}{\sqrt{x+x^2}}\right) + \frac{\int \frac{-1+\left(-1-\sqrt{2}\right)x}{(1+x^2)\sqrt{x+x^2}}\,dx}{\sqrt{2}} - \frac{\int \frac{-1+\left(-1+\sqrt{2}\right.}{(1+x^2)\sqrt{x+x}}}{\sqrt{2}}$$

$$= \frac{1}{4}(5+2x)\sqrt{x+x^2} - \frac{5}{4}\tanh^{-1}\left(\frac{x}{\sqrt{x+x^2}}\right) + \left(-2+\sqrt{2}\right)\text{Subst}\left(\int \frac{1}{2\left(1-\sqrt{2}\right)-}\right.$$

$$= \frac{1}{4}(5+2x)\sqrt{x+x^2} + \sqrt{1+\sqrt{2}}\tan^{-1}\left(\frac{1+\sqrt{2}-x}{\sqrt{2\left(1+\sqrt{2}\right)}\sqrt{x+x^2}}\right) - \sqrt{-1+\sqrt{2}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.12, size = 117, normalized size = 0.90

$$\frac{\sqrt{x}\sqrt{1+x}\left(\sqrt{x}\sqrt{1+x}(5+2x) - 5\tanh^{-1}\left(\sqrt{\frac{x}{1+x}}\right) + 8\text{RootSum}\left[16+32\#1+16\#1^2+\#1^4\&, \frac{\log\left(-2x+2\sqrt{x}\sqrt{1+x}+\#1\right)\#1^2}{8+8\#1+\#1^3}\&\right]\right)}{4\sqrt{x(1+x)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x + x^2)^(3/2)/(1 + x^2),x]
```

```
[Out] (Sqrt[x]*Sqrt[1 + x]*(Sqrt[x]*Sqrt[1 + x]*(5 + 2*x) - 5*ArcTanh[Sqrt[x/(1 +
 x)]] + 8*RootSum[16 + 32*#1 + 16*#1^2 + #1^4 & , (Log[-2*x + 2*Sqrt[x]*Sqr
t[1 + x] + #1]*#1^2)/(8 + 8*#1 + #1^3) & ]))/(4*Sqrt[x*(1 + x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 788 vs. 2(98) = 196.
time = 0.51, size = 789, normalized size = 6.07 Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2+x)^(3/2)/(x^2+1),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*x*(x^2+x)^(1/2)+5/4*(x^2+x)^(1/2)-5/8*ln(x+1/2+(x^2+x)^(1/2))+1/2*(4*(-
2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2-3*2^(1/2)*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2
+4+3*2^(1/2))^(1/2)*2^(1/2)*((-2+2*2^(1/2))^(1/2)*arctan(1/2*((3*2^(1/2)-4)
*(-(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2+12*2^(1/2)+17))^(1/2)*(-2+2*2^(1/2))^(
1/2)*(24*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2+17*2^(1/2)*(-2^(1/2)-1+x)^2/(-2^
(1/2)+1-x)^2-2^(1/2))*(-2^(1/2)-1+x)/(-2^(1/2)+1-x)*(3*2^(1/2)-4)/((-2^(1/2
)-1+x)^4/(-2^(1/2)+1-x)^4-34*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2+1))*(1+2^(1/
2))^(1/2)*2^(1/2)-2*(-2+2*2^(1/2))^(1/2)*arctan(1/2*((3*2^(1/2)-4)*(-(-2^(1
/2)-1+x)^2/(-2^(1/2)+1-x)^2+12*2^(1/2)+17))^(1/2)*(-2+2*2^(1/2))^(1/2)*(24*
(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2+17*2^(1/2)*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x
)^2-2^(1/2))*(-2^(1/2)-1+x)/(-2^(1/2)+1-x)*(3*2^(1/2)-4)/((-2^(1/2)-1+x)^4/
(-2^(1/2)+1-x)^4-34*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2+1))*(1+2^(1/2))^(1/2)
-4*arctanh(1/2*(4*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2-3*2^(1/2)*(-2^(1/2)-1+x
)^2/(-2^(1/2)+1-x)^2+4+3*2^(1/2))^(1/2)/(1+2^(1/2))^(1/2))*2^(1/2)+6*arctan
h(1/2*(4*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2-3*2^(1/2)*(-2^(1/2)-1+x)^2/(-2^(
1/2)+1-x)^2+4+3*2^(1/2))^(1/2)/(1+2^(1/2))^(1/2)))/(-(3*2^(1/2)*(-2^(1/2)-1
+x)^2/(-2^(1/2)+1-x)^2-4*(-2^(1/2)-1+x)^2/(-2^(1/2)+1-x)^2-3*2^(1/2)-4)/(1+
(-2^(1/2)-1+x)/(-2^(1/2)+1-x))^2)^(1/2)/(1+(-2^(1/2)-1+x)/(-2^(1/2)+1-x))/(
3*2^(1/2)-4)/(1+2^(1/2))^(1/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2+x)^(3/2)/(x^2+1),x, algorithm="maxima")
```

```
[Out] integrate((x^2 + x)^(3/2)/(x^2 + 1), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 777 vs. $2(96) = 192$.
time = 0.37, size = 777, normalized size = 5.98

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2+x)^(3/2)/(x^2+1),x, algorithm="fricas")
```

```
[Out] -1/8*8^(1/4)*sqrt(2*sqrt(2) + 4)*(sqrt(2) - 2)*log(8*x^2 - 8*sqrt(x^2 + x)*
x + 2*(8^(1/4)*sqrt(x^2 + x)*(sqrt(2) - 1) - 8^(1/4)*(sqrt(2)*x - x - 1))*s
qrt(2*sqrt(2) + 4) + 4*x + 4*sqrt(2) + 4) + 1/8*8^(1/4)*sqrt(2*sqrt(2) + 4)
```

```
*(sqrt(2) - 2)*log(8*x^2 - 8*sqrt(x^2 + x)*x - 2*(8^(1/4)*sqrt(x^2 + x)*(sq
rt(2) - 1) - 8^(1/4)*(sqrt(2)*x - x - 1))*sqrt(2*sqrt(2) + 4) + 4*x + 4*sqr
t(2) + 4) + 1/2*8^(1/4)*sqrt(2)*sqrt(2*sqrt(2) + 4)*arctan(1/7*sqrt(2)*(sqr
t(2)*(5*x + 1) + 6*x + 4) + 1/112*sqrt(8*x^2 - 8*sqrt(x^2 + x)*x - 2*(8^(1/
4)*sqrt(x^2 + x)*(sqrt(2) - 1) - 8^(1/4)*(sqrt(2)*x - x - 1))*sqrt(2*sqrt(2
) + 4) + 4*x + 4*sqrt(2) + 4)*(8*sqrt(2)*(5*sqrt(2) + 6) + (8^(3/4)*(5*sqrt
(2) + 6) + 8*8^(1/4)*(2*sqrt(2) + 1))*sqrt(2*sqrt(2) + 4) + 64*sqrt(2) + 32
) - 1/7*sqrt(x^2 + x)*(sqrt(2)*(5*sqrt(2) + 6) + 8*sqrt(2) + 4) + 1/7*sqrt(
2)*(8*x + 3) + 1/56*(8^(3/4)*(sqrt(2)*(5*x + 1) + 6*x + 4) - sqrt(x^2 + x)*
(8^(3/4)*(5*sqrt(2) + 6) + 8*8^(1/4)*(2*sqrt(2) + 1)) + 8*8^(1/4)*(sqrt(2)*
(2*x - 1) + x + 3))*sqrt(2*sqrt(2) + 4) + 4/7*x + 5/7) + 1/2*8^(1/4)*sqrt(2
)*sqrt(2*sqrt(2) + 4)*arctan(-1/7*sqrt(2)*(sqrt(2)*(5*x + 1) + 6*x + 4) - 1
/112*sqrt(8*x^2 - 8*sqrt(x^2 + x)*x + 2*(8^(1/4)*sqrt(x^2 + x)*(sqrt(2) - 1
) - 8^(1/4)*(sqrt(2)*x - x - 1))*sqrt(2*sqrt(2) + 4) + 4*x + 4*sqrt(2) + 4)
*(8*sqrt(2)*(5*sqrt(2) + 6) - (8^(3/4)*(5*sqrt(2) + 6) + 8*8^(1/4)*(2*sqrt(
2) + 1))*sqrt(2*sqrt(2) + 4) + 64*sqrt(2) + 32) + 1/7*sqrt(x^2 + x)*(sqrt(2
)*(5*sqrt(2) + 6) + 8*sqrt(2) + 4) - 1/7*sqrt(2)*(8*x + 3) + 1/56*(8^(3/4)*
(sqrt(2)*(5*x + 1) + 6*x + 4) - sqrt(x^2 + x)*(8^(3/4)*(5*sqrt(2) + 6) + 8*
8^(1/4)*(2*sqrt(2) + 1)) + 8*8^(1/4)*(sqrt(2)*(2*x - 1) + x + 3))*sqrt(2*sq
rt(2) + 4) - 4/7*x - 5/7) + 1/4*sqrt(x^2 + x)*(2*x + 5) + 5/8*log(-2*x + 2*
sqrt(x^2 + x) - 1)
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{(x(x+1))^{\frac{3}{2}}}{x^2+1}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x**2+x)**(3/2)/(x**2+1),x)
```

```
[Out] Integral((x*(x + 1))**(3/2)/(x**2 + 1), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((x^2+x)^(3/2)/(x^2+1),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Unable to divide, perhaps due to roun
ding error%%%{%%{poly1[293782586339316530197997184853348485491135961433723б
965430
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\left(x^2+x\right)^{3/2}}{x^2+1}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x + x^2)^(3/2)/(x^2 + 1),x)
```

```
[Out] int((x + x^2)^(3/2)/(x^2 + 1), x)
```

## 3.94 $\int \frac{x^4}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=369

$$\frac{3b\sqrt{a+bx+cx^2}}{4c^2f}-\frac{x\sqrt{a+bx+cx^2}}{2cf}-\frac{d\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2}-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{b+2}{2\sqrt{c}\,\sqrt{a+}}\right.}{8c^{5/2}f}$$

```
[Out] -1/8*(-4*a*c+3*b^2)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(5
/2)/f-d*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f^2/c^(1/2)+3/4*
b*(c*x^2+b*x+a)^(1/2)/c^2/f-1/2*x*(c*x^2+b*x+a)^(1/2)/c/f+1/2*d^(3/2)*arcta
nh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2
)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/f^2/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)+
1/2*d^(3/2)*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(
c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/f^2/(c*d+a*f+b*d^(1/2
)*f^(1/2))^(1/2)
```

**Rubi** [A]

time = 0.51, antiderivative size = 369, normalized size of antiderivative = 1.00, number of steps used = 13, number of rules used = 7, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {6857, 635, 212, 756, 654, 998, 738}

$$-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{8c^{5/2}f}+\frac{3b\sqrt{a+bx+cx^2}}{4c^2f}+\frac{d^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^2\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{d^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f^2\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{d\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2}-\frac{x\sqrt{a+bx+cx^2}}{2cf}$$

Antiderivative was successfully verified.

```
[In] Int[x^4/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] (3*b*Sqrt[a + b*x + c*x^2])/(4*c^2*f) - (x*Sqrt[a + b*x + c*x^2])/(2*c*f) -
 (d*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(Sqrt[c]*f^2) -
 ((3*b^2 - 4*a*c)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/(
8*c^(5/2)*f) + (d^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b
*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]
)/(2*f^2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + (d^(3/2)*ArcTanh[(b*Sqrt[d]
 + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[
f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*
f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 756

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m - 1)*((a + b*x + c*x^2)^(p + 1)/(c*(m + 2*p +
 1))), x] + Dist[1/(c*(m + 2*p + 1)), Int[(d + e*x)^(m - 2)*Simp[c*d^2*(m +
 2*p + 1) - e*(a*e*(m - 1) + b*d*(p + 1)) + e*(2*c*d - b*e)*(m + p)*x, x]*(
a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, m, p}, x] && NeQ[b^2 -
4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && If[Rat
ionalQ[m], GtQ[m, 1], SumSimplerQ[m, -2]] && NeQ[m + 2*p + 1, 0] && IntQuad
raticQ[a, b, c, d, e, m, p, x]
```

Rule 998

```
Int[1/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Sy
mbol] :> Dist[1/2, Int[1/((a - Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x],
 x] + Dist[1/2, Int[1/((a + Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x], x]
 /; FreeQ[{a, c, d, e, f}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{x^4}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \int \left(-\frac{d}{f^2\sqrt{a+bx+cx^2}} - \frac{x^2}{f\sqrt{a+bx+cx^2}} + \frac{d^2}{f^2\sqrt{a+bx+cx^2}\,(d-\right.$$

$$= -\frac{d\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f^2} + \frac{d^2\int \frac{1}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f^2} - \frac{\int \frac{x^2}{\sqrt{a+bx}}}{}$$

$$= -\frac{x\sqrt{a+bx+cx^2}}{2cf} - \frac{(2d)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f^2} + \frac{d^2}{}$$

$$= \frac{3b\sqrt{a+bx+cx^2}}{4c^2f} - \frac{x\sqrt{a+bx+cx^2}}{2cf} - \frac{d\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right.}{\sqrt{c}\,f^2}$$

$$= \frac{3b\sqrt{a+bx+cx^2}}{4c^2f} - \frac{x\sqrt{a+bx+cx^2}}{2cf} - \frac{d\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right.}{\sqrt{c}\,f^2}$$

$$= \frac{3b\sqrt{a+bx+cx^2}}{4c^2f} - \frac{x\sqrt{a+bx+cx^2}}{2cf} - \frac{d\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right.}{\sqrt{c}\,f^2}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.72, size = 252, normalized size = 0.68

$$\frac{2\sqrt{c}\,f(3b-2cx)\sqrt{a+x(b+cx)} + (8c^2d+3b^2f-4acf)\log\left(c^2f^2\left(b+2cx-2\sqrt{c}\,\sqrt{a+x(b+cx)}\right)\right) - 4c^{5/2}d^2\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{8c^{5/2}f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^4/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] (2*Sqrt[c]*f*(3*b - 2*c*x)*Sqrt[a + x*(b + c*x)] + (8*c^2*d + 3*b^2*f - 4*a
*c*f)*Log[c^2*f^2*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])] - 4*c^(5/2
)*d^2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 -
f*#1^4 & , (b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1)/(b*Sqrt[c]*d - 2*c*d*#1 -
a*f*#1 + f*#1^3) & ])/(8*c^(5/2)*f^2)
```

**Maple** [A]
time = 0.14, size = 513, normalized size = 1.39

| method | result |
|---|---|
| risch | $\frac{(-2cx+3b)\sqrt{c\,x^2+bx+a}}{4c^2 f}+\frac{a\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{2c^{\frac{3}{2}}f}-\frac{3\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)b^2}{8c^{\frac{5}{2}}f}-\frac{d\ln\left(\frac{\frac{b}{2}+c}{\sqrt{c}}\right.}{}$ |
| default | $-\frac{\frac{x\sqrt{c\,x^2+bx+a}}{2c}-\frac{3b\left(\frac{\sqrt{c\,x^2+bx+a}}{c}-\frac{b\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{2c^{\frac{3}{2}}}\right)}{4c}-\frac{a\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{2c^{\frac{3}{2}}}}{f}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/2*x/c*(c*x^2+b*x+a)^(1/2)-3/4*b/c*(1/c*(c*x^2+b*x+a)^(1/2)-1/2*b/c^
(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))-1/2*a/c^(3/2)*ln((1/2*b+
c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))-d/f^2*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+
a)^(1/2))/c^(1/2)-1/2/f^2*d^2/(d*f)^(1/2)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1
/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1
/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-
2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))
/(x+(d*f)^(1/2)/f))+1/2/f^2*d^2/(d*f)^(1/2)/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/
2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f
)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/
2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2
)/f))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
```

```
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^4}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**4/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
[Out] -Integral(x**4/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)),
 x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^4}{\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
[Out] int(x^4/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

## 3.95 $\int \frac{x^3}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=287

$$-\frac{\sqrt{a+bx+cx^2}}{cf}+\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2c^{3/2}f}-\frac{d\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2f^{3/2}\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}}$$

```
[Out] 1/2*b*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f-(c*x^2+b
*x+a)^(1/2)/c/f-1/2*d*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f
^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/f^(3/2)/(c*
d+a*f-b*d^(1/2)*f^(1/2))^(1/2)+1/2*d*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(
2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/
2))/f^(3/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.41, antiderivative size = 287, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 6, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.214$, Rules used = {6857, 654, 635, 212, 1047, 738}

$$\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{2c^{3/2}f}-\frac{d\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f^{3/2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{d\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}\right)}{2f^{3/2}\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}-\frac{\sqrt{a+bx+cx^2}}{cf}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/(c*f)) + (b*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a +
 b*x + c*x^2])])/(2*c^(3/2)*f) - (d*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c
*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*
x + c*x^2])])/(2*f^(3/2)*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + (d*ArcTanh[
(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqr
t[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f^(3/2)*Sqrt[c*d + b*Sqrt[d
]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{x^3}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \int \left(-\frac{x}{f\sqrt{a+bx+cx^2}} + \frac{dx}{f\sqrt{a+bx+cx^2}\,(d-fx^2)}\right) dx$$

$$= -\frac{\int \frac{x}{\sqrt{a+bx+cx^2}}\,dx}{f} + \frac{d\int \frac{x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{cf} + \frac{b\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{2cf} + \frac{d\int \frac{1}{\left(-\sqrt{d}\sqrt{f}-fx\right)\sqrt{a+b}}}{2f}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{cf} + \frac{b\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{cf} - \frac{d\text{Subst}\left(\right.}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{cf} + \frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2c^{3/2}f} - \frac{d\tanh^{-1}\left(\frac{}{2\sqrt{c}}\right.}{2f^3}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.39, size = 210, normalized size = 0.73

$$-\frac{\frac{2\sqrt{a+x(b+cx)}}{c} + \frac{b\log\left(cf\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)}{c^{3/2}} + d\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{a\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-b\sqrt{c}\,d+2cd\#1+af\#1-f\#1^3}\&\right]}{2f}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*((2*Sqrt[a + x*(b + c*x)])/c + (b*Log[c*f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[
a + x*(b + c*x)])])/c^(3/2) + d*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 +
4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (a*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1] - Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(-(b*Sqr
t[c]*d) + 2*c*d*#1 + a*f*#1 - f*#1^3) & ])/f
```

**Maple** [A]
time = 0.13, size = 409, normalized size = 1.43

| method | result |
|---|---|

| | |
|---|---|
| default | $-\frac{\frac{\sqrt{c\,x^2+bx+a}}{c}-\frac{b\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{2c^{\frac{3}{2}}}}{f}+\frac{d\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\cdots}{\cdots}\right)}{\cdots}$ |
| risch | $-\frac{\sqrt{c\,x^2+bx+a}}{cf}+\frac{b\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{2c^{\frac{3}{2}}f}+\frac{d\ln\left(\frac{\frac{2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(2c\sqrt{df}+bf\right)\left(x-\frac{\sqrt{df}}{f}\right)}{f}+2\cdots}{\cdots}\right)}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(1/c*(c*x^2+b*x+a)^(1/2)-1/2*b/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b
*x+a)^(1/2)))+1/2*d/f^2/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d
*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*
(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*
f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)
)+1/2*d/f^2/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)
/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^
(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(
d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^3}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**3/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)),
 x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int(x^3/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

# 3.96 $\int \frac{x^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=266

$$-\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2f\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}+\frac{\sqrt{d}\,\tanh^{-1}\cdots}{\cdots}$$

```
[Out] -arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f/c^(1/2)+1/2*arctanh(1
/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c
*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)/f/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2
)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+
b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)/f/(c*d+a*f+b*d^(1/2
)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.13, antiderivative size = 266, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {1093, 635, 212, 998, 738}

$$\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -(ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])]/(Sqrt[c]*f)) + (Sq
rt[d]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sq
rt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*f*Sqrt[c*d -
b*Sqrt[d]*Sqrt[f] + a*f]) + (Sqrt[d]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*
c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b
*x + c*x^2])])/(2*f*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 998

```
Int[1/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Sy
mbol] :> Dist[1/2, Int[1/((a - Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x],
 x] + Dist[1/2, Int[1/((a + Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x], x]
 /; FreeQ[{a, c, d, e, f}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1093

```
Int[((A_.) + (C_.)*(x_)^2)/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) +
 (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f*x^2], x], x]
 + Dist[(A*c - a*C)/c, Int[1/((a + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /;
 FreeQ[{a, c, d, e, f, A, C}, x] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\begin{aligned}
\int \frac{x^2}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx &= -\frac{\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f}+\frac{d\int \frac{1}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{f}\\
&= -\frac{2\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f}+\frac{d\int \frac{1}{\left(d-\sqrt{d}\,\sqrt{f}\,x\right)\sqrt{a+bx+\cdots}}}{2f}\\
&= -\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}-\frac{d\text{Subst}\left(\int \frac{1}{4cd^2-4bd^{3/2}\sqrt{f}+4adf-x^2}\,dx\cdots\right.}{\cdots}\\
&= -\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\cdots\right.}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+\cdots}}\right.}{2f\sqrt{cd-b\sqrt{d}\,\sqrt{f}\cdots}}
\end{aligned}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.

time = 0.29, size = 194, normalized size = 0.73

$$\frac{\frac{2\log\left(f\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)}{\sqrt{c}}-d\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2f}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] ((2*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])])/Sqrt[c] - d*RootS
um[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & ,
(b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1]*#1)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*
#1^3) & ])/(2*f)
```

**Maple [A]**
time = 0.13, size = 399, normalized size = 1.50

| method | result |
|---|---|
| default | $-\frac{\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{f\sqrt{c}}-\frac{d\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}\cdots}{x+\cdots}\right)}{2\sqrt{df}\,f\sqrt{\frac{-b\cdots}{\cdots}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-1/2*d/(d*f)^(1/2)/
f/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f
*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^
(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1
/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))+1/2*d/(d*f)^(1/2)/f/
((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f
)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d
*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f
*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^2}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**2/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)),
 x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int(x^2/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

# 3.97 $\int \frac{x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=220

$$-\frac{\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{f}\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}+\frac{\tanh^{-1}\left(\frac{b\sqrt{d}+2a\sqrt{f}+\left(2c\sqrt{d}+b\sqrt{f}\right)x}{2\sqrt{cd+b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{f}\sqrt{cd+b\sqrt{d}\sqrt{f}+af}}$$

```
[Out] -1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b
*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/f^(1/2)/(c*d+a*f-b*d^(1/2)*f
^(1/2))^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/
2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/f^(1/2)/(c*d+a*
f+b*d^(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.08, antiderivative size = 220, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.115$, Rules used = {1047, 738, 212}

$$\frac{\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{f}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{f}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}$$

Antiderivative was successfully verified.

```
[In] Int[x/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqr
t[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]/(Sqrt[f]*Sqrt[c*d
- b*Sqrt[d]*Sqrt[f] + a*f]) + ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[
d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*
x^2])]/(2*Sqrt[f]*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{x}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \frac{1}{2}\int \frac{1}{\left(-\sqrt{d}\,\sqrt{f}-fx\right)\sqrt{a+bx+cx^2}}\,dx + \frac{1}{2}\int \frac{1}{\left(\sqrt{d}\,\sqrt{f}-fx\right)\sqrt{a}$$

$$= -\text{Subst}\left(\int \frac{1}{4cdf-4b\sqrt{d}\,f^{3/2}+4af^2-x^2}\,dx, x, \frac{b\sqrt{d}\,\sqrt{f}-2af-\left(-\right.}{\sqrt{a+bx}}\right.$$

$$= -\frac{\tanh^{-1}\left(\frac{b\sqrt{d}\,\sqrt{f}-2af-\left(-2c\sqrt{d}\,\sqrt{f}+bf\right)x}{2\sqrt{f}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{f}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}} - \frac{\tanh^{-1}\left(\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.24, size = 149, normalized size = 0.68

$$-\frac{1}{2}\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{a\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-b\sqrt{c}\,d+2cd\#1+af\#1-f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[x/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f
*#1^4 & , (a*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(-(b*Sqrt[c]*d) + 2*c*d*#1 + a*f*#1
 - f*#1^3) & ]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 353 vs. 2(164) = 328.
time = 0.13, size = 354, normalized size = 1.61

| method | result |
|---|---|
| default | $\frac{\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2}c+\frac{\left(-2c\sqrt{df}+bf\right.}{f}}}{x+\frac{\sqrt{df}}{f}}\right.}{2f\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2/f/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*
d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/
f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))+1/2/f/((b*(d*f)^
(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*
f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/
f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)
^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 2753 vs. $2(164) = 328$.
time = 0.73, size = 2753, normalized size = 12.51

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt((c*d + a*f + (c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*
d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c +
 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 -
 (b^2 - 2*a*c)*d*f^2))*log((2*b*c*d*x + b^2*d + 2*(b^2*d*f - (c^3*d^3*f + a
^3*f^4 - (b^2*c - 3*a*c^2)*d^2*f^2 - (a*b^2 - 3*a^2*c)*d*f^3)*sqrt(b^2*d/(c
^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a
^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))*sqrt(c*x^2 + b*x + a)*sqrt
((c*d + a*f + (c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*d/(c^4*d
^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c
^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 - (b^2 -
2*a*c)*d*f^2)) - (2*a*c^2*d^2*f + 2*a^3*f^3 - 2*(a*b^2 - 2*a^2*c)*d*f^2 + (
b*c^2*d^2*f + a^2*b*f^3 - (b^3 - 2*a*b*c)*d*f^2)*x)*sqrt(b^2*d/(c^4*d^4*f +
 a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^
2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/x) - 1/4*sqrt((c*d + a*f + (c^2*d^2*
f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2
*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^
2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2))*log((2*b
*c*d*x + b^2*d - 2*(b^2*d*f - (c^3*d^3*f + a^3*f^4 - (b^2*c - 3*a*c^2)*d^2*
f^2 - (a*b^2 - 3*a^2*c)*d*f^3)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2
 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 -
2*a^3*c)*d*f^4)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d + a*f + (c^2*d^2*f + a^2*
f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2
*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^
3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)) - (2*a*c^2*d^2*f
 + 2*a^3*f^3 - 2*(a*b^2 - 2*a^2*c)*d*f^2 + (b*c^2*d^2*f + a^2*b*f^3 - (b^3
- 2*a*b*c)*d*f^2)*x)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3
)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d
*f^4)))/x) + 1/4*sqrt((c*d + a*f - (c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f
^2)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4
- 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*
f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2))*log((2*b*c*d*x + b^2*d + 2*(b^2*d*f + (
c^3*d^3*f + a^3*f^4 - (b^2*c - 3*a*c^2)*d^2*f^2 - (a*b^2 - 3*a^2*c)*d*f^3)*
sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*
a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))*sqrt(c*x^2 +
b*x + a)*sqrt((c*d + a*f - (c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt
(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^
2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*
f^3 - (b^2 - 2*a*c)*d*f^2)) + (2*a*c^2*d^2*f + 2*a^3*f^3 - 2*(a*b^2 - 2*a^2
*c)*d*f^2 + (b*c^2*d^2*f + a^2*b*f^3 - (b^3 - 2*a*b*c)*d*f^2)*x)*sqrt(b^2*d
/(c^4*d^4*f + a^4*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c +
6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/x) - 1/4*sqrt((c*d + a*
f - (c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*d/(c^4*d^4*f + a^4
*f^5 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^
3 - 2*(a^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f
^2))*log((2*b*c*d*x + b^2*d - 2*(b^2*d*f + (c^3*d^3*f + a^3*f^4 - (b^2*c -
```

```
3*a*c^2)*d^2*f^2 - (a*b^2 - 3*a^2*c)*d*f^3)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5
 - 2*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 -
2*(a^2*b^2 - 2*a^3*c)*d*f^4)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d + a*f - (c^2
*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2
*(b^2*c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a
^2*b^2 - 2*a^3*c)*d*f^4)))/(c^2*d^2*f + a^2*f^3 - (b^2 - 2*a*c)*d*f^2)) + (
2*a*c^2*d^2*f + 2*a^3*f^3 - 2*(a*b^2 - 2*a^2*c)*d*f^2 + (b*c^2*d^2*f + a^2*
b*f^3 - (b^3 - 2*a*b*c)*d*f^2)*x)*sqrt(b^2*d/(c^4*d^4*f + a^4*f^5 - 2*(b^2*
c^2 - 2*a*c^3)*d^3*f^2 + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^2*f^3 - 2*(a^2*b^2
 - 2*a^3*c)*d*f^4)))/x)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
[Out] -Integral(x/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueWarning, integration
of abs
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
[Out] int(x/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

# 3.98 $\int \frac{1}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=220

$$\frac{\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{d}\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}+\frac{\tanh^{-1}\left(\frac{b\sqrt{d}+2a\sqrt{f}+\left(2c\sqrt{d}+b\sqrt{f}\right)x}{2\sqrt{cd+b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{d}\sqrt{cd+b\sqrt{d}\sqrt{f}+af}}$$

```
[Out] 1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*
x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d^(1/2)/(c*d+a*f-b*d^(1/2)*f^
(1/2))^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2
)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/d^(1/2)/(c*d+a*f
+b*d^(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.09, antiderivative size = 220, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.120$, Rules used = {998, 738, 212}

$$\frac{\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{d}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d
 - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]/(2*Sqrt[d]*Sqrt[c*d - b
*Sqrt[d]*Sqrt[f] + a*f]) + ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d]
+ b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2
])]/(2*Sqrt[d]*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 998

```
Int[1/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Sy
mbol] :> Dist[1/2, Int[1/((a - Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x],
 x] + Dist[1/2, Int[1/((a + Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x], x]
 /; FreeQ[{a, c, d, e, f}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{1}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \frac{1}{2}\int \frac{1}{\left(d-\sqrt{d}\,\sqrt{f}\,x\right)\sqrt{a+bx+cx^2}}\,dx + \frac{1}{2}\int \frac{1}{\left(d+\sqrt{d}\,\sqrt{f}\,x\right)\sqrt{a}\cdots}$$

$$= -\text{Subst}\left(\int \frac{1}{4cd^2-4bd^{3/2}\sqrt{f}+4adf-x^2}\,dx, x, \frac{-bd+2a\sqrt{d}\,\sqrt{f}-\cdots}{\sqrt{a+bx}\cdots}\right.$$

$$= -\frac{\tanh^{-1}\left(\frac{-bd+2a\sqrt{d}\,\sqrt{f}-\left(2cd-b\sqrt{d}\,\sqrt{f}\right)x}{2\sqrt{d}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{d}\,\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}} - \frac{\tanh^{-1}\left(\cdots\right.}{\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.25, size = 151, normalized size = 0.69

$$-\frac{1}{2}\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f
*#1^4 & , (b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*Log
[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1)/(b*Sqrt[c]*d - 2*c*d*#1 - a
*f*#1 + f*#1^3) & ]
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 357 vs. $2(164) = 328$.
time = 0.12, size = 358, normalized size = 1.63

| method | result |
|---|---|
| default | $-\frac{\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2}c+\frac{\left(-2c\sqrt{df}+bf \cdots\right.}{f}}}{x+\frac{\sqrt{df}}{f}}\right.}{2\sqrt{df}\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/(d*f)^(1/2)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/
2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(
1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(
d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))+1/2/(
d*f)^(1/2)/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/
f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(
1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d
*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 2641 vs. $2(164) = 328$.
time = 0.78, size = 2641, normalized size = 12.00

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt((c*d + a*f + (c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(b^2*
f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6
*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f^2 -
 (b^2 - 2*a*c)*d^2*f))*log((2*b*c*x + b^2 + 2*(b*c*d + a*b*f - (b*c^2*d^3 +
 a^2*b*d*f^2 - (b^3 - 2*a*b*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(
b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b
^2 - 2*a^3*c)*d^2*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d + a*f + (c^2*d^3 +
 a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*
c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 -
 2*a^3*c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)) - (2*a*c^
2*d^2 + 2*a^3*f^2 - 2*(a*b^2 - 2*a^2*c)*d*f + (b*c^2*d^2 + a^2*b*f^2 - (b^3
 - 2*a*b*c)*d*f)*x)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)
*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*
f^3)))/x) - 1/4*sqrt((c*d + a*f + (c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*
f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4
*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/(c^2*d^3 +
 a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f))*log((2*b*c*x + b^2 - 2*(b*c*d + a*b*f -
(b*c^2*d^3 + a^2*b*d*f^2 - (b^3 - 2*a*b*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4
*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^
2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d + a*f
+ (c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f
^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 -
2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f
)) - (2*a*c^2*d^2 + 2*a^3*f^2 - 2*(a*b^2 - 2*a^2*c)*d*f + (b*c^2*d^2 + a^2*
b*f^2 - (b^3 - 2*a*b*c)*d*f)*x)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^
2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2
*a^3*c)*d^2*f^3)))/x) + 1/4*sqrt((c*d + a*f - (c^2*d^3 + a^2*d*f^2 - (b^2 -
 2*a*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*
f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3))
)/(c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f))*log((2*b*c*x + b^2 + 2*(b*c*
d + a*b*f + (b*c^2*d^3 + a^2*b*d*f^2 - (b^3 - 2*a*b*c)*d^2*f)*sqrt(b^2*f/(c
^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2
*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt
((c*d + a*f - (c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(b^2*f/(c^4*d
^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2
)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f^2 - (b^2 -
2*a*c)*d^2*f)) + (2*a*c^2*d^2 + 2*a^3*f^2 - 2*(a*b^2 - 2*a^2*c)*d*f + (b*c^
2*d^2 + a^2*b*f^2 - (b^3 - 2*a*b*c)*d*f)*x)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4
 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*
(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/x) - 1/4*sqrt((c*d + a*f - (c^2*d^3 + a^2*d*
f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2
*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*
c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f))*log((2*b*c*x + b
^2 - 2*(b*c*d + a*b*f + (b*c^2*d^3 + a^2*b*d*f^2 - (b^3 - 2*a*b*c)*d^2*f)*s
qrt(b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b
```

```
^2*c + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))*sqrt(c*x^2 + b
*x + a)*sqrt((c*d + a*f - (c^2*d^3 + a^2*d*f^2 - (b^2 - 2*a*c)*d^2*f)*sqrt(
b^2*f/(c^4*d^5 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c
 + 6*a^2*c^2)*d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/(c^2*d^3 + a^2*d*f
^2 - (b^2 - 2*a*c)*d^2*f)) + (2*a*c^2*d^2 + 2*a^3*f^2 - 2*(a*b^2 - 2*a^2*c)
*d*f + (b*c^2*d^2 + a^2*b*f^2 - (b^3 - 2*a*b*c)*d*f)*x)*sqrt(b^2*f/(c^4*d^5
 + a^4*d*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^4*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*
d^3*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^2*f^3)))/x)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-d\sqrt{a+bx+cx^2}+fx^2\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-d*sqrt(a + b*x + c*x**2) + f*x**2*sqrt(a + b*x + c*x**2)), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueWarning, integration
of abs
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int(1/((d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

## 3.99 $\int \frac{1}{x\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=267

$$-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d}-\frac{\sqrt{f}\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2d\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}+\frac{\sqrt{f}\tanh^{-1}}{}$$

[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d/a^(1/2)-1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)/d/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)/d/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)

**Rubi** [A]
time = 0.42, antiderivative size = 267, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 4, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {6857, 738, 212, 1047}

$$-\frac{\sqrt{f}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{\sqrt{f}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d}$$

Antiderivative was successfully verified.

[In] Int[1/(x*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]

[Out] -(ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])]/(Sqrt[a]*d)) - (Sqrt[f]*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + (Sqrt[f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f])

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/((((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
```

```
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{1}{x\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \int \left(\frac{1}{dx\sqrt{a+bx+cx^2}} - \frac{fx}{d\sqrt{a+bx+cx^2}\,(-d+fx^2)}\right)dx$$

$$= \frac{\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{d} - \frac{f\int \frac{x}{\sqrt{a+bx+cx^2}\,(-d+fx^2)}\,dx}{d}$$

$$= -\frac{2\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx, x, \frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{d} - \frac{f\int \frac{1}{\left(-\sqrt{d}\sqrt{f}+fx\right)\sqrt{a+}}}{2d}$$

$$= -\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d} + \frac{f\text{Subst}\left(\int \frac{1}{4cdf-4b\sqrt{d}\,f^{3/2}+4af^2-x^2}\right.}{}$$

$$= -\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d} - \frac{\sqrt{f}\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+}{2\sqrt{cd-b\sqrt{d}\sqrt{f}}}\right.}{2d\sqrt{cd-b\sqrt{d}\sqrt{f}}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.27, size = 193, normalized size = 0.72

$$\frac{\frac{4\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{\sqrt{a}} - f\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{a\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-b\sqrt{c}\,d+2cd\#1+af\#1-f\#1^3}\&\right]}{2d}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] ((4*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]])/Sqrt[a] - f*RootS
um[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & ,
(a*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1]*#1^2)/(-(b*Sqrt[c]*d) + 2*c*d*#1 + a*f*#1 - f*#1^3)
 & ])/(2*d)
```

**Maple** [A]
time = 0.13, size = 391, normalized size = 1.46

| method | result |
|---|---|
| default | $\frac{\ln\left(\frac{\frac{-2b\sqrt{df}+2fa+2cd}{f}+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+2\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2}c+\frac{\left(-2c\sqrt{df}+bf\right.}{f}}}{x+\frac{\sqrt{df}}{f}}\right.}{2d\sqrt{\frac{-b\sqrt{df}+fa+cd}{f}}}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2/d/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)
+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*
d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/
f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f))-1/d/a^(1/2)*ln((
2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x)+1/2/d/((b*(d*f)^(1/2)+f*a+c*d)/f)
^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/
2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)
^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^
(1/2)/f))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 - d)*x), x)
```

**Fricas** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 2993 vs. $2(203) = 406$.
time = 39.31, size = 5995, normalized size = 22.45

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(a*d*sqrt((c*d*f + a*f^2 + (c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*
f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4
 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d
^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f))*log((2*b*c*f^2*x + b^2*f^2 + 2*(b^
2*d*f^2 - (c^3*d^5 + a^3*d^2*f^3 - (b^2*c - 3*a*c^2)*d^4*f - (a*b^2 - 3*a^2
*c)*d^3*f^2)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^
6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3
)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f + a*f^2 + (c^2*d^4 + a^2*d^2*f^2 - (b
^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c
^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d
^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)) - (2*a*c^2*d^3*f +
 2*a^3*d*f^3 - 2*(a*b^2 - 2*a^2*c)*d^2*f^2 + (b*c^2*d^3*f + a^2*b*d*f^3 - (
b^3 - 2*a*b*c)*d^2*f^2)*x)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2
 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*
a^3*c)*d^4*f^3)))/x) - a*d*sqrt((c*d*f + a*f^2 + (c^2*d^4 + a^2*d^2*f^2 - (
b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*
c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*
d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f))*log((2*b*c*f^2*x
+ b^2*f^2 - 2*(b^2*d*f^2 - (c^3*d^5 + a^3*d^2*f^3 - (b^2*c - 3*a*c^2)*d^4*f
 - (a*b^2 - 3*a^2*c)*d^3*f^2)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*
c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 -
 2*a^3*c)*d^4*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f + a*f^2 + (c^2*d^4 +
 a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2
*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2
*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)) -
 (2*a*c^2*d^3*f + 2*a^3*d*f^3 - 2*(a*b^2 - 2*a^2*c)*d^2*f^2 + (b*c^2*d^3*f
+ a^2*b*d*f^3 - (b^3 - 2*a*b*c)*d^2*f^2)*x)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3
*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2
- 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/x) + a*d*sqrt((c*d*f + a*f^2 - (c^2*d^4
+ a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 -
2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^
2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f))*
log((2*b*c*f^2*x + b^2*f^2 + 2*(b^2*d*f^2 + (c^3*d^5 + a^3*d^2*f^3 - (b^2*c
 - 3*a*c^2)*d^4*f - (a*b^2 - 3*a^2*c)*d^3*f^2)*sqrt(b^2*f^3/(c^4*d^7 + a^4*
d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f
^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f + a
```

```
*f^2 - (c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7
+ a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)
*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 -
 2*a*c)*d^3*f)) + (2*a*c^2*d^3*f + 2*a^3*d*f^3 - 2*(a*b^2 - 2*a^2*c)*d^2*f^
2 + (b*c^2*d^3*f + a^2*b*d*f^3 - (b^3 - 2*a*b*c)*d^2*f^2)*x)*sqrt(b^2*f^3/(
c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*
a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/x) - a*d*sqrt((c*d*f +
a*f^2 - (c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7
 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2
)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2
- 2*a*c)*d^3*f))*log((2*b*c*f^2*x + b^2*f^2 - 2*(b^2*d*f^2 + (c^3*d^5 + a^3
*d^2*f^3 - (b^2*c - 3*a*c^2)*d^4*f - (a*b^2 - 3*a^2*c)*d^3*f^2)*sqrt(b^2*f^
3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c +
 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))*sqrt(c*x^2 + b*x + a
)*sqrt((c*d*f + a*f^2 - (c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(
b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b
^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2
*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)) + (2*a*c^2*d^3*f + 2*a^3*d*f^3 - 2*(a*b^2
- 2*a^2*c)*d^2*f^2 + (b*c^2*d^3*f + a^2*b*d*f^3 - (b^3 - 2*a*b*c)*d^2*f^2)*
x)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4
 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))/x) + 2
*sqrt(a)*log(-(8*a*b*x + (b^2 + 4*a*c)*x^2 - 4*sqrt(c*x^2 + b*x + a)*(b*x +
 2*a)*sqrt(a) + 8*a^2)/x^2))/(a*d), 1/4*(a*d*sqrt((c*d*f + a*f^2 + (c^2*d^4
 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f)*sqrt(b^2*f^3/(c^4*d^7 + a^4*d^3*f^4 -
 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*f^2 - 2*(a
^2*b^2 - 2*a^3*c)*d^4*f^3)))/(c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*d^3*f))
*log((2*b*c*f^2*x + b^2*f^2 + 2*(b^2*d*f^2 - (c^3*d^5 + a^3*d^2*f^3 - (b^2*
c - 3*a*c^2)*d^4*f - (a*b^2 - 3*a^2*c)*d^3*f^2)*sqrt(b^2*f^3/(c^4*d^7 + a^4
*d^3*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^6*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^5*
f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^4*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f +
a*f^2 + (c^2*d^4 + a^2*d^2*f^2 - (b^2 - 2*a*c)*...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-dx\sqrt{a+bx+cx^2}+fx^3\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-d*x*sqrt(a + b*x + c*x**2) + f*x**3*sqrt(a + b*x + c*x**2)),
x)
```

**Giac [F]**

time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] sage2
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\,\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int(1/(x*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

# 3.100 $\int \frac{1}{x^2\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=291

$$-\frac{\sqrt{a+bx+cx^2}}{adx}+\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d}+\frac{f\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right.}{2d^{3/2}\sqrt{cd-b\sqrt{d}\sqrt{f}+af}}$$

```
[Out] 1/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)/d-(c*x^2+b
*x+a)^(1/2)/a/d/x+1/2*f*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b
*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d^(3/2)/(
c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2)+1/2*f*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x
*(2*c*d^(1/2)+b*f^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(
1/2))/d^(3/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]

time = 0.42, antiderivative size = 291, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 5, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.179$, Rules used = {6857, 744, 738, 212, 998}

$$\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d}+\frac{f\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^{3/2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{f\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^{3/2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{\sqrt{a+bx+cx^2}}{adx}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/(a*d*x)) + (b*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a
 + b*x + c*x^2])])/(2*a^(3/2)*d) + (f*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2
*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a +
b*x + c*x^2])])/(2*d^(3/2)*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + (f*ArcTan
h[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*S
qrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^(3/2)*Sqrt[c*d + b*Sqrt
[d]*Sqrt[f] + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 744

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[(2*c*d - b*e)/(2*(c*d^2 - b*d*e + a*e^2)),
 Int[(d + e*x)^(m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e,
 m, p}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2
*c*d - b*e, 0] && EqQ[m + 2*p + 3, 0]
```

Rule 998

```
Int[1/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Sy
mbol] :> Dist[1/2, Int[1/((a - Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x],
 x] + Dist[1/2, Int[1/((a + Rt[(-a)*c, 2]*x)*Sqrt[d + e*x + f*x^2]), x], x]
 /; FreeQ[{a, c, d, e, f}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \int \left(\frac{1}{dx^2\sqrt{a+bx+cx^2}} + \frac{f}{d\sqrt{a+bx+cx^2}\,(d-fx^2)}\right)dx$$

$$= \frac{\int \frac{1}{x^2\sqrt{a+bx+cx^2}}\,dx}{d} + \frac{f\int \frac{1}{\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx}{d}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} - \frac{b\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{2ad} + \frac{f\int \frac{1}{\left(d-\sqrt{d}\,\sqrt{f}\,x\right)\sqrt{a \cdots}}}{2d}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} + \frac{b\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx, x, \frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{ad} - \frac{f\text{Sub}\cdots}{\cdots}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} + \frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d} + \frac{f\tanh^{-1}\left(\cdots\right.}{\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.47, size = 216, normalized size = 0.74

$$-\frac{\frac{2\sqrt{a+x(b+cx)}}{ax} + \frac{2b\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{3/2}} + f\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&, \frac{b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2d}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*((2*Sqrt[a + x*(b + c*x)])/(a*x) + (2*b*ArcTanh[(Sqrt[c]*x - Sqrt[a +
x*(b + c*x)])/Sqrt[a]])/a^(3/2) + f*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#
1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
 + c*x^2] - #1] - 2*Sqrt[c]*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*
#1)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/d
```

**Maple** [A]
time = 0.14, size = 426, normalized size = 1.46

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{-\frac{\sqrt{c\,x^2+bx+a}}{ax}+\frac{b\ln\left(\frac{2a+bx+2\sqrt{a}\,\sqrt{c\,x^2+bx+a}}{x}\right)}{2a^{\frac{3}{2}}}}{d}-\frac{f\ln\left(\frac{\frac{-2b\sqrt{df}}{f}+2fa+2cd+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}}{}\right)}{}$ |
| risch | $-\frac{\sqrt{c\,x^2+bx+a}}{adx}+\frac{b\ln\left(\frac{2a+bx+2\sqrt{a}\,\sqrt{c\,x^2+bx+a}}{x}\right)}{2a^{\frac{3}{2}}d}+\frac{f\ln\left(\frac{\frac{2b\sqrt{df}}{f}+2fa+2cd+\frac{\left(2c\sqrt{df}+bf\right)\left(x-\frac{\sqrt{df}}{f}\right)}{f}}{}\right)}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/a/x*(c*x^2+b*x+a)^(1/2)+1/2*b/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+
b*x+a)^(1/2))/x))-1/2*f/d/(d*f)^(1/2)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*
ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/
f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*
(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+
(d*f)^(1/2)/f))+1/2*f/d/(d*f)^(1/2)/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2
*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*
(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/
f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 - d)*x^2), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 3005 vs. 2(223) = 446.
time = 170.98, size = 6018, normalized size = 20.68

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] [1/4*(a^2*d*x*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c
)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f
+ (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/
(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f))*log((2*b*c*f^3*x + b^2*f^3 +
 2*(b*c*d^2*f^2 + a*b*d*f^3 - (b*c^2*d^6 + a^2*b*d^4*f^2 - (b^3 - 2*a*b*c)*
d^5*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f +
(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))*sq
rt(c*x^2 + b*x + a)*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 + a^2*d^3*f^2 - (b^2 -
 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*
d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f
^3)))/(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)) - (2*a*c^2*d^4*f + 2*a
^3*d^2*f^3 - 2*(a*b^2 - 2*a^2*c)*d^3*f^2 + (b*c^2*d^4*f + a^2*b*d^2*f^3 - (
b^3 - 2*a*b*c)*d^3*f^2)*x)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2
 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*
a^3*c)*d^6*f^3)))/x) - a^2*d*x*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 + a^2*d^3*f
^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2
- 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a
^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f))*log((2*b*c*
f^3*x + b^2*f^3 - 2*(b*c*d^2*f^2 + a*b*d*f^3 - (b*c^2*d^6 + a^2*b*d^4*f^2 -
 (b^3 - 2*a*b*c)*d^5*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 -
2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3
*c)*d^6*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 + a^2
*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^
2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2
 - 2*a^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)) - (2*
a*c^2*d^4*f + 2*a^3*d^2*f^3 - 2*(a*b^2 - 2*a^2*c)*d^3*f^2 + (b*c^2*d^4*f +
a^2*b*d^2*f^3 - (b^3 - 2*a*b*c)*d^3*f^2)*x)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5
*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2
- 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/x) + a^2*d*x*sqrt((c*d*f^2 + a*f^3 - (c^
2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*
f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 -
 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^
4*f))*log((2*b*c*f^3*x + b^2*f^3 + 2*(b*c*d^2*f^2 + a*b*d*f^3 + (b*c^2*d^6
+ a^2*b*d^4*f^2 - (b^3 - 2*a*b*c)*d^5*f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^
4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2
*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt((c*d*f^2 + a*f^3
 - (c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c^4*d^9 + a^
4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^7
*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a
*c)*d^4*f)) + (2*a*c^2*d^4*f + 2*a^3*d^2*f^3 - 2*(a*b^2 - 2*a^2*c)*d^3*f^2
+ (b*c^2*d^4*f + a^2*b*d^2*f^3 - (b^3 - 2*a*b*c)*d^3*f^2)*x)*sqrt(b^2*f^5/(
```

```
c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*
a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/x) - a^2*d*x*sqrt((c*d*
f^2 + a*f^3 - (c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*f^5/(c
^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a
^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3*f^2 -
 (b^2 - 2*a*c)*d^4*f))*log((2*b*c*f^3*x + b^2*f^3 - 2*(b*c*d^2*f^2 + a*b*d*
f^3 + (b*c^2*d^6 + a^2*b*d^4*f^2 - (b^3 - 2*a*b*c)*d^5*f)*sqrt(b^2*f^5/(c^4
*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c + 6*a^2
*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))*sqrt(c*x^2 + b*x + a)*sqrt
((c*d*f^2 + a*f^3 - (c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f)*sqrt(b^2*
f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 - 4*a*b^2*c
 + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/(c^2*d^5 + a^2*d^3
*f^2 - (b^2 - 2*a*c)*d^4*f)) + (2*a*c^2*d^4*f + 2*a^3*d^2*f^3 - 2*(a*b^2 -
2*a^2*c)*d^3*f^2 + (b*c^2*d^4*f + a^2*b*d^2*f^3 - (b^3 - 2*a*b*c)*d^3*f^2)*
x)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4
 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/x) + s
qrt(a)*b*x*log(-(8*a*b*x + (b^2 + 4*a*c)*x^2 + 4*sqrt(c*x^2 + b*x + a)*(b*x
 + 2*a)*sqrt(a) + 8*a^2)/x^2) - 4*sqrt(c*x^2 + b*x + a)*a)/(a^2*d*x), 1/4*(
a^2*d*x*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*
f)*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4
 - 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))/(c^2*d
^5 + a^2*d^3*f^2 - (b^2 - 2*a*c)*d^4*f))*log((2*b*c*f^3*x + b^2*f^3 + 2*(b*
c*d^2*f^2 + a*b*d*f^3 - (b*c^2*d^6 + a^2*b*d^4*f^2 - (b^3 - 2*a*b*c)*d^5*f)
*sqrt(b^2*f^5/(c^4*d^9 + a^4*d^5*f^4 - 2*(b^2*c^2 - 2*a*c^3)*d^8*f + (b^4 -
 4*a*b^2*c + 6*a^2*c^2)*d^7*f^2 - 2*(a^2*b^2 - 2*a^3*c)*d^6*f^3)))*sqrt(c*x
^2 + b*x + a)*sqrt((c*d*f^2 + a*f^3 + (c^2*d^5 ...
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-dx^2\sqrt{a+bx+cx^2}+fx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-d*x**2*sqrt(a + b*x + c*x**2) + f*x**4*sqrt(a + b*x + c*x**2)
), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueEvaluation time: 0.77
Done
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^2\,\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
[Out] int(1/(x^2*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

# 3.101 $\int \frac{1}{x^3\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx$

**Optimal**. Leaf size=376

$$-\frac{\sqrt{a+bx+cx^2}}{2adx^2}+\frac{3b\sqrt{a+bx+cx^2}}{4a^2dx}-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{8a^{5/2}d}-\frac{f\tanh^{-1}\left(\frac{2a+}{2\sqrt{a}\,\sqrt{a+}}\right.}{\sqrt{a}\,d^2}$$

```
[Out] -1/8*(-4*a*c+3*b^2)*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(5
/2)/d-f*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d^2/a^(1/2)-1/2*
(c*x^2+b*x+a)^(1/2)/a/d/x^2+3/4*b*(c*x^2+b*x+a)^(1/2)/a^2/d/x-1/2*f^(3/2)*a
rctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^
(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d^2/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1
/2)+1/2*f^(3/2)*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)
))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/d^2/(c*d+a*f+b*d^
(1/2)*f^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.49, antiderivative size = 376, normalized size of antiderivative = 1.00, number of steps used = 13, number of rules used = 6, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.214$, Rules used = {6857, 758, 820, 738, 212, 1047}

$$-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{8a^{5/2}d}+\frac{3b\sqrt{a+bx+cx^2}}{4a^2dx}-\frac{f^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^2\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}+\frac{f^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^2\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}-\frac{f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d^2}-\frac{\sqrt{a+bx+cx^2}}{2adx^2}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^3*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] -1/2*Sqrt[a + b*x + c*x^2]/(a*d*x^2) + (3*b*Sqrt[a + b*x + c*x^2])/(4*a^2*d
*x) - ((3*b^2 - 4*a*c)*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2]
)])/(8*a^(5/2)*d) - (f*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2]
)])/(Sqrt[a]*d^2) - (f^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d
] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x
^2])])/(2*d^2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]) + (f^(3/2)*ArcTanh[(b*Sq
rt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*
Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f]
 + a*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 758

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[1/((m + 1)*(c*d^2 - b*d*e + a*e^2)), Int[(
d + e*x)^(m + 1)*Simp[c*d*(m + 1) - b*e*(m + p + 2) - c*e*(m + 2*p + 3)*x,
x]*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, m, p}, x] && NeQ[b^
2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && Ne
Q[m, -1] && ((LtQ[m, -1] && IntQuadraticQ[a, b, c, d, e, m, p, x]) || (SumS
implerQ[m, 1] && IntegerQ[p]) || ILtQ[Simplify[m + 2*p + 3], 0])
```

Rule 820

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(e*f - d*g))*(d + e*x)^(m + 1)*((a +
 b*x + c*x^2)^(p + 1)/(2*(p + 1)*(c*d^2 - b*d*e + a*e^2))), x] - Dist[(b*(e
*f + d*g) - 2*(c*d*f + a*e*g))/(2*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(
m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && EqQ[Simplify[m
+ 2*p + 3], 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{1}{x^3\sqrt{a+bx+cx^2}\,(d-fx^2)}\,dx = \int \left(\frac{1}{dx^3\sqrt{a+bx+cx^2}} + \frac{f}{d^2x\sqrt{a+bx+cx^2}} + \frac{f^2x}{d^2\sqrt{a+bx+cx^2}\,(d\right.$$

$$= \frac{\int \frac{1}{x^3\sqrt{a+bx+cx^2}}\,dx}{d} + \frac{f\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{d^2} + \frac{f^2\int \frac{x}{\sqrt{a+bx+c}}}{d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} - \frac{\int \frac{\frac{3b}{2}+cx}{x^2\sqrt{a+bx+cx^2}}\,dx}{2ad} - \frac{(2f)\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx,\right.}{d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} - \frac{f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+c}}\right.}{\sqrt{a}\,d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} - \frac{f\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+c}}\right.}{\sqrt{a}\,d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} - \frac{(3b^2-4ac)\tanh^{-1}\left(\frac{}{2\sqrt{a}\sqrt{}}\right.}{8a^{5/2}d}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.75, size = 241, normalized size = 0.64

$$\frac{\frac{d(-2a+3bx)\sqrt{a+x(b+cx)}}{a^2x^2} - \frac{(3b^2d-4acd+8a^2f)\tanh^{-1}\left(\frac{-\sqrt{c}\,x+\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{5/2}} - 2f^2\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{a\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-b\sqrt{c}\,d+2cd\#1+af\#1-f\#1^3}\&\right]}{4d^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^3*Sqrt[a + b*x + c*x^2]*(d - f*x^2)),x]
```

```
[Out] ((d*(-2*a + 3*b*x)*Sqrt[a + x*(b + c*x)])/(a^2*x^2) - ((3*b^2*d - 4*a*c*d +
 8*a^2*f)*ArcTanh[(-(Sqrt[c]*x) + Sqrt[a + x*(b + c*x)])/Sqrt[a]])/a^(5/2)
- 2*f^2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2
- f*#1^4 & , (a*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(-(b*Sqrt[c]*d) + 2*c*d*#1 + a*f
*#1 - f*#1^3) & ])/(4*d^2)
```

**Maple** [A]
time = 0.14, size = 516, normalized size = 1.37

| method | result |
|---|---|
| risch | $-\frac{\sqrt{c\,x^{2}+b x+a}\,(-3 b x+2 a)}{4 a^{2} d\,x^{2}}-\frac{f \ln\left(\frac{2 a+b x+2 \sqrt{a}\,\sqrt{c\,x^{2}+b x+a}}{x}\right)}{d^{2} \sqrt{a}}+\frac{\ln\left(\frac{2 a+b x+2 \sqrt{a}\,\sqrt{c\,x^{2}+b x+a}}{x}\right)}{2 d\,a^{\frac{3}{2}}}$ |
| default | $\frac{-\frac{\sqrt{c\,x^{2}+b x+a}}{2 a\,x^{2}}-\frac{3 b\left(-\frac{\sqrt{c\,x^{2}+b x+a}}{a x}+\frac{b \ln\left(\frac{2 a+b x+2 \sqrt{a}\,\sqrt{c\,x^{2}+b x+a}}{x}\right)}{2 a^{\frac{3}{2}}}\right)}{4 a}+\frac{c \ln\left(\frac{2 a+b x+2 \sqrt{a}\,\sqrt{c\,x^{2}}}{x}\right)}{2 a^{\frac{3}{2}}}}{d}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/2/a/x^2*(c*x^2+b*x+a)^(1/2)-3/4*b/a*(-1/a/x*(c*x^2+b*x+a)^(1/2)+1/2
*b/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))+1/2*c/a^(3/2)*ln(
(2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))+1/2/d^2*f/(1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)
*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f
)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+
c*d))^(1/2))/(x+(d*f)^(1/2)/f))-f/d^2/a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+
b*x+a)^(1/2))/x)+1/2/d^2*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)
^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/
2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f
)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 - d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-dx^3\sqrt{a+bx+cx^2}+fx^5\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**3/(c*x**2+b*x+a)**(1/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-d*x**3*sqrt(a + b*x + c*x**2) + f*x**5*sqrt(a + b*x + c*x**2)
), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueEvaluation time: 0.8D
one
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^3\,\left(d-f\,x^2\right)\,\sqrt{c\,x^2+b\,x+a}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^3*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)),x)
```

```
[Out] int(1/(x^3*(d - f*x^2)*(a + b*x + c*x^2)^(1/2)), x)
```

## 3.102 $\int \frac{x^4}{\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=466

$$-\frac{2x(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}}+\frac{2d(b+2cx)}{\left(b^2-4ac\right)f^2\sqrt{a+bx+cx^2}}-\frac{2d^2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2}{\left(b^2-4ac\right)f^2\left(b^2df-(cd+af)^2\right)\sqrt{a+bx}}$$

```
[Out] -arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f+1/2*d^(3/2)*a
rctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^
(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/f/(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2
)+1/2*d^(3/2)*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))
/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/f/(c*d+a*f+b*d^(1/2
)*f^(1/2))^(3/2)-2*x*(b*x+2*a)/(-4*a*c+b^2)/f/(c*x^2+b*x+a)^(1/2)+2*d*(2*c*
x+b)/(-4*a*c+b^2)/f^2/(c*x^2+b*x+a)^(1/2)-2*d^2*(b*(b^2*f-c*(3*a*f+c*d))-c*
(2*a*c*f-b^2*f+2*c^2*d)*x)/(-4*a*c+b^2)/f^2/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*
x+a)^(1/2)+2*b*(c*x^2+b*x+a)^(1/2)/c/(-4*a*c+b^2)/f
```

**Rubi** [A]
time = 0.85, antiderivative size = 466, normalized size of antiderivative = 1.00, number of steps used = 13, number of rules used = 9, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.321$, Rules used = {6857, 627, 752, 654, 635, 212, 989, 1047, 738}

$$-\frac{2d^2\left(b\left(b^2f-c(3af+cd)\right)-cx\left(2acf+b^2(-f)+2c^2d\right)\right)}{f^2\left(b^2-4ac\right)\sqrt{a+bx+cx^2}\left(b^2df-(af+cd)^2\right)}+\frac{2d(b+2cx)}{f^2\left(b^2-4ac\right)\sqrt{a+bx+cx^2}}+\frac{2b\sqrt{a+bx+cx^2}}{cf\left(b^2-4ac\right)}-\frac{2x(2a+bx)}{f\left(b^2-4ac\right)\sqrt{a+bx+cx^2}}-\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{c^{3/2}f}+\frac{d^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2f\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{d^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2f\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^4/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-2*x*(2*a + b*x))/((b^2 - 4*a*c)*f*Sqrt[a + b*x + c*x^2]) + (2*d*(b + 2*c*
x))/((b^2 - 4*a*c)*f^2*Sqrt[a + b*x + c*x^2]) - (2*d^2*(b*(b^2*f - c*(c*d +
 3*a*f)) - c*(2*c^2*d - b^2*f + 2*a*c*f)*x))/((b^2 - 4*a*c)*f^2*(b^2*d*f -
(c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) + (2*b*Sqrt[a + b*x + c*x^2])/(c*(b^2
 - 4*a*c)*f) - ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])]/(c^(3
/2)*f) + (d^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[
f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]])/(2*f
*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)) + (d^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a
*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a
*f]*Sqrt[a + b*x + c*x^2])]])/(2*f*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 627

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-3/2), x_Symbol] :> Simp[-2*((b +
2*c*x)/((b^2 - 4*a*c)*Sqrt[a + b*x + c*x^2])), x] /; FreeQ[{a, b, c}, x] &&
 NeQ[b^2 - 4*a*c, 0]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 752

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m - 1)*(d*b - 2*a*e + (2*c*d - b*e)*x)*((a + b*x
+ c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c))), x] + Dist[1/((p + 1)*(b^2 - 4*a*
c)), Int[(d + e*x)^(m - 2)*Simp[e*(2*a*e*(m - 1) + b*d*(2*p - m + 4)) - 2*c
*d^2*(2*p + 3) + e*(b*e - 2*d*c)*(m + 2*p + 2)*x, x]*(a + b*x + c*x^2)^(p +
 1), x], x] /; FreeQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^
2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p, -1] && GtQ[m, 1] &&
IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 989

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b^3*f + b*c*(c*d - 3*a*f) + c*(2*c^2*d + b^2*f - c*(2*a*f
))*x)*(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^(q + 1)/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1))), x] - Dist[1/((b^2 - 4*a*c)*(b^2*d*f + (c*d -
a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Simp[2*c*(b^2
```

```
*d*f + (c*d - a*f)^2)*(p + 1) - (2*c^2*d + b^2*f - c*(2*a*f))*(a*f*(p + 1)
- c*d*(p + 2)) + (2*f*(b^3*f + b*c*(c*d - 3*a*f))*(p + q + 2) - (2*c^2*d +
b^2*f - c*(2*a*f))*(b*f*(p + 1)))*x + c*f*(2*c^2*d + b^2*f - c*(2*a*f))*(2*
p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x] && NeQ[b^2 - 4
*a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[
p] && ILtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{x^4}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx = \int \left(-\frac{d}{f^2(a+bx+cx^2)^{3/2}} - \frac{x^2}{f(a+bx+cx^2)^{3/2}} + \frac{d^2}{f^2(a+bx+cx^2)^{3/2}\right.$$

$$= -\frac{d\int \frac{1}{(a+bx+cx^2)^{3/2}}\,dx}{f^2} + \frac{d^2\int \frac{1}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx}{f^2} - \frac{\int \frac{x^2}{(a+bx+cx^2)^{3/2}}\,dx}{f}$$

$$= -\frac{2x(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2d(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} - \frac{2d^2(b(b}{(b^2-}$$

$$= -\frac{2x(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2d(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} - \frac{2d^2(b(b}{(b^2-}$$

$$= -\frac{2x(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2d(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} - \frac{2d^2(b(b}{(b^2-}$$

$$= -\frac{2x(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2d(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} - \frac{2d^2(b(b}{(b^2-}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.25, size = 453, normalized size = 0.97

$$\frac{2(b^4dx+ab^2d(b-4cx)-a^3f(b-2cx)-a^2(3bcd-2c^2dx+b^2fx))}{c(-b^2+4ac)(c^2d^2+2acdf+f(-b^2d+a^2f))\sqrt{a+x(b+cx)}} + \frac{\log\left(cf\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)}{c^{3/2}f} - \frac{d^2\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)+2abf\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1-bf\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}d-2cd\#1-af\#1+f\#1^3}\&\right]}{2f(c^2d^2+2acdf+f(-b^2d+a^2f))}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^4/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (2*(b^4*d*x + a*b^2*d*(b - 4*c*x) - a^3*f*(b - 2*c*x) - a^2*(3*b*c*d - 2*c^
2*d*x + b^2*f*x)))/(c*(-b^2 + 4*a*c)*(c^2*d^2 + 2*a*c*d*f + f*(-(b^2*d) + a
^2*f))*Sqrt[a + x*(b + c*x)]) + Log[c*f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(
b + c*x)])]/(c^(3/2)*f) - (d^2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4
*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
+ c*x^2] - #1] + 2*a*b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2
*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*
f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - b*f*Log[-(Sqrt[c]*x)
+ Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1
^3) & ])/(2*f*(c^2*d^2 + 2*a*c*d*f + f*(-(b^2*d) + a^2*f)))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1063 vs. $2(394) = 788$.

time = 0.13, size = 1064, normalized size = 2.28

| method | result |
|---|---|
| default | $-\frac{-\frac{x}{c\sqrt{c\,x^{2}+b x +a}}-\frac{b \left(-\frac{1}{c\sqrt{c\,x^{2}+b x +a}}-\frac{b \left(2 c x +b \right)}{c \left(4 a c -b^{2}\right) \sqrt{c\,x^{2}+b x +a}}\right)}{2 c}+\frac{\ln \left(\frac{\frac{b}{2}+c x}{\sqrt{c}}+\sqrt{c\,x^{2}+b x +a}\right)}{c^{\frac{3}{2}}}}{f}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(-x/c/(c*x^2+b*x+a)^(1/2)-1/2*b/c*(-1/c/(c*x^2+b*x+a)^(1/2)-b/c*(2*c*x
+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2))+1/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^
2+b*x+a)^(1/2)))-2*d/f^2*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)+1/2/f^2*
d^2/(d*f)^(1/2)*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*
c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-
2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-
2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/
2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)
/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b
*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*
f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+
(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*
f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2/f^2*d^2/(d*f)^(1/2)*(1/(b
*(d*f)^(1/2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(
d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*
f)^(1/2)+f*a+c*d)*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(
d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(
2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1
/(b*(d*f)^(1/2)+f*a+c*d)*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)
^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/
2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f
)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^4}{-ad\sqrt{a+bx+cx^2}+afx^2\sqrt{a+bx+cx^2}-bdx\sqrt{a+bx+cx^2}+bfx^3\sqrt{a+bx+cx^2}-cdx^2\sqrt{a+bx+cx^2}+cfx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**4/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**4/(-a*d*sqrt(a + b*x + c*x**2) + a*f*x**2*sqrt(a + b*x + c*x**
2) - b*d*x*sqrt(a + b*x + c*x**2) + b*f*x**3*sqrt(a + b*x + c*x**2) - c*d*x
**2*sqrt(a + b*x + c*x**2) + c*f*x**4*sqrt(a + b*x + c*x**2)), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^4}{(d - f\,x^2)\,\left(c\,x^2 + b\,x + a\right)^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
[Out] int(x^4/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.103 $\int \frac{x^3}{\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=341

$$-\frac{2(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}}-\frac{2d\left(a\left(2c^2d-b^2f+2acf\right)+bc(cd-af)x\right)}{\left(b^2-4ac\right)f\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}}-\frac{d\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{\ }}{2\sqrt{cd-b\sqrt{d}}\ \sqrt{\ }}\right.}{2\sqrt{f}\ \left(cd-b\sqrt{\ }\right.}$$

```
[Out] -1/2*d*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2
+b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/f^(1/2)/(c*d+a*f-b*d^(1/2)
*f^(1/2))^(3/2)+1/2*d*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f
^(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/f^(1/2)/(c*
d+a*f+b*d^(1/2)*f^(1/2))^(3/2)-2*(b*x+2*a)/(-4*a*c+b^2)/f/(c*x^2+b*x+a)^(1/
2)-2*d*(a*(2*a*c*f-b^2*f+2*c^2*d)+b*c*(-a*f+c*d)*x)/(-4*a*c+b^2)/f/(b^2*d*f
-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]

time = 0.67, antiderivative size = 341, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.214$, Rules used = {6857, 650, 1032, 1047, 738, 212}

$$-\frac{2d\left(a\left(2acf+b^2(-f)+2c^2d\right)+bcx(cd-af)\right)}{f\left(b^2-4ac\right)\sqrt{a+bx+cx^2}\left(b^2df-(af+cd)^2\right)}-\frac{2(2a+bx)}{f\left(b^2-4ac\right)\sqrt{a+bx+cx^2}}-\frac{d\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{f}\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{d\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\sqrt{f}\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-2*(2*a + b*x))/((b^2 - 4*a*c)*f*Sqrt[a + b*x + c*x^2]) - (2*d*(a*(2*c^2*d
 - b^2*f + 2*a*c*f) + b*c*(c*d - a*f)*x))/((b^2 - 4*a*c)*f*(b^2*d*f - (c*d
+ a*f)^2)*Sqrt[a + b*x + c*x^2]) - (d*ArcTanh[(b*Sqrt[d] - 2*a*Sqrt[f] + (2
*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a +
b*x + c*x^2])]])/(2*Sqrt[f]*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)) + (d*ArcT
anh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*Sqrt[c*d + b
*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]])/(2*Sqrt[f]*(c*d + b*Sqrt[d
]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 650

```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x] && NeQ[2*c*d - b*e, 0] && NeQ[b
^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{x^3}{\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx = \int \left(-\frac{x}{f\left(a+bx+cx^2\right)^{3/2}} + \frac{dx}{f\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\right) dx$$

$$= -\frac{\int \frac{x}{(a+bx+cx^2)^{3/2}}\,dx}{f} + \frac{d\int \frac{x}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx}{f}$$

$$= -\frac{2(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}} - \frac{2d\left(a\left(2c^2d-b^2f+2acf\right)+bc(cd-\right.}{\left(b^2-4ac\right)f\left(b^2df-(cd+af)^2\right)\sqrt{a+}}$$

$$= -\frac{2(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}} - \frac{2d\left(a\left(2c^2d-b^2f+2acf\right)+bc(cd-\right.}{\left(b^2-4ac\right)f\left(b^2df-(cd+af)^2\right)\sqrt{a+}}$$

$$= -\frac{2(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}} - \frac{2d\left(a\left(2c^2d-b^2f+2acf\right)+bc(cd-\right.}{\left(b^2-4ac\right)f\left(b^2df-(cd+af)^2\right)\sqrt{a+}}$$

$$= -\frac{2(2a+bx)}{\left(b^2-4ac\right)f\sqrt{a+bx+cx^2}} - \frac{2d\left(a\left(2c^2d-b^2f+2acf\right)+bc(cd-\right.}{\left(b^2-4ac\right)f\left(b^2df-(cd+af)^2\right)\sqrt{a+}}$$

**<u>Mathematica</u>** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.77, size = 410, normalized size = 1.20

$$\frac{8a^3f-4b^3dx-4abd(b-3cx)+4a^2(2cd+bfx)-\left(b^2-4ac\right)d\sqrt{a+x(b+cx)}\,\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+acd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+a^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-cd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-af\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2\left(b^2-4ac\right)\left(-c^2d^2-2acdf+f\left(b^2d-a^2f\right)\right)\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (8*a^3*f - 4*b^3*d*x - 4*a*b*d*(b - 3*c*x) + 4*a^2*(2*c*d + b*f*x) - (b^2 -
 4*a*c)*d*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 +
4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x
 + c*x^2] - #1] + a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^
2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c*d*Log[-(Sqrt[c]*x) + Sqrt[a +
 b*x + c*x^2] - #1]*#1^2 - a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #
1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(2*(b^2 - 4*a*c)*(
-(c^2*d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*Sqrt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 959 vs. $2(281) = 562$.

time = 0.13, size = 960, normalized size = 2.82

| method | result |
|---|---|
| default | $-\frac{-\frac{1}{c\sqrt{c\,x^{2}+b x+a}}-\frac{b(2cx+b)}{c\left(4ac-b^{2}\right)\sqrt{c\,x^{2}+b x+a}}}{f}-\frac{d\left(\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^{2}c+\frac{\left(-2c\sqrt{df}\right.}{}}}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/f*(-1/c/(c*x^2+b*x+a)^(1/2)-b/c*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2
))-1/2*d/f^2*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(
d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-2*c
*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c
*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+
b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)
+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d
*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^
(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*
f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^
(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2*d/f^2*(1/(b*(d*f)^(1/2)+f*a+
c*d)*f/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*
(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)
*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*
d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b
*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^(1/2)+f
*a+c*d)*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f
+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1
/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*
f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [F(-2)]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 17339 vs. $2(281) = 562$.
time = 15.12, size = 17339, normalized size = 50.85

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*(((a*b^2*c^2 - 4*a^2*c^3)*d^2 - (a*b^4 - 6*a^2*b^2*c + 8*a^3*c^2)*d*f +
 (a^3*b^2 - 4*a^4*c)*f^2 + ((b^2*c^3 - 4*a*c^4)*d^2 - (b^4*c - 6*a*b^2*c^2
+ 8*a^2*c^3)*d*f + (a^2*b^2*c - 4*a^3*c^2)*f^2)*x^2 + ((b^3*c^2 - 4*a*b*c^3
)*d^2 - (b^5 - 6*a*b^3*c + 8*a^2*b*c^2)*d*f + (a^2*b^3 - 4*a^3*b*c)*f^2)*x)
*sqrt((c^3*d^5 + a^3*d^2*f^3 + 3*(b^2*c + a*c^2)*d^4*f + 3*(a*b^2 + a^2*c)*
d^3*f^2 + (c^6*d^6*f + a^6*f^7 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f^2 + 3*(b^4*c^2
 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^3 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 2
0*a^3*c^3)*d^3*f^4 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^5 - 3*(a^4
*b^2 - 2*a^5*c)*d*f^6)*sqrt((9*b^2*c^4*d^9 + 9*a^4*b^2*d^5*f^4 + 6*(b^4*c^2
 + 6*a*b^2*c^3)*d^8*f + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^7*f^2 + 6*(a^
2*b^4 + 6*a^3*b^2*c)*d^6*f^3)/(c^12*d^12*f + a^12*f^13 - 6*(b^2*c^10 - 2*a*
c^11)*d^11*f^2 + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^3 - 10*(
2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^4 + 15*(b^8*c
^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^5 -
6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2
*c^6 - 132*a^5*c^7)*d^7*f^6 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^
3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^7 - 6*
(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c
^4 - 132*a^7*c^5)*d^5*f^8 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48
*a^7*b^2*c^3 + 33*a^8*c^4)*d^4*f^9 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*
b^2*c^2 - 22*a^9*c^3)*d^3*f^10 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)
*d^2*f^11 - 6*(a^10*b^2 - 2*a^11*c)*d*f^12)))/(c^6*d^6*f + a^6*f^7 - 3*(b^2
*c^4 - 2*a*c^5)*d^5*f^2 + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^3 - (
b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^4 + 3*(a^2*b^4 - 4*a^3
```

```
*b^2*c + 5*a^4*c^2)*d^2*f^5 - 3*(a^4*b^2 - 2*a^5*c)*d*f^6))*log((3*b^2*c^2*
d^6 + 3*a^2*b^2*d^4*f^2 + (b^4 + 6*a*b^2*c)*d^5*f + 2*(3*b*c^3*d^6 + 3*a^2*
b*c*d^4*f^2 + (b^3*c + 6*a*b*c^2)*d^5*f)*x + 2*(6*b^2*c^3*d^6*f + 6*a^3*b^2
*d^3*f^4 + 2*(b^4*c + 9*a*b^2*c^2)*d^5*f^2 + 2*(a*b^4 + 9*a^2*b^2*c)*d^4*f^
3 - (c^8*d^8*f + a^8*f^9 - 2*(b^2*c^6 - 4*a*c^7)*d^7*f^2 - 4*(3*a*b^2*c^5 -
 7*a^2*c^6)*d^6*f^3 + 2*(b^6*c^2 - 15*a^2*b^2*c^4 + 28*a^3*c^5)*d^5*f^4 - (
b^8 - 4*a*b^6*c + 40*a^3*b^2*c^3 - 70*a^4*c^4)*d^4*f^5 + 2*(a^2*b^6 - 15*a^
4*b^2*c^2 + 28*a^5*c^3)*d^3*f^6 - 4*(3*a^5*b^2*c - 7*a^6*c^2)*d^2*f^7 - 2*(
a^6*b^2 - 4*a^7*c)*d*f^8)*sqrt((9*b^2*c^4*d^9 + 9*a^4*b^2*d^5*f^4 + 6*(b^4*
c^2 + 6*a*b^2*c^3)*d^8*f + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^7*f^2 + 6*
(a^2*b^4 + 6*a^3*b^2*c)*d^6*f^3)/(c^12*d^12*f + a^12*f^13 - 6*(b^2*c^10 - 2
*a*c^11)*d^11*f^2 + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^3 - 1
0*(2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^4 + 15*(b^
8*c^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^5
 - 6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*
b^2*c^6 - 132*a^5*c^7)*d^7*f^6 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400
*a^3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^7 -
 6*(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^
2*c^4 - 132*a^7*c^5)*d^5*f^8 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 -
 48*a^7*b^2*c^3 + 33*a^8*c^4)*d^4*f^9 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a
^8*b^2*c^2 - 22*a^9*c^3)*d^3*f^10 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c
^2)*d^2*f^11 - 6*(a^10*b^2 - 2*a^11*c)*d*f^12)))*sqrt(c*x^2 + b*x + a)*sqrt
((c^3*d^5 + a^3*d^2*f^3 + 3*(b^2*c + a*c^2)*d^4*f + 3*(a*b^2 + a^2*c)*d^3*f
^2 + (c^6*d^6*f + a^6*f^7 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f^2 + 3*(b^4*c^2 - 4*
a*b^2*c^3 + 5*a^2*c^4)*d^4*f^3 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3
*c^3)*d^3*f^4 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^5 - 3*(a^4*b^2
- 2*a^5*c)*d*f^6)*sqrt((9*b^2*c^4*d^9 + 9*a^4*b^2*d^5*f^4 + 6*(b^4*c^2 + 6*
a*b^2*c^3)*d^8*f + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^7*f^2 + 6*(a^2*b^4
 + 6*a^3*b^2*c)*d^6*f^3)/(c^12*d^12*f + a^12*f^13 - 6*(b^2*c^10 - 2*a*c^11)
*d^11*f^2 + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^3 - 10*(2*b^6
*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^4 + 15*(b^8*c^4 -
8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^5 - 6*(b^
10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6
- 132*a^5*c^7)*d^7*f^6 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^6
*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^7 - 6*(a^2*
b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 -
132*a^7*c^5)*d^5*f^8 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*
b^2*c^3 + 33*a^8*c^4)*d^4*f^9 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c
^2 - 22*a^9*c^3)*d^3*f^10 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^2*
f^11 - 6*(a^10*b^2 - 2*a^11*c)*d*f^12)))/(c^6*d^6*f + a^6*f^7 - 3*(b^2*c^4
- 2*a*c^5)*d^5*f^2 + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^3 - (b^6 -
 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^4 + 3*(a^2*b^4 - 4*a^3*b^2*
c + 5*a^4*c^2)*d^2*f^5 - 3*(a^4*b^2 - 2*a^5*c)*...
```

**Sympy** [**F**]

time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^3}{-ad\sqrt{a+bx+cx^2}+afx^2\sqrt{a+bx+cx^2}-bdx\sqrt{a+bx+cx^2}+bfx^3\sqrt{a+bx+cx^2}-cdx^2\sqrt{a+bx+cx^2}+cfx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**3/(-a*d*sqrt(a + b*x + c*x**2) + a*f*x**2*sqrt(a + b*x + c*x**
2) - b*d*x*sqrt(a + b*x + c*x**2) + b*f*x**3*sqrt(a + b*x + c*x**2) - c*d*x
**2*sqrt(a + b*x + c*x**2) + c*f*x**4*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueDone
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(x^3/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.104 $\int \frac{x^2}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx$

**Optimal**. Leaf size=297

$$\frac{2(ab(cd-af)+c(b^2d-2a(cd+af))\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\left(cd-b\sqrt{d}\,\sqrt{f}+af\right)^{3/2}}+$$

```
[Out] 1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*
x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)/(c*d+a*f-b*d^(1/2)*f^
(1/2))^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2
)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*d^(1/2)/(c*d+a*f
+b*d^(1/2)*f^(1/2))^(3/2)+2*(a*b*(-a*f+c*d)+c*(b^2*d-2*a*(a*f+c*d))*x)/(-4*
a*c+b^2)/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]
time = 0.28, antiderivative size = 297, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {1079, 1047, 738, 212}

$$\frac{2(cx(b^2d-2a(af+cd))+ab(cd-af))}{(b^2-4ac)\sqrt{a+bx+cx^2}\,(b^2df-(af+cd)^2)}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{\sqrt{d}\,\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}\right)}{2\left(af+b\sqrt{d}\,\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (2*(a*b*(c*d - a*f) + c*(b^2*d - 2*a*(c*d + a*f))*x))/((b^2 - 4*a*c)*(b^2*d
*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) + (Sqrt[d]*ArcTanh[(b*Sqrt[d] -
2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f]
+ a*f]*Sqrt[a + b*x + c*x^2])])/(2*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)) +
 (Sqrt[d]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(
2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*(c*d + b*
Sqrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 1079

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) +
(f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2
)^(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((A*c - a*C)*(
(-b)*(c*d + a*f)) + (A*b)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(A*(2*c^2*d + b
^2*f - c*(2*a*f)) + C*(b^2*d - 2*a*(c*d - a*f)))*x), x] + Dist[1/((b^2 - 4*
a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d +
 f*x^2)^q*Simp[(-2*A*c - 2*a*C)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) +
(b^2*(C*d + A*f) + 2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(a*f*(p + 1) -
c*d*(p + 2)) - (2*f*((A*c - a*C)*((-b)*(c*d + a*f)) + (A*b)*(2*c^2*d + b^2*
f - c*(2*a*f)))*(p + q + 2) - (b^2*(C*d + A*f) + 2*(A*c*(c*d - a*f) - a*(c*
C*d - a*C*f)))*(b*f*(p + 1)))*x - c*f*(b^2*(C*d + A*f) + 2*(A*c*(c*d - a*f)
 - a*(c*C*d - a*C*f)))*(2*p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d
, f, A, C, q}, x] && NeQ[b^2 - 4*a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*
d - a*f)^2, 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rubi steps

$$\int \frac{x^2}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx = \frac{2(ab(cd-af)+c(b^2d-2a(cd+af))\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}} + \frac{2\int \frac{-\frac{1}{2}(b^2-4ac)\,d(cd+af)\cdots}{\sqrt{a+bx+cx^2\cdots}}}{(b^2-4ac)\,(b^2df-\cdots}$$

$$= \frac{2(ab(cd-af)+c(b^2d-2a(cd+af))\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}} - \frac{\left(\sqrt{d}\,\sqrt{f}\right)\int \frac{\cdots}{\left(-\sqrt{d}\cdots\right.}}{2\left(cd-\cdots\right.}$$

$$= \frac{2(ab(cd-af)+c(b^2d-2a(cd+af))\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}} + \frac{\left(\sqrt{d}\,\sqrt{f}\right)\mathrm{Subst}\left(\cdots\right.}{\cdots}$$

$$= \frac{2(ab(cd-af)+c(b^2d-2a(cd+af))\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}} + \frac{\sqrt{d}\,\tanh^{-1}\left(\frac{b\cdots}{2\sqrt{cd\cdots}}\right.}{2\left(cd\cdots\right.}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.72, size = 371, normalized size = 1.25

$$\frac{4b^2cdx+4acd(b-2cx)-4a^2f(b+2cx)+(b^2-4ac)\,d\sqrt{a+x(b+cx)}\,\mathrm{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+2abf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}\,f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-bf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2(b^2-4ac)\left(-c^2d^2-2acdf+f\left(b^2d-a^2f\right)\right)\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (4*b^2*c*d*x + 4*a*c*d*(b - 2*c*x) - 4*a^2*f*(b + 2*c*x) + (b^2 - 4*a*c)*d*
Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2
 + 2*a*f*#1^2 - f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1] + 2*a*b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*c^(3/2)*
d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*f*Log[-(S
qrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - b*f*Log[-(Sqrt[c]*x) + Sqrt[a
+ b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/
(2*(b^2 - 4*a*c)*(-(c^2*d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*Sqrt[a + x*(b
 + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 945 vs. 2(239) = 478.
time = 0.14, size = 946, normalized size = 3.19

| method | result |
|---|---|

| | |
|---|---|
| default | $-\frac{2(2cx+b)}{f(4ac-b^2)\sqrt{c\,x^2+bx+a}}+\frac{d\left(\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\cdots}{}}}\cdots\right.}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -2/f*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)+1/2*d/(d*f)^(1/2)/f*(f/(-b*(
d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d
*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-2*c*(d*f)^(1/2)+b*f)/(-b
*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/(4
*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/((x+(d*f)^(1/
2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(
1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(
1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(
-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)
)/(x+(d*f)^(1/2)/f)))-1/2*d/(d*f)^(1/2)/f*(1/(b*(d*f)^(1/2)+f*a+c*d)*f/((x-
(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)
+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x-(d
*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(
d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d
*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^(1/2)+f*a+c*d)*f/(
(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)
^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*
f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*
a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 17285 vs. $2(239) = 478$.
time = 13.21, size = 17285, normalized size = 58.20

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*(((a*b^2*c^2 - 4*a^2*c^3)*d^2 - (a*b^4 - 6*a^2*b^2*c + 8*a^3*c^2)*d*f +
 (a^3*b^2 - 4*a^4*c)*f^2 + ((b^2*c^3 - 4*a*c^4)*d^2 - (b^4*c - 6*a*b^2*c^2
+ 8*a^2*c^3)*d*f + (a^2*b^2*c - 4*a^3*c^2)*f^2)*x^2 + ((b^3*c^2 - 4*a*b*c^3
)*d^2 - (b^5 - 6*a*b^3*c + 8*a^2*b*c^2)*d*f + (a^2*b^3 - 4*a^3*b*c)*f^2)*x)
*sqrt((c^3*d^4 + a^3*d*f^3 + 3*(b^2*c + a*c^2)*d^3*f + 3*(a*b^2 + a^2*c)*d^
2*f^2 + (c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a
*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*
c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 -
 2*a^5*c)*d*f^5)*sqrt((9*b^2*c^4*d^7*f + 9*a^4*b^2*d^3*f^5 + 6*(b^4*c^2 + 6
*a*b^2*c^3)*d^6*f^2 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^5*f^3 + 6*(a^2*
b^4 + 6*a^3*b^2*c)*d^4*f^4)/(c^12*d^12 + a^12*f^12 - 6*(b^2*c^10 - 2*a*c^11
)*d^11*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^2 - 10*(2*b^6*
c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^3 + 15*(b^8*c^4 - 8
*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^4 - 6*(b^1
0*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6 -
 132*a^5*c^7)*d^7*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^6*
c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^6 - 6*(a^2*b
^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 - 1
32*a^7*c^5)*d^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*b
^2*c^3 + 33*a^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c^
2 - 22*a^9*c^3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^2*f^
10 - 6*(a^10*b^2 - 2*a^11*c)*d*f^11)))/(c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*
a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b
^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*
a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 - 2*a^5*c)*d*f^5))*log((3*b^2*c^2*d^4 + 3*a^2
*b^2*d^2*f^2 + (b^4 + 6*a*b^2*c)*d^3*f + 2*(3*b*c^3*d^4 + 3*a^2*b*c*d^2*f^2
 + (b^3*c + 6*a*b*c^2)*d^3*f)*x + 2*(3*b*c^4*d^5 + 3*a^4*b*d*f^4 + 4*(b^3*c
^2 + 3*a*b*c^3)*d^4*f + (b^5 + 8*a*b^3*c + 18*a^2*b*c^2)*d^3*f^2 + 4*(a^2*b
^3 + 3*a^3*b*c)*d^2*f^3 - 2*(b*c^7*d^7 + a^7*b*f^7 - (3*b^3*c^5 - 7*a*b*c^6
)*d^6*f + 3*(b^5*c^3 - 5*a*b^3*c^4 + 7*a^2*b*c^5)*d^5*f^2 - (b^7*c - 9*a*b^
```

```
5*c^2 + 30*a^2*b^3*c^3 - 35*a^3*b*c^4)*d^4*f^3 - (a*b^7 - 9*a^2*b^5*c + 30*
a^3*b^3*c^2 - 35*a^4*b*c^3)*d^3*f^4 + 3*(a^3*b^5 - 5*a^4*b^3*c + 7*a^5*b*c^
2)*d^2*f^5 - (3*a^5*b^3 - 7*a^6*b*c)*d*f^6)*sqrt((9*b^2*c^4*d^7*f + 9*a^4*b
^2*d^3*f^5 + 6*(b^4*c^2 + 6*a*b^2*c^3)*d^6*f^2 + (b^6 + 12*a*b^4*c + 54*a^2
*b^2*c^2)*d^5*f^3 + 6*(a^2*b^4 + 6*a^3*b^2*c)*d^4*f^4)/(c^12*d^12 + a^12*f^
12 - 6*(b^2*c^10 - 2*a*c^11)*d^11*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*
c^10)*d^10*f^2 - 10*(2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9
)*d^9*f^3 + 15*(b^8*c^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 3
3*a^4*c^8)*d^8*f^4 - 6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*
b^4*c^5 + 210*a^4*b^2*c^6 - 132*a^5*c^7)*d^7*f^5 + (b^12 - 12*a*b^10*c + 90
*a^2*b^8*c^2 - 400*a^3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*
a^6*c^6)*d^6*f^6 - 6*(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^
4*c^3 + 210*a^6*b^2*c^4 - 132*a^7*c^5)*d^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c
+ 28*a^6*b^4*c^2 - 48*a^7*b^2*c^3 + 33*a^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 1
2*a^7*b^4*c + 27*a^8*b^2*c^2 - 22*a^9*c^3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*
b^2*c + 22*a^10*c^2)*d^2*f^10 - 6*(a^10*b^2 - 2*a^11*c)*d*f^11)))*sqrt(c*x^
2 + b*x + a)*sqrt((c^3*d^4 + a^3*d*f^3 + 3*(b^2*c + a*c^2)*d^3*f + 3*(a*b^2
 + a^2*c)*d^2*f^2 + (c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f + 3*(b
^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c
^2 - 20*a^3*c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^4 -
3*(a^4*b^2 - 2*a^5*c)*d*f^5)*sqrt((9*b^2*c^4*d^7*f + 9*a^4*b^2*d^3*f^5 + 6*
(b^4*c^2 + 6*a*b^2*c^3)*d^6*f^2 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^5*f
^3 + 6*(a^2*b^4 + 6*a^3*b^2*c)*d^4*f^4)/(c^12*d^12 + a^12*f^12 - 6*(b^2*c^1
0 - 2*a*c^11)*d^11*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^2
- 10*(2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^3 + 15*
(b^8*c^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*
f^4 - 6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a
^4*b^2*c^6 - 132*a^5*c^7)*d^7*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 -
400*a^3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^
6 - 6*(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6
*b^2*c^4 - 132*a^7*c^5)*d^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^
2 - 48*a^7*b^2*c^3 + 33*a^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 2
7*a^8*b^2*c^2 - 22*a^9*c^3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10
*c^2)*d^2*f^10 - 6*(a^10*b^2 - 2*a^11*c)*d*f^11)))/(c^6*d^6 + a^6*f^6 - 3*(
b^2*c^4 - 2*a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 -
(b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^
3*b^2*c + 5*a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 - 2*a...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x^2}{-ad\sqrt{a+bx+cx^2}+afx^2\sqrt{a+bx+cx^2}-bdx\sqrt{a+bx+cx^2}+bfx^3\sqrt{a+bx+cx^2}-cdx^2\sqrt{a+bx+cx^2}+cfx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x**2/(-a*d*sqrt(a + b*x + c*x**2) + a*f*x**2*sqrt(a + b*x + c*x**
2) - b*d*x*sqrt(a + b*x + c*x**2) + b*f*x**3*sqrt(a + b*x + c*x**2) - c*d*x
**2*sqrt(a + b*x + c*x**2) + c*f*x**4*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueDone
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(x^2/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.105 $\int \frac{x}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx$

**Optimal**. Leaf size=299

$$-\frac{2(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}}-\frac{\sqrt{f}\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\sqrt{f}+af}\sqrt{a+bx+cx^2}}\right)}{2\left(cd-b\sqrt{d}\sqrt{f}+af\right)^{3/2}}$$

```
[Out] -1/2*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b
*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)/(c*d+a*f-b*d^(1/2)*f
^(1/2))^(3/2)+1/2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/
2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))*f^(1/2)/(c*d+a*
f+b*d^(1/2)*f^(1/2))^(3/2)-2*(a*(2*a*c*f-b^2*f+2*c^2*d)+b*c*(-a*f+c*d)*x)/(
-4*a*c+b^2)/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]

time = 0.24, antiderivative size = 299, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 26, $\frac{\text{number of rules}}{\text{integrand size}} = 0.154$, Rules used = {1032, 1047, 738, 212}

$$-\frac{2(a(2acf+b^2(-f)+2c^2d)+bcx(cd-af))}{(b^2-4ac)\sqrt{a+bx+cx^2}\left(b^2df-(af+cd)^2\right)}-\frac{\sqrt{f}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{\sqrt{f}\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[x/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-2*(a*(2*c^2*d - b^2*f + 2*a*c*f) + b*c*(c*d - a*f)*x))/((b^2 - 4*a*c)*(b^
2*d*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) - (Sqrt[f]*ArcTanh[(b*Sqrt[d]
 - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[
f] + a*f]*Sqrt[a + b*x + c*x^2])]])/(2*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)
) + (Sqrt[f]*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x
)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])]])/(2*(c*d +
 b*Sqrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{x}{\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx = -\frac{2(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}} + \frac{2\int \frac{\frac{1}{2}b(b^2-4ac)df-\frac{1}{2}(b^2-}{\sqrt{a+bx+ca}}}{(b^2-4ac)\left(b^2df-\right.}$$

$$= -\frac{2(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}} + \frac{f\int \frac{1}{\left(-\sqrt{d}\sqrt{f}-fx\right)}}{2\left(cd-b\sqrt{d}\right.}$$

$$= -\frac{2(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}} - \frac{f\text{Subst}\left(\int \frac{}{4cdf-4b\sqrt{c}}\right.}{}$$

$$= -\frac{2(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}} - \frac{\sqrt{f}\tanh^{-1}\left(\frac{}{2\sqrt{cd}}\right.}{2\left(cd\right.}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.74, size = 407, normalized size = 1.36

$$\frac{-8a^2cf-4bc^2dx+4a(-2c^2d+b^2f+bcfx)-(b^2-4ac)f\sqrt{a+x(b+cx)}\,\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)+acd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)+a^2f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1-cd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-af\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}d-2cd\#1-af\#1+f\#1^3}\&\right]}{2(b^2-4ac)\left(-c^2d^2-2acdf+f(b^2d-a^2f)\right)\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-8*a^2*c*f - 4*b*c^2*d*x + 4*a*(-2*c^2*d + b^2*f + b*c*f*x) - (b^2 - 4*a*c
)*f*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*
#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1] + a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*f*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1]*#1^2 - a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^
2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(2*(b^2 - 4*a*c)*(-(c^2*
d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*Sqrt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 898 vs. 2(241) = 482.
time = 0.12, size = 899, normalized size = 3.01

| method | result |
|---|---|

| | |
|---|---|
| default | $-\frac{\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}}-\frac{}{\left(-b\sqrt{df}+fa+cd\right.}}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2/f*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(
1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-2*c*(d*f)
^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)
^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2
)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(
-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1
/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+
b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/
2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+
f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2/f*(1/(b*(d*f)^(1/2)+f*a+c*d)*f/((x
-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2
)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x-(
d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*
(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(
d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^(1/2)+f*a+c*d)*f/
((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f
)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d
*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f
*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
```

```
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 17258 vs. $2(241) = 482$.
time = 14.93, size = 17258, normalized size = 57.72

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*(((a*b^2*c^2 - 4*a^2*c^3)*d^2 - (a*b^4 - 6*a^2*b^2*c + 8*a^3*c^2)*d*f +
 (a^3*b^2 - 4*a^4*c)*f^2 + ((b^2*c^3 - 4*a*c^4)*d^2 - (b^4*c - 6*a*b^2*c^2
+ 8*a^2*c^3)*d*f + (a^2*b^2*c - 4*a^3*c^2)*f^2)*x^2 + ((b^3*c^2 - 4*a*b*c^3
)*d^2 - (b^5 - 6*a*b^3*c + 8*a^2*b*c^2)*d*f + (a^2*b^3 - 4*a^3*b*c)*f^2)*x)
*sqrt((c^3*d^3*f + a^3*f^4 + 3*(b^2*c + a*c^2)*d^2*f^2 + 3*(a*b^2 + a^2*c)*
d*f^3 + (c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a
*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*
c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 -
 2*a^5*c)*d*f^5)*sqrt((9*b^2*c^4*d^5*f^3 + 9*a^4*b^2*d*f^7 + 6*(b^4*c^2 + 6
*a*b^2*c^3)*d^4*f^4 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^3*f^5 + 6*(a^2*
b^4 + 6*a^3*b^2*c)*d^2*f^6)/(c^12*d^12 + a^12*f^12 - 6*(b^2*c^10 - 2*a*c^11
)*d^11*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^2 - 10*(2*b^6*
c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^3 + 15*(b^8*c^4 - 8
*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^4 - 6*(b^1
0*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6 -
 132*a^5*c^7)*d^7*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^6*
c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^6 - 6*(a^2*b
^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 - 1
32*a^7*c^5)*d^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*b
^2*c^3 + 33*a^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c^
2 - 22*a^9*c^3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^2*f^
10 - 6*(a^10*b^2 - 2*a^11*c)*d*f^11)))/(c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*
a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b
^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*
a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 - 2*a^5*c)*d*f^5))*log((3*b^2*c^2*d^3*f^2 + 3
*a^2*b^2*d*f^4 + (b^4 + 6*a*b^2*c)*d^2*f^3 + 2*(3*b*c^3*d^3*f^2 + 3*a^2*b*c
*d*f^4 + (b^3*c + 6*a*b*c^2)*d^2*f^3)*x + 2*(6*b^2*c^3*d^4*f^2 + 6*a^3*b^2*
d*f^5 + 2*(b^4*c + 9*a*b^2*c^2)*d^3*f^3 + 2*(a*b^4 + 9*a^2*b^2*c)*d^2*f^4 -
 (c^8*d^8 + a^8*f^8 - 2*(b^2*c^6 - 4*a*c^7)*d^7*f - 4*(3*a*b^2*c^5 - 7*a^2*
c^6)*d^6*f^2 + 2*(b^6*c^2 - 15*a^2*b^2*c^4 + 28*a^3*c^5)*d^5*f^3 - (b^8 - 4
*a*b^6*c + 40*a^3*b^2*c^3 - 70*a^4*c^4)*d^4*f^4 + 2*(a^2*b^6 - 15*a^4*b^2*c
^2 + 28*a^5*c^3)*d^3*f^5 - 4*(3*a^5*b^2*c - 7*a^6*c^2)*d^2*f^6 - 2*(a^6*b^2
 - 4*a^7*c)*d*f^7)*sqrt((9*b^2*c^4*d^5*f^3 + 9*a^4*b^2*d*f^7 + 6*(b^4*c^2 +
```

```
6*a*b^2*c^3)*d^4*f^4 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^3*f^5 + 6*(a^
2*b^4 + 6*a^3*b^2*c)*d^2*f^6)/(c^12*d^12 + a^12*f^12 - 6*(b^2*c^10 - 2*a*c^
11)*d^11*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^2 - 10*(2*b^
6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^3 + 15*(b^8*c^4 -
 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^4 - 6*(b
^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6
 - 132*a^5*c^7)*d^7*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^
6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^6 - 6*(a^2
*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 -
 132*a^7*c^5)*d^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7
*b^2*c^3 + 33*a^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*
c^2 - 22*a^9*c^3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^2*
f^10 - 6*(a^10*b^2 - 2*a^11*c)*d*f^11)))*sqrt(c*x^2 + b*x + a)*sqrt((c^3*d^
3*f + a^3*f^4 + 3*(b^2*c + a*c^2)*d^2*f^2 + 3*(a*b^2 + a^2*c)*d*f^3 + (c^6*
d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*
a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^3*f^3
+ 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*f^4 - 3*(a^4*b^2 - 2*a^5*c)*d*f
^5)*sqrt((9*b^2*c^4*d^5*f^3 + 9*a^4*b^2*d*f^7 + 6*(b^4*c^2 + 6*a*b^2*c^3)*d
^4*f^4 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^3*f^5 + 6*(a^2*b^4 + 6*a^3*b
^2*c)*d^2*f^6)/(c^12*d^12 + a^12*f^12 - 6*(b^2*c^10 - 2*a*c^11)*d^11*f + 3*
(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^10*f^2 - 10*(2*b^6*c^6 - 12*a*b^
4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^9*f^3 + 15*(b^8*c^4 - 8*a*b^6*c^5 +
28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^8*f^4 - 6*(b^10*c^2 - 10*a*
b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6 - 132*a^5*c^7)
*d^7*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^6*c^3 + 1050*a^
4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^6*f^6 - 6*(a^2*b^10 - 10*a^3*
b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 - 132*a^7*c^5)*d
^5*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*b^2*c^3 + 33*a
^8*c^4)*d^4*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c^2 - 22*a^9*c^
3)*d^3*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^2*f^10 - 6*(a^10*
b^2 - 2*a^11*c)*d*f^11)))/(c^6*d^6 + a^6*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^5*f
+ 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^4*f^2 - (b^6 - 6*a*b^4*c + 18*a^2
*b^2*c^2 - 20*a^3*c^3)*d^3*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^2*
f^4 - 3*(a^4*b^2 - 2*a^5*c)*d*f^5)) - (2*a*c^6*...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{x}{-ad\sqrt{a+bx+cx^2}+afx^2\sqrt{a+bx+cx^2}-bdx\sqrt{a+bx+cx^2}+bfx^3\sqrt{a+bx+cx^2}-cdx^2\sqrt{a+bx+cx^2}+cfx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(x/(-a*d*sqrt(a + b*x + c*x**2) + a*f*x**2*sqrt(a + b*x + c*x**2)
- b*d*x*sqrt(a + b*x + c*x**2) + b*f*x**3*sqrt(a + b*x + c*x**2) - c*d*x**2
*sqrt(a + b*x + c*x**2) + c*f*x**4*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueDone
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(x/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

## 3.106 $\int \frac{1}{\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=310

$$-\frac{2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)\,x)}{(b^2-4ac)\,(b^2df-(cd+af)^2)\,\sqrt{a+bx+cx^2}}+\frac{f\tanh^{-1}\left(\frac{b\sqrt{d}-2a\sqrt{f}+\left(2c\sqrt{d}-b\sqrt{f}\right)x}{2\sqrt{cd-b\sqrt{d}\,\sqrt{f}+af}\,\sqrt{a+bx+cx^2}}\right)}{2\sqrt{d}\,\left(cd-b\sqrt{d}\,\sqrt{f}+af\right)^{3/2}}$$

```
[Out] 1/2*f*arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+
b*x+a)^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d^(1/2)/(c*d+a*f-b*d^(1/2)*
f^(1/2))^(3/2)+1/2*f*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^
(1/2)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/d^(1/2)/(c*d
+a*f+b*d^(1/2)*f^(1/2))^(3/2)-2*(b*(b^2*f-c*(3*a*f+c*d))-c*(2*a*c*f-b^2*f+2
*c^2*d)*x)/(-4*a*c+b^2)/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]

time = 0.25, antiderivative size = 310, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 25, $\frac{\text{number of rules}}{\text{integrand size}} = 0.160$, Rules used = {989, 1047, 738, 212}

$$-\frac{2(b(b^2f-c(3af+cd))-cx(2acf+b^2(-f)+2c^2d))}{(b^2-4ac)\sqrt{a+bx+cx^2}\,(b^2df-(af+cd)^2)}+\frac{f\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2\sqrt{d}\,\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{f\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\,\sqrt{af+b\sqrt{d}\,\sqrt{f}+cd}}\right)}{2\sqrt{d}\,\left(af+b\sqrt{d}\,\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-2*(b*(b^2*f - c*(c*d + 3*a*f)) - c*(2*c^2*d - b^2*f + 2*a*c*f)*x))/((b^2
- 4*a*c)*(b^2*d*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) + (f*ArcTanh[(b*S
qrt[d] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]
*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*Sqrt[d]*(c*d - b*Sqrt[d]*Sqrt[f
] + a*f)^(3/2)) + (f*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sq
rt[f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(
2*Sqrt[d]*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 989

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b^3*f + b*c*(c*d - 3*a*f) + c*(2*c^2*d + b^2*f - c*(2*a*f
))*x)*(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^(q + 1)/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1))), x] - Dist[1/((b^2 - 4*a*c)*(b^2*d*f + (c*d -
a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Simp[2*c*(b^2
*d*f + (c*d - a*f)^2)*(p + 1) - (2*c^2*d + b^2*f - c*(2*a*f))*(a*f*(p + 1)
- c*d*(p + 2)) + (2*f*(b^3*f + b*c*(c*d - 3*a*f))*(p + q + 2) - (2*c^2*d +
b^2*f - c*(2*a*f))*(b*f*(p + 1)))*x + c*f*(2*c^2*d + b^2*f - c*(2*a*f))*(2*
p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x] && NeQ[b^2 - 4
*a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[
p] && ILtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rubi steps

$$\int \frac{1}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx = -\frac{2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)x)}{(b^2-4ac)(b^2df-(cd+af)^2)\sqrt{a+bx+cx^2}} - \frac{2\int \frac{\frac{1}{2}(b^2-4ac)f(cd}{\sqrt{a+bx}}}{(b^2-4ac)(b}$$

$$= -\frac{2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)x)}{(b^2-4ac)(b^2df-(cd+af)^2)\sqrt{a+bx+cx^2}} - \frac{f^{3/2}\int \frac{}{\left(-\sqrt{d}\,\sqrt{}\right.}}{2\sqrt{d}\left(c\right.}$$

$$= -\frac{2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)x)}{(b^2-4ac)(b^2df-(cd+af)^2)\sqrt{a+bx+cx^2}} + \frac{f^{3/2}\mathrm{Subst}\left(\int \frac{}{4}\right.}{}$$

$$= -\frac{2(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)x)}{(b^2-4ac)(b^2df-(cd+af)^2)\sqrt{a+bx+cx^2}} + \frac{f\tanh^{-1}\left(\frac{}{2\sqrt{}}\right.}{2\sqrt{d}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.72, size = 379, normalized size = 1.22

$$\frac{-4b^3f+4bc(cd+3af)-4b^2cfx+8c^2(cd+af)x+(b^2-4ac)f\sqrt{a+x(b+cx)}\,\mathrm{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+2abf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2a\sqrt{c}\,f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-bf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2(b^2-4ac)(-c^2d^2-2acdf+f(b^2d-a^2f))\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/((a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-4*b^3*f + 4*b*c*(c*d + 3*a*f) - 4*b^2*c*f*x + 8*c^2*(c*d + a*f)*x + (b^2
- 4*a*c)*f*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 +
 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1] + 2*a*b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] -
 2*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c
]*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - b*f*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*
#1^3) & ])/(2*(b^2 - 4*a*c)*(-(c^2*d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*Sq
rt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 902 vs. $2(252) = 504$.
time = 0.13, size = 903, normalized size = 2.91

| method | result |
|---|---|
| default | $\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}}-\frac{}{\left(-b\sqrt{df}+fa+cd\right)\left(\frac{4c}{}\right.}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2/(d*f)^(1/2)*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*
c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-
2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-
2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/
2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)
/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b
*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*
f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+
(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*
f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2/(d*f)^(1/2)*(1/(b*(d*f)^(
1/2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/
2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)
+f*a+c*d)*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/
2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f
)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f
)^(1/2)+f*a+c*d)*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f
*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c
*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/
f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(((c*sqrt(4*d*f))/(2*f^2)>0)', see `
assume?
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 17397 vs. $2(252) = 504$.
time = 13.36, size = 17397, normalized size = 56.12

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] 1/4*(((a*b^2*c^2 - 4*a^2*c^3)*d^2 - (a*b^4 - 6*a^2*b^2*c + 8*a^3*c^2)*d*f +
 (a^3*b^2 - 4*a^4*c)*f^2 + ((b^2*c^3 - 4*a*c^4)*d^2 - (b^4*c - 6*a*b^2*c^2
+ 8*a^2*c^3)*d*f + (a^2*b^2*c - 4*a^3*c^2)*f^2)*x^2 + ((b^3*c^2 - 4*a*b*c^3
)*d^2 - (b^5 - 6*a*b^3*c + 8*a^2*b*c^2)*d*f + (a^2*b^3 - 4*a^3*b*c)*f^2)*x)
*sqrt((c^3*d^3*f^2 + a^3*f^5 + 3*(b^2*c + a*c^2)*d^2*f^3 + 3*(a*b^2 + a^2*c
)*d*f^4 + (c^6*d^7 + a^6*d*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^6*f + 3*(b^4*c^2 -
 4*a*b^2*c^3 + 5*a^2*c^4)*d^5*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*
a^3*c^3)*d^4*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)*d^3*f^4 - 3*(a^4*b
^2 - 2*a^5*c)*d^2*f^5)*sqrt((9*b^2*c^4*d^4*f^5 + 9*a^4*b^2*f^9 + 6*(b^4*c^2
 + 6*a*b^2*c^3)*d^3*f^6 + (b^6 + 12*a*b^4*c + 54*a^2*b^2*c^2)*d^2*f^7 + 6*(
a^2*b^4 + 6*a^3*b^2*c)*d*f^8)/(c^12*d^13 + a^12*d*f^12 - 6*(b^2*c^10 - 2*a*
c^11)*d^12*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 + 22*a^2*c^10)*d^11*f^2 - 10*(2*
b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*a^3*c^9)*d^10*f^3 + 15*(b^8*c^
4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2*c^7 + 33*a^4*c^8)*d^9*f^4 - 6
*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*
c^6 - 132*a^5*c^7)*d^8*f^5 + (b^12 - 12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3
*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^5 + 924*a^6*c^6)*d^7*f^6 - 6*(
a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^
4 - 132*a^7*c^5)*d^6*f^7 + 15*(a^4*b^8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*
a^7*b^2*c^3 + 33*a^8*c^4)*d^5*f^8 - 10*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b
^2*c^2 - 22*a^9*c^3)*d^4*f^9 + 3*(5*a^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d
^3*f^10 - 6*(a^10*b^2 - 2*a^11*c)*d^2*f^11)))/(c^6*d^7 + a^6*d*f^6 - 3*(b^2
*c^4 - 2*a*c^5)*d^6*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^5*f^2 - (b^
6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^4*f^3 + 3*(a^2*b^4 - 4*a^3*b
^2*c + 5*a^4*c^2)*d^3*f^4 - 3*(a^4*b^2 - 2*a^5*c)*d^2*f^5))*log((3*b^2*c^2*
d^2*f^3 + 3*a^2*b^2*f^5 + (b^4 + 6*a*b^2*c)*d*f^4 + 2*(3*b*c^3*d^2*f^3 + 3*
a^2*b*c*f^5 + (b^3*c + 6*a*b*c^2)*d*f^4)*x + 2*(3*b*c^4*d^4*f^2 + 3*a^4*b*f
^6 + 4*(b^3*c^2 + 3*a*b*c^3)*d^3*f^3 + (b^5 + 8*a*b^3*c + 18*a^2*b*c^2)*d^2
*f^4 + 4*(a^2*b^3 + 3*a^3*b*c)*d*f^5 - 2*(b*c^7*d^8 + a^7*b*d*f^7 - (3*b^3*
c^5 - 7*a*b*c^6)*d^7*f + 3*(b^5*c^3 - 5*a*b^3*c^4 + 7*a^2*b*c^5)*d^6*f^2 -
(b^7*c - 9*a*b^5*c^2 + 30*a^2*b^3*c^3 - 35*a^3*b*c^4)*d^5*f^3 - (a*b^7 - 9*
```

```
a^2*b^5*c + 30*a^3*b^3*c^2 - 35*a^4*b*c^3)*d^4*f^4 + 3*(a^3*b^5 - 5*a^4*b^3
*c + 7*a^5*b*c^2)*d^3*f^5 - (3*a^5*b^3 - 7*a^6*b*c)*d^2*f^6)*sqrt((9*b^2*c^
4*d^4*f^5 + 9*a^4*b^2*f^9 + 6*(b^4*c^2 + 6*a*b^2*c^3)*d^3*f^6 + (b^6 + 12*a
*b^4*c + 54*a^2*b^2*c^2)*d^2*f^7 + 6*(a^2*b^4 + 6*a^3*b^2*c)*d*f^8)/(c^12*d
^13 + a^12*d*f^12 - 6*(b^2*c^10 - 2*a*c^11)*d^12*f + 3*(5*b^4*c^8 - 20*a*b^
2*c^9 + 22*a^2*c^10)*d^11*f^2 - 10*(2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c
^8 - 22*a^3*c^9)*d^10*f^3 + 15*(b^8*c^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48
*a^3*b^2*c^7 + 33*a^4*c^8)*d^9*f^4 - 6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^
6*c^4 - 140*a^3*b^4*c^5 + 210*a^4*b^2*c^6 - 132*a^5*c^7)*d^8*f^5 + (b^12 -
12*a*b^10*c + 90*a^2*b^8*c^2 - 400*a^3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^
5*b^2*c^5 + 924*a^6*c^6)*d^7*f^6 - 6*(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*
c^2 - 140*a^5*b^4*c^3 + 210*a^6*b^2*c^4 - 132*a^7*c^5)*d^6*f^7 + 15*(a^4*b^
8 - 8*a^5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*b^2*c^3 + 33*a^8*c^4)*d^5*f^8 - 1
0*(2*a^6*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c^2 - 22*a^9*c^3)*d^4*f^9 + 3*(5*a
^8*b^4 - 20*a^9*b^2*c + 22*a^10*c^2)*d^3*f^10 - 6*(a^10*b^2 - 2*a^11*c)*d^2
*f^11)))*sqrt(c*x^2 + b*x + a)*sqrt((c^3*d^3*f^2 + a^3*f^5 + 3*(b^2*c + a*c
^2)*d^2*f^3 + 3*(a*b^2 + a^2*c)*d*f^4 + (c^6*d^7 + a^6*d*f^6 - 3*(b^2*c^4 -
 2*a*c^5)*d^6*f + 3*(b^4*c^2 - 4*a*b^2*c^3 + 5*a^2*c^4)*d^5*f^2 - (b^6 - 6*
a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*d^4*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c +
 5*a^4*c^2)*d^3*f^4 - 3*(a^4*b^2 - 2*a^5*c)*d^2*f^5)*sqrt((9*b^2*c^4*d^4*f^
5 + 9*a^4*b^2*f^9 + 6*(b^4*c^2 + 6*a*b^2*c^3)*d^3*f^6 + (b^6 + 12*a*b^4*c +
 54*a^2*b^2*c^2)*d^2*f^7 + 6*(a^2*b^4 + 6*a^3*b^2*c)*d*f^8)/(c^12*d^13 + a^
12*d*f^12 - 6*(b^2*c^10 - 2*a*c^11)*d^12*f + 3*(5*b^4*c^8 - 20*a*b^2*c^9 +
22*a^2*c^10)*d^11*f^2 - 10*(2*b^6*c^6 - 12*a*b^4*c^7 + 27*a^2*b^2*c^8 - 22*
a^3*c^9)*d^10*f^3 + 15*(b^8*c^4 - 8*a*b^6*c^5 + 28*a^2*b^4*c^6 - 48*a^3*b^2
*c^7 + 33*a^4*c^8)*d^9*f^4 - 6*(b^10*c^2 - 10*a*b^8*c^3 + 50*a^2*b^6*c^4 -
140*a^3*b^4*c^5 + 210*a^4*b^2*c^6 - 132*a^5*c^7)*d^8*f^5 + (b^12 - 12*a*b^1
0*c + 90*a^2*b^8*c^2 - 400*a^3*b^6*c^3 + 1050*a^4*b^4*c^4 - 1512*a^5*b^2*c^
5 + 924*a^6*c^6)*d^7*f^6 - 6*(a^2*b^10 - 10*a^3*b^8*c + 50*a^4*b^6*c^2 - 14
0*a^5*b^4*c^3 + 210*a^6*b^2*c^4 - 132*a^7*c^5)*d^6*f^7 + 15*(a^4*b^8 - 8*a^
5*b^6*c + 28*a^6*b^4*c^2 - 48*a^7*b^2*c^3 + 33*a^8*c^4)*d^5*f^8 - 10*(2*a^6
*b^6 - 12*a^7*b^4*c + 27*a^8*b^2*c^2 - 22*a^9*c^3)*d^4*f^9 + 3*(5*a^8*b^4 -
 20*a^9*b^2*c + 22*a^10*c^2)*d^3*f^10 - 6*(a^10*b^2 - 2*a^11*c)*d^2*f^11)))
/(c^6*d^7 + a^6*d*f^6 - 3*(b^2*c^4 - 2*a*c^5)*d^6*f + 3*(b^4*c^2 - 4*a*b^2*
c^3 + 5*a^2*c^4)*d^5*f^2 - (b^6 - 6*a*b^4*c + 18*a^2*b^2*c^2 - 20*a^3*c^3)*
d^4*f^3 + 3*(a^2*b^4 - 4*a^3*b^2*c + 5*a^4*c^2)...
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-ad\sqrt{a+bx+cx^2}+afx^2\sqrt{a+bx+cx^2}-bdx\sqrt{a+bx+cx^2}+bfx^3\sqrt{a+bx+cx^2}-cdx^2\sqrt{a+bx+cx^2}+cfx^4\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-a*d*sqrt(a + b*x + c*x**2) + a*f*x**2*sqrt(a + b*x + c*x**2)
- b*d*x*sqrt(a + b*x + c*x**2) + b*f*x**3*sqrt(a + b*x + c*x**2) - c*d*x**2
*sqrt(a + b*x + c*x**2) + c*f*x**4*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueDone
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{(d-f\,x^2)\,\left(c\,x^2+b\,x+a\right)^{3/2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(1/((d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.107 $\int \frac{1}{x(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx$

**Optimal**. Leaf size=394

$$\frac{2(b^2-2ac+bcx)}{a(b^2-4ac)\,d\sqrt{a+bx+cx^2}}-\frac{2f(a(2c^2d-b^2f+2acf)+bc(cd-af)x)}{(b^2-4ac)\,d\,(b^2df-(cd+af)^2)\sqrt{a+bx+cx^2}}-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+c}}\right.}{a^{3/2}d}$$

```
[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)/d-1/2*f^(3/2)*a
rctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)^
(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d/(c*d+a*f-b*d^(1/2)*f^(1/2))^(3/2
)+1/2*f^(3/2)*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2)))
/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/d/(c*d+a*f+b*d^(1/2
)*f^(1/2))^(3/2)+2*(b*c*x-2*a*c+b^2)/a/(-4*a*c+b^2)/d/(c*x^2+b*x+a)^(1/2)-2
*f*(a*(2*a*c*f-b^2*f+2*c^2*d)+b*c*(-a*f+c*d)*x)/(-4*a*c+b^2)/d/(b^2*d*f-(a*
f+c*d)^2)/(c*x^2+b*x+a)^(1/2)
```

**Rubi** [A]

time = 0.72, antiderivative size = 394, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 7, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {6857, 754, 12, 738, 212, 1032, 1047}

$$-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{a^{3/2}d}-\frac{2f(a(2acf+b^2(-f)+2c^2d)+bcx(cd-af))}{d(b^2-4ac)\sqrt{a+bx+cx^2}\,(b^2df-(af+cd)^2)}+\frac{2(-2ac+b^2+bcx)}{ad(b^2-4ac)\sqrt{a+bx+cx^2}}-\frac{f^{3/2}\tanh^{-1}\left(\frac{-2a\sqrt{f}+x(2c\sqrt{d}-b\sqrt{f})+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b(-\sqrt{d})\sqrt{f}+cd}}\right)}{2d\left(af+b(-\sqrt{d})\sqrt{f}+cd\right)^{3/2}}+\frac{f^{3/2}\tanh^{-1}\left(\frac{2a\sqrt{f}+x(b\sqrt{f}+2c\sqrt{d})+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*(a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (2*(b^2 - 2*a*c + b*c*x))/(a*(b^2 - 4*a*c)*d*Sqrt[a + b*x + c*x^2]) - (2*f*
(a*(2*c^2*d - b^2*f + 2*a*c*f) + b*c*(c*d - a*f)*x))/((b^2 - 4*a*c)*d*(b^2*
d*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) - ArcTanh[(2*a + b*x)/(2*Sqrt[a
]*Sqrt[a + b*x + c*x^2])]/(a^(3/2)*d) - (f^(3/2)*ArcTanh[(b*Sqrt[d] - 2*a*S
qrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt[f] + a*f
]*Sqrt[a + b*x + c*x^2])])/(2*d*(c*d - b*Sqrt[d]*Sqrt[f] + a*f)^(3/2)) + (f
^(3/2)*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[f])*x)/(2*S
qrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d*(c*d + b*S
qrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] && !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 754

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*(b*c*d - b^2*e + 2*a*c*e + c*(2*c*d - b*e)
*x)*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^
2))), x] + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d +
 e*x)^m*Simp[b*c*d*e*(2*p - m + 2) + b^2*e^2*(m + p + 2) - 2*c^2*d^2*(2*p +
 3) - 2*a*c*e^2*(m + 2*p + 3) - c*e*(2*c*d - b*e)*(m + 2*p + 4)*x, x]*(a +
b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4
*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p,
-1] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 1032

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (f
_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^
(q + 1)/((b^2 - 4*a*c)*(b^2*d*f + (c*d - a*f)^2)*(p + 1)))*((g*c)*((-b)*(c*
d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)) + c*(g*(2*c^2*d + b^2
*f - c*(2*a*f)) - h*(b*c*d + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Si
mp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d)*((-b)*f))*(p + 1) + (b^2*(g*f) - b*
(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(a*f*(p + 1) - c*d*(p + 2)) - (2*f*(
(g*c)*((-b)*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(2*a*f)))*(p +
q + 2) - (b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(b*f*(p + 1)
))*x - c*f*(b^2*(g*f) - b*(h*c*d + a*h*f) + 2*(g*c*(c*d - a*f)))*(2*p + 2*q
 + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, g, h, q}, x] && NeQ[b^2 - 4*
a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[p
] && ILtQ[q, -1])
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
```

```
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{1}{x\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx = \int \left(\frac{1}{dx\left(a+bx+cx^2\right)^{3/2}} - \frac{fx}{d\left(a+bx+cx^2\right)^{3/2}\left(-d+fx^2\right)}\right) dx$$

$$= \frac{\int \frac{1}{x(a+bx+cx^2)^{3/2}}\,dx}{d} - \frac{f\int \frac{x}{(a+bx+cx^2)^{3/2}(-d+fx^2)}\,dx}{d}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} - \frac{2f(a(2c^2d-b^2f+2acf)+bc(cd\ -}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\sqrt{a\ +}}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} - \frac{2f(a(2c^2d-b^2f+2acf)+bc(cd\ -}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\sqrt{a\ +}}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} - \frac{2f(a(2c^2d-b^2f+2acf)+bc(cd\ -}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\sqrt{a\ +}}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} - \frac{2f(a(2c^2d-b^2f+2acf)+bc(cd\ -}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\sqrt{a\ +}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.34, size = 486, normalized size = 1.23

$$\frac{2(b^4f+2ac^2(cd+af)-b^2c(cd+4af)+b^3cfx-bc^2(cd+3af)x)}{a\left(-b^2+4ac\right)\left(c^2d^2+2acdf+f\left(-b^2d+a^2f\right)\right)\sqrt{a+x(b+cx)}}+\frac{2\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{3/2}d}+\frac{f^2\text{RootSum}\left[b^2d-a^2f-4b\sqrt{c}\,d\#1+4cd\#1^2+2af\#1^2-f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+acd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+a^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-cd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-af\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{b\sqrt{c}\,d-2cd\#1-af\#1+f\#1^3}\&\right]}{2c^2d^3-2b^2d^2f+4acd^2f+2a^2df^2}$$

Antiderivative was successfully verified.

[In] Integrate[1/(x*(a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]

[Out] (2*(b^4*f + 2*a*c^2*(c*d + a*f) - b^2*c*(c*d + 4*a*f) + b^3*c*f*x - b*c^2*(c*d + 3*a*f)*x))/(a*(-b^2 + 4*a*c)*(c^2*d^2 + 2*a*c*d*f + f*(-(b^2*d) + a^2*f))*Sqrt[a + x*(b + c*x)]) + (2*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]])/(a^(3/2)*d) + (f^2*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^3) & ])/(2*c^2*d^3 - 2*b^2*d^2*f + 4*a*c*d^2*f + 2*a^2*d*f^2)

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 989 vs. $2(326) = 652$.
time = 0.12, size = 990, normalized size = 2.51

| method | result |
|---|---|
| default | $-\frac{\frac{f}{\left(-b\sqrt{df}+fa+cd\right)\sqrt{\left(x+\frac{\sqrt{df}}{f}\right)^2 c+\frac{\left(-2c\sqrt{df}+bf\right)\left(x+\frac{\sqrt{df}}{f}\right)}{f}+\frac{-b\sqrt{df}+fa+cd}{f}}}-\frac{\cdots}{\left(-b\sqrt{df}+fa+cd\right)\cdots}}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

[In] int(1/x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)

[Out] -1/2/d*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-2*c*(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))+1/d*(1/a/(c*x^2+b*x+a)^(1/2)-b/a*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)-1/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))-1/2/d*(1/(b*(d*f)^(1/2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)^2*

```
c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2
)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x-(d*f)^(1/2)/f)+(2*c
*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2
/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(
d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f)^(1/2)+f*a+c*d)*f/((b*(d*f)^(1/2)+f*
a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-
(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(
2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/
(x-(d*f)^(1/2)/f)))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
[Out] -integrate(1/((c*x^2 + b*x + a)^(3/2)*(f*x^2 - d)*x), x)
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-adx\sqrt{a+bx+cx^2} + afx^3\sqrt{a+bx+cx^2} - bdx^2\sqrt{a+bx+cx^2} + bfx^4\sqrt{a+bx+cx^2} - cdx^3\sqrt{a+bx+cx^2} + cfx^5\sqrt{a+bx+cx^2}}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
[Out] -Integral(1/(-a*d*x*sqrt(a + b*x + c*x**2) + a*f*x**3*sqrt(a + b*x + c*x**2
) - b*d*x**2*sqrt(a + b*x + c*x**2) + b*f*x**4*sqrt(a + b*x + c*x**2) - c*d
*x**3*sqrt(a + b*x + c*x**2) + c*f*x**5*sqrt(a + b*x + c*x**2)), x)
```

**Giac [F]**
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] sage2
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\,\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(1/(x*(d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.108 $\int \frac{1}{x^2\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx$

**Optimal**. Leaf size=454

$$\frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)dx\sqrt{a+bx+cx^2}}-\frac{2f(b(b^2f-c(cd+3af))-c(2c^2d-b^2f+2acf)\,x)}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\sqrt{a+bx+cx^2}}-\frac{\left(3b^2-8ac\right)\sqrt{a+bx+}}{a^2\left(b^2-4ac\right)dx}$$

```
[Out] 3/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(5/2)/d+1/2*f^2*
arctanh(1/2*(b*d^(1/2)-2*a*f^(1/2)+x*(2*c*d^(1/2)-b*f^(1/2)))/(c*x^2+b*x+a)
^(1/2)/(c*d+a*f-b*d^(1/2)*f^(1/2))^(1/2))/d^(3/2)/(c*d+a*f-b*d^(1/2)*f^(1/2
))^(3/2)+1/2*f^2*arctanh(1/2*(b*d^(1/2)+2*a*f^(1/2)+x*(2*c*d^(1/2)+b*f^(1/2
)))/(c*x^2+b*x+a)^(1/2)/(c*d+a*f+b*d^(1/2)*f^(1/2))^(1/2))/d^(3/2)/(c*d+a*f
+b*d^(1/2)*f^(1/2))^(3/2)+2*(b*c*x-2*a*c+b^2)/a/(-4*a*c+b^2)/d/x/(c*x^2+b*x
+a)^(1/2)-2*f*(b*(b^2*f-c*(3*a*f+c*d))-c*(2*a*c*f-b^2*f+2*c^2*d)*x)/(-4*a*c
+b^2)/d/(b^2*d*f-(a*f+c*d)^2)/(c*x^2+b*x+a)^(1/2)-(-8*a*c+3*b^2)*(c*x^2+b*x
+a)^(1/2)/a^2/(-4*a*c+b^2)/d/x
```

**Rubi** [A]
time = 0.76, antiderivative size = 454, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 7, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.250$, Rules used = {6857, 754, 820, 738, 212, 989, 1047}

$$\frac{3b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{5/2}d}-\frac{\left(3b^2-8ac\right)\sqrt{a+bx+cx^2}}{a^2dx\left(b^2-4ac\right)}-\frac{2f(b(b^2f-c(3af+cd))-cx(2acf+b^2(-f)+2c^2d))}{d\left(b^2-4ac\right)\sqrt{a+bx+cx^2}\left(b^2df-(af+cd)^2\right)}+\frac{2(-2ac+b^2+bcx)}{adx\left(b^2-4ac\right)\sqrt{a+bx+cx^2}}+\frac{f^2\tanh^{-1}\left(\frac{-2a\sqrt{f}+x\left(2c\sqrt{d}-b\sqrt{f}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\left(-\sqrt{d}\right)\sqrt{f}+cd}}\right)}{2d^{3/2}\left(af+b\left(-\sqrt{d}\right)\sqrt{f}+cd\right)^{3/2}}+\frac{f^2\tanh^{-1}\left(\frac{2a\sqrt{f}+x\left(b\sqrt{f}+2c\sqrt{d}\right)+b\sqrt{d}}{2\sqrt{a+bx+cx^2}\sqrt{af+b\sqrt{d}\sqrt{f}+cd}}\right)}{2d^{3/2}\left(af+b\sqrt{d}\sqrt{f}+cd\right)^{3/2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*(a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (2*(b^2 - 2*a*c + b*c*x))/(a*(b^2 - 4*a*c)*d*x*Sqrt[a + b*x + c*x^2]) - (2*
f*(b*(b^2*f - c*(c*d + 3*a*f)) - c*(2*c^2*d - b^2*f + 2*a*c*f)*x))/((b^2 -
4*a*c)*d*(b^2*d*f - (c*d + a*f)^2)*Sqrt[a + b*x + c*x^2]) - ((3*b^2 - 8*a*c
)*Sqrt[a + b*x + c*x^2])/(a^2*(b^2 - 4*a*c)*d*x) + (3*b*ArcTanh[(2*a + b*x)
/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(2*a^(5/2)*d) + (f^2*ArcTanh[(b*Sqrt[d
] - 2*a*Sqrt[f] + (2*c*Sqrt[d] - b*Sqrt[f])*x)/(2*Sqrt[c*d - b*Sqrt[d]*Sqrt
[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d^(3/2)*(c*d - b*Sqrt[d]*Sqrt[f] + a
*f)^(3/2)) + (f^2*ArcTanh[(b*Sqrt[d] + 2*a*Sqrt[f] + (2*c*Sqrt[d] + b*Sqrt[
f])*x)/(2*Sqrt[c*d + b*Sqrt[d]*Sqrt[f] + a*f]*Sqrt[a + b*x + c*x^2])])/(2*d
^(3/2)*(c*d + b*Sqrt[d]*Sqrt[f] + a*f)^(3/2))
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 754

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*(b*c*d - b^2*e + 2*a*c*e + c*(2*c*d - b*e)
*x)*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^
2))), x] + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d +
 e*x)^m*Simp[b*c*d*e*(2*p - m + 2) + b^2*e^2*(m + p + 2) - 2*c^2*d^2*(2*p +
 3) - 2*a*c*e^2*(m + 2*p + 3) - c*e*(2*c*d - b*e)*(m + 2*p + 4)*x, x]*(a +
b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4
*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p,
-1] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 820

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(e*f - d*g))*(d + e*x)^(m + 1)*((a +
 b*x + c*x^2)^(p + 1)/(2*(p + 1)*(c*d^2 - b*d*e + a*e^2))), x] - Dist[(b*(e
*f + d*g) - 2*(c*d*f + a*e*g))/(2*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(
m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && EqQ[Simplify[m
+ 2*p + 3], 0]
```

Rule 989

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (f_.)*(x_)^2)^(q_), x
_Symbol] :> Simp[(b^3*f + b*c*(c*d - 3*a*f) + c*(2*c^2*d + b^2*f - c*(2*a*f
))*x)*(a + b*x + c*x^2)^(p + 1)*((d + f*x^2)^(q + 1)/((b^2 - 4*a*c)*(b^2*d*
f + (c*d - a*f)^2)*(p + 1))), x] - Dist[1/((b^2 - 4*a*c)*(b^2*d*f + (c*d -
a*f)^2)*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + f*x^2)^q*Simp[2*c*(b^2
*d*f + (c*d - a*f)^2)*(p + 1) - (2*c^2*d + b^2*f - c*(2*a*f))*(a*f*(p + 1)
- c*d*(p + 2)) + (2*f*(b^3*f + b*c*(c*d - 3*a*f))*(p + q + 2) - (2*c^2*d +
b^2*f - c*(2*a*f))*(b*f*(p + 1)))*x + c*f*(2*c^2*d + b^2*f - c*(2*a*f))*(2*
p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, f, q}, x] && NeQ[b^2 - 4
*a*c, 0] && LtQ[p, -1] && NeQ[b^2*d*f + (c*d - a*f)^2, 0] &&  !( !IntegerQ[
p] && ILtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rule 1047

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (
f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[(-a)*c, 2]}, Dist[h/2 + c*(g/(2*q)
), Int[1/((-q + c*x)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/2 - c*(g/(2*q)
), Int[1/((q + c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, c, d, e, f
, g, h}, x] && NeQ[e^2 - 4*d*f, 0] && PosQ[(-a)*c]
```

Rule 6857

```
Int[(u_)/((a_) + (b_.)*(x_)^(n_)), x_Symbol] :> With[{v = RationalFunctionE
xpand[u/(a + b*x^n), x]}, Int[v, x] /; SumQ[v]] /; FreeQ[{a, b}, x] && IGtQ
[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\,dx = \int \left(\frac{1}{dx^2\left(a+bx+cx^2\right)^{3/2}}+\frac{f}{d\left(a+bx+cx^2\right)^{3/2}\left(d-fx^2\right)}\right)dx$$

$$= \frac{\int \frac{1}{x^2(a+bx+cx^2)^{3/2}}\,dx}{d}+\frac{f\int \frac{1}{(a+bx+cx^2)^{3/2}(d-fx^2)}\,dx}{d}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)dx\sqrt{a+bx+cx^2}}-\frac{2f\left(b(b^2f-c(cd+3af))-c(2c^2d-\cdots\right.}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\cdots}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)dx\sqrt{a+bx+cx^2}}-\frac{2f\left(b(b^2f-c(cd+3af))-c(2c^2d-\cdots\right.}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\cdots}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)dx\sqrt{a+bx+cx^2}}-\frac{2f\left(b(b^2f-c(cd+3af))-c(2c^2d-\cdots\right.}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\cdots}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)dx\sqrt{a+bx+cx^2}}-\frac{2f\left(b(b^2f-c(cd+3af))-c(2c^2d-\cdots\right.}{\left(b^2-4ac\right)d\left(b^2df-(cd+af)^2\right)\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 2.27, size = 620, normalized size = 1.37

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*(a + b*x + c*x^2)^(3/2)*(d - f*x^2)),x]
```

```
[Out] (-2*Sqrt[a]*(4*a^4*c*f^2 + 3*b^2*d*(-(c^2*d) + b^2*f)*x*(b + c*x) + a^3*f*(
-(b^2*f) + 4*b*c*f*x + 4*c^2*(2*d + f*x^2)) + a^2*(18*b*c^2*d*f*x - b^3*f^2
*x - b^2*c*f*(6*d + f*x^2) + 4*c^3*d*(d + 3*f*x^2)) + a*d*(b^4*f + 10*b*c^3
*d*x - 16*b^3*c*f*x + 8*c^4*d*x^2 - b^2*c^2*(d + 14*f*x^2))) - 6*b*(b^2 - 4
*a*c)*(-(c^2*d^2) - 2*a*c*d*f + f*(b^2*d - a^2*f))*x*Sqrt[a + x*(b + c*x)]*
ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]] + a^(5/2)*(b^2 - 4*a*c
)*f^2*x*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a^2*f - 4*b*Sqrt[c]*d*#1 + 4*
c*d*#1^2 + 2*a*f*#1^2 - f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1] + 2*a*b*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*
c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*a*Sqrt[c]*f
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - b*f*Log[-(Sqrt[c]*x) +
 Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(b*Sqrt[c]*d - 2*c*d*#1 - a*f*#1 + f*#1^
3) & ])/(2*a^(5/2)*(-b^2 + 4*a*c)*d*(c^2*d^2 + 2*a*c*d*f + f*(-(b^2*d) + a^
2*f))*x*Sqrt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1064 vs. $2(382) = 764$.
time = 0.14, size = 1065, normalized size = 2.35

| method | result |
|---|---|
| default | $\frac{-\frac{1}{a x \sqrt{c x^2+bx+a}}-\frac{3b\left(\frac{1}{a\sqrt{c x^2+bx+a}}-\frac{b(2cx+b)}{a\left(4ac-b^2\right)\sqrt{c x^2+bx+a}}-\frac{\ln\left(\frac{2a+bx+2\sqrt{a}\,\sqrt{c x^2+bx+a}}{x}\right.}{a^{\frac{3}{2}}}\right.}{2a}}{d}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/d*(-1/a/x/(c*x^2+b*x+a)^(1/2)-3/2*b/a*(1/a/(c*x^2+b*x+a)^(1/2)-b/a*(2*c*x
+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)-1/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+
b*x+a)^(1/2))/x))-4*c/a*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2))+1/2*f/d/
```

```
(d*f)^(1/2)*(f/(-b*(d*f)^(1/2)+f*a+c*d)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d
*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-(-2*c*
(d*f)^(1/2)+b*f)/(-b*(d*f)^(1/2)+f*a+c*d)*(2*c*(x+(d*f)^(1/2)/f)+1/f*(-2*c*
(d*f)^(1/2)+b*f))/(4*c/f*(-b*(d*f)^(1/2)+f*a+c*d)-1/f^2*(-2*c*(d*f)^(1/2)+b
*f)^2)/((x+(d*f)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+
1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)-f/(-b*(d*f)^(1/2)+f*a+c*d)/(1/f*(-b*(d*
f)^(1/2)+f*a+c*d))^(1/2)*ln((2/f*(-b*(d*f)^(1/2)+f*a+c*d)+1/f*(-2*c*(d*f)^(
1/2)+b*f)*(x+(d*f)^(1/2)/f)+2*(1/f*(-b*(d*f)^(1/2)+f*a+c*d))^(1/2)*((x+(d*f
)^(1/2)/f)^2*c+1/f*(-2*c*(d*f)^(1/2)+b*f)*(x+(d*f)^(1/2)/f)+1/f*(-b*(d*f)^(
1/2)+f*a+c*d))^(1/2))/(x+(d*f)^(1/2)/f)))-1/2*f/d/(d*f)^(1/2)*(1/(b*(d*f)^(
1/2)+f*a+c*d)*f/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/
2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-(2*c*(d*f)^(1/2)+b*f)/(b*(d*f)^(1/2)
+f*a+c*d)*(2*c*(x-(d*f)^(1/2)/f)+(2*c*(d*f)^(1/2)+b*f)/f)/(4*c*(b*(d*f)^(1/
2)+f*a+c*d)/f-(2*c*(d*f)^(1/2)+b*f)^2/f^2)/((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f
)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)-1/(b*(d*f
)^(1/2)+f*a+c*d)*f/((b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2)*ln((2*(b*(d*f)^(1/2)+f
*a+c*d)/f+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/f)+2*((b*(d*f)^(1/2)+f*a+c
*d)/f)^(1/2)*((x-(d*f)^(1/2)/f)^2*c+(2*c*(d*f)^(1/2)+b*f)/f*(x-(d*f)^(1/2)/
f)+(b*(d*f)^(1/2)+f*a+c*d)/f)^(1/2))/(x-(d*f)^(1/2)/f)))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="maxima")
```

```
[Out] -integrate(1/((c*x^2 + b*x + a)^(3/2)*(f*x^2 - d)*x^2), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{1}{-adx^2\sqrt{a+bx+cx^2}+afx^4\sqrt{a+bx+cx^2}-bdx^3\sqrt{a+bx+cx^2}+bfx^5\sqrt{a+bx+cx^2}-cdx^4\sqrt{a+bx+cx^2}+cfx^6\sqrt{a+bx+cx^2}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(c*x**2+b*x+a)**(3/2)/(-f*x**2+d),x)
```

```
[Out] -Integral(1/(-a*d*x**2*sqrt(a + b*x + c*x**2) + a*f*x**4*sqrt(a + b*x + c*x
**2) - b*d*x**3*sqrt(a + b*x + c*x**2) + b*f*x**5*sqrt(a + b*x + c*x**2) -
c*d*x**4*sqrt(a + b*x + c*x**2) + c*f*x**6*sqrt(a + b*x + c*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(3/2)/(-f*x^2+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument ValueEvaluation time: 0.47
Done
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^2\,\left(d-f\,x^2\right)\,{\left(c\,x^2+b\,x+a\right)}^{3/2}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(d - f*x^2)*(a + b*x + c*x^2)^(3/2)),x)
```

```
[Out] int(1/(x^2*(d - f*x^2)*(a + b*x + c*x^2)^(3/2)), x)
```

# 3.109 $\int \frac{x^2\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=761

$$-\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{(b^2f^2+4cf(be-af)-8c^2(e^2-df))\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{8c^{3/2}f^3}\cdot$$

```
[Out] -1/8*(b^2*f^2+4*c*f*(-a*f+b*e)-8*c^2*(-d*f+e^2))*arctanh(1/2*(2*c*x+b)/c^(1
/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f^3-1/4*(-2*c*f*x-b*f+4*c*e)*(c*x^2+b*x+a)
^(1/2)/c/f^2-1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e
-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*
f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(c*(e^4-4*d*e^2*f+2*d^2*f^2-e^3*(
-4*d*f+e^2)^(1/2)+2*d*e*f*(-4*d*f+e^2)^(1/2))+f*(a*f*(e^2-2*d*f-e*(-4*d*f+e
^2)^(1/2))-b*(e^3-3*d*e*f-e^2*(-4*d*f+e^2)^(1/2)+d*f*(-4*d*f+e^2)^(1/2))))/
f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(f*(2*a*f-b*(e-(-4*d*f+e^2)^(1/2)))+c*(e^2-2
*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^
(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e
^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(c*(e^4-4*d*
e^2*f+2*d^2*f^2+e^3*(-4*d*f+e^2)^(1/2)-2*d*e*f*(-4*d*f+e^2)^(1/2))+f*(a*f*(
e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))-b*(e^3-3*d*e*f+e^2*(-4*d*f+e^2)^(1/2)-d*f*(
-4*d*f+e^2)^(1/2))))/f^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*(e^2-2*d*f+e*(-4*d*f
+e^2)^(1/2))+f*(2*a*f-b*(e+(-4*d*f+e^2)^(1/2))))^(1/2)
```

**Rubi** [A]

time = 1.93, antiderivative size = 761, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {1081, 1090, 635, 212, 1046, 738}

$$-\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)(4cf(be-af)+b^2f^2-8c^2(e^2-df))}{8c^{3/2}f^3}-\frac{\left(f\left(af\left(-e\sqrt{e^2-4df}-2df+e^2\right)-b\left(-e^2\sqrt{e^2-4df}+df\sqrt{e^2-4df}-3def+e^3\right)\right)+c\left(2d^2f^2-4de^2f+2def\sqrt{e^2-4df}-e^3\sqrt{e^2-4df}+e^4\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f^3\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(f\left(af\left(e\sqrt{e^2-4df}-2df+e^2\right)-b\left(e^2\sqrt{e^2-4df}-df\sqrt{e^2-4df}-3def+e^3\right)\right)+c\left(2d^2f^2-4de^2f-2def\sqrt{e^2-4df}+e^3\sqrt{e^2-4df}+e^4\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f^3\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{a+bx+cx^2}(-bf+4ce-2cfx)}{4cf^2}$$

Antiderivative was successfully verified.

```
[In] Int[(x^2*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] -1/4*((4*c*e - b*f - 2*c*f*x)*Sqrt[a + b*x + c*x^2])/(c*f^2) - ((b^2*f^2 +
4*c*f*(b*e - a*f) - 8*c^2*(e^2 - d*f))*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[
a + b*x + c*x^2])])/(8*c^(3/2)*f^3) - ((c*(e^4 - 4*d*e^2*f + 2*d^2*f^2 - e^
3*Sqrt[e^2 - 4*d*f] + 2*d*e*f*Sqrt[e^2 - 4*d*f]) + f*(a*f*(e^2 - 2*d*f - e*
Sqrt[e^2 - 4*d*f]) - b*(e^3 - 3*d*e*f - e^2*Sqrt[e^2 - 4*d*f] + d*f*Sqrt[e^
2 - 4*d*f])))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e -
Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 -
(c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f^3*Sqrt[e
```

```
^2 - 4*d*f]*Sqrt[c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e -
Sqrt[e^2 - 4*d*f]))]]) + ((c*(e^4 - 4*d*e^2*f + 2*d^2*f^2 + e^3*Sqrt[e^2 - 4
*d*f] - 2*d*e*f*Sqrt[e^2 - 4*d*f]) + f*(a*f*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d
*f]) - b*(e^3 - 3*d*e*f + e^2*Sqrt[e^2 - 4*d*f] - d*f*Sqrt[e^2 - 4*d*f])))*
ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d
*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sq
rt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f^3*Sqrt[e^2 - 4*d*f]*Sq
rt[c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e + Sqrt[e^2 - 4*d
*f]))])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1081

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(C*(b*f*p - c*e*(2*p + q
 + 2)) + 2*c*C*f*(p + q + 1)*x)*(a + b*x + c*x^2)^p*((d + e*x + f*x^2)^(q +
 1)/(2*c*f^2*(p + q + 1)*(2*p + 2*q + 3))), x] - Dist[1/(2*c*f^2*(p + q + 1
)*(2*p + 2*q + 3)), Int[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[
p*(b*d - a*e)*(C*(c*e - b*f)*(q + 1) - c*(C*e)*(2*p + 2*q + 3)) + (p + q +
1)*(b^2*C*d*f*p + a*c*(C*(2*d*f - e^2*(2*p + q + 2)) + f*(-2*A*f)*(2*p + 2*
```

```
q + 3))) + (2*p*(c*d - a*f)*(C*(c*e - b*f)*(q + 1) - c*(C*e)*(2*p + 2*q + 3
)) + (p + q + 1)*(C*e*f*p*(b^2 - 4*a*c) - b*c*(C*(e^2 - 4*d*f)*(2*p + q + 2
) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x + (p*(c*e - b*f)*(C*(c*e - b*f)*
(q + 1) - c*(C*e)*(2*p + 2*q + 3)) + (p + q + 1)*(C*f^2*p*(b^2 - 4*a*c) - c
^2*(C*(e^2 - 4*d*f)*(2*p + q + 2) + f*(2*C*d + 2*A*f)*(2*p + 2*q + 3))))*x^
2, x], x], x] /; FreeQ[{a, b, c, d, e, f, A, C, q}, x] && NeQ[b^2 - 4*a*c,
0] && NeQ[e^2 - 4*d*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0] && NeQ[2*p + 2*
q + 3, 0] &&  !IGtQ[p, 0] &&  !IGtQ[q, 0]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\begin{aligned}\int \frac{x^2\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx &= -\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{\int \frac{-\frac{1}{4}d(4bce-b^2f-4acf)-\frac{1}{4}(8c^2de-b^2ef-4acef+4bc(e}{\sqrt{a+bx+cx}}}{2cf}\\ &= -\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{\int \frac{-\frac{1}{4}df(4bce-b^2f-4acf)-\frac{1}{4}d(b^2f^2+4cf(be-af)-8c^2(}{}}{}\\ &= -\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{(b^2f^2+4cf(be-af)-8c^2(e^2-df))\,\mathrm{Sub}}{4cf^3}\\ &= -\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{(b^2f^2+4cf(be-af)-8c^2(e^2-df))\tanh}{8c^{3/2}f^3}\\ &= -\frac{(4ce-bf-2cfx)\sqrt{a+bx+cx^2}}{4cf^2}-\frac{(b^2f^2+4cf(be-af)-8c^2(e^2-df))\tanh}{8c^{3/2}f^3}\end{aligned}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.10, size = 863, normalized size = 1.13

Antiderivative was successfully verified.

```
[In] Integrate[(x^2*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (2*Sqrt[c]*f*(-4*c*e + b*f + 2*c*f*x)*Sqrt[a + x*(b + c*x)] + (b^2*f^2 + 4*
c*f*(b*e - a*f) - 8*c^2*(e^2 - d*f))*Log[c*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a +
x*(b + c*x)])] - 8*c^(3/2)*RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1
 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3
 + f*#1^4 & , (-(b*c*d*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1])
+ a*c*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b*c*d^2*f*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^2*d*e*f*Log[-(Sqrt[c]*x) + Sqr
t[a + b*x + c*x^2] - #1] - 2*a*c*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1] - a*b*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2
*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*c^(3/2)*d*e^2*Log
[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*c^(3/2)*d^2*f*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*b*Sqrt[c]*d*e*f*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*d*f^2*Log[-(Sqrt[c]*x) + S
qrt[a + b*x + c*x^2] - #1]*#1 - c*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1]*#1^2 + 2*c*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#
1^2 + b*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - b*d*f^2
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*e*f^2*Log[-(Sqrt[c
]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c
*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/(8*c^(3/2)*f^
3)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1665 vs. $2(680) = 1360$.
time = 0.20, size = 1666, normalized size = 2.19

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1666 |
| risch | Expression too large to display | 9381 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+
c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2)))+1/2*(-e*(-4*d*f+e^2)^(1/2)+2*d*f-e^2)/f^
2/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(
-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f
+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+
1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-
c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c
```

```
*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)
+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(
-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1
/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(
1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+
e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(
1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d
*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^
2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(
1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2*(c*
(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f+c*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))
^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2
*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2
)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(
1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2
)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*
d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/
2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e
)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2*(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**2*sqrt(a + b*x + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2\,\sqrt{c\,x^2+b\,x+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x^2*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2),x)
```

```
[Out] int((x^2*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2), x)
```

## 3.110 $\int \frac{x\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=549

$$\frac{\sqrt{a+bx+cx^2}}{f}-\frac{(2ce-bf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2}-\frac{\left(2df(ce-bf)+\left(e-\sqrt{e^2-4df}\right)(f(be-a\ldots}{\sqrt{2}\,f^2\ldots}$$

```
[Out] -1/2*(-b*f+2*c*e)*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f^2/c^
(1/2)+(c*x^2+b*x+a)^(1/2)/f-1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^
2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c
*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*f*(-b*f+c*e)+
(f*(-a*f+b*e)-c*(-d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2
)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)+1
/2*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(
1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)
*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*f*(-b*f+c*e)+(f*(-a*f+b*e)-c*(-d*f+e^2))*(
e+(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+
2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 4.55, antiderivative size = 549, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 6, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.214$, Rules used = {1033, 1090, 635, 212, 1046, 738}

$$-\frac{\left(\left(e-\sqrt{e^2-4df}\right)(f(be-af)-c(e^2-df))+2df(ce-bf)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}+\frac{\left(\left(\sqrt{e^2-4df}+e\right)(f(be-af)-c(e^2-df))+2df(ce-bf)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{(2ce-bf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2}+\frac{\sqrt{a+bx+cx^2}}{f}$$

Antiderivative was successfully verified.

```
[In] Int[(x*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] Sqrt[a + b*x + c*x^2]/f - ((2*c*e - b*f)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqr
t[a + b*x + c*x^2])])/(2*Sqrt[c]*f^2) - ((2*d*f*(c*e - b*f) + (e - Sqrt[e^2
 - 4*d*f])*(f*(b*e - a*f) - c*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e - Sqrt[e^
2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2
- 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x +
 c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*
a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]) + ((2*d*f*(c*e - b*f) + (e + Sqrt[e
^2 - 4*d*f])*(f*(b*e - a*f) - c*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e + Sqrt[
e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^
2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x
 + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f +
2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1033

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d
+ e*x + f*x^2)^(q + 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), I
nt[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[h*p*(b*d - a*e) + a*(
h*e - 2*g*f)*(p + q + 1) + (2*h*p*(c*d - a*f) + b*(h*e - 2*g*f)*(p + q + 1)
)*x + (h*p*(c*e - b*f) + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /; Fr
eeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d
*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\int \frac{x\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx = \frac{\sqrt{a+bx+cx^2}}{f} - \frac{\int \frac{\frac{bd}{2}+\frac{1}{2}(2cd+be-2af)x+\frac{1}{2}(2ce-bf)x^2}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{f}$$

$$= \frac{\sqrt{a+bx+cx^2}}{f} - \frac{\int \frac{\frac{bdf}{2}-\frac{1}{2}d(2ce-bf)+\left(\frac{1}{2}f(2cd+be-2af)-\frac{1}{2}e(2ce-bf)\right)x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{f^2} - \frac{(2ce-bf)\int -}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{f} - \frac{(2ce-bf)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f^2} - \frac{\left(2f\left(\frac{bdf}{2}-\right.\right.}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{f} - \frac{(2ce-bf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2} + \frac{\left(2\left(2f\left(\frac{bdf}{2}-\frac{1}{2}d(\right.\right.\right.}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{f} - \frac{(2ce-bf)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,f^2} - \frac{\left(2f(cde-bdf)+\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.70, size = 642, normalized size = 1.17

Antiderivative was successfully verified.

```
[In] Integrate[(x*Sqrt[a + b*x + c*x^2])/(d + e*x + f*x^2),x]
```

```
[Out] (2*f*Sqrt[a + x*(b + c*x)] + ((2*c*e - b*f)*Log[b + 2*c*x - 2*Sqrt[c]*Sqrt[
a + x*(b + c*x)]])/Sqrt[c] + 2*RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*
d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*
#1^3 + f*#1^4 & , (-(b*c*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]
) + a*c*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^2*d*f*Log[-(
Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] - a*b*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1] + a^2*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*c^(3/2)*d
*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - 2*b*Sqrt[c]*d*f*Log[
-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c*e^2*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1]*#1^2 + c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
```

```
x^2] - #1]*#1^2 + b*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2
 - a*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*
d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^
3) & ])/(2*f^2)
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1579 vs. $2(490) = 980$.
time = 0.19, size = 1580, normalized size = 2.88

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1580 |
| risch | Expression too large to display | 6947 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(1/2*(4*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f
-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2/f*
(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)
+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*
(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4*d*
f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2
)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*
e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4
*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/
2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*
f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(-e+(-4*d*f+e^2
)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c
+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*
f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^
2)^(1/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^(1/2)
+b*f-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e
*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(
1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*
f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^
2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^
```

```
(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^
2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(
1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x\sqrt{a + bx + cx^2}}{d + ex + fx^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x*sqrt(a + b*x + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument Value
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x\sqrt{c\,x^2+b\,x+a}}{f\,x^2+e\,x+d}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2),x)
```

```
[Out] int((x*(a + b*x + c*x^2)^(1/2))/(d + e*x + f*x^2), x)
```

## 3.111 $\int \frac{\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx$

**Optimal**. Leaf size=431

$$\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f}-\frac{\sqrt{c\left(e^2-2df-e\sqrt{e^2-4df}\right)+f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)}\ \tan}{\sqrt{2}}$$

```
[Out] arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))*c^(1/2)/f-1/2*arctanh(1/
4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/
2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)
^(1/2))^(1/2))*(f*(2*a*f-b*(e-(-4*d*f+e^2)^(1/2)))+c*(e^2-2*d*f-e*(-4*d*f+e
^2)^(1/2)))^(1/2)/f*2^(1/2)/(-4*d*f+e^2)^(1/2)+1/2*arctanh(1/4*(4*a*f-b*(e+
(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+
a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))
*(c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))+f*(2*a*f-b*(e+(-4*d*f+e^2)^(1/2))))^(1
/2)/f*2^(1/2)/(-4*d*f+e^2)^(1/2)
```

**Rubi** [A]
time = 0.37, antiderivative size = 431, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.185$, Rules used = {1003, 635, 212, 1046, 738}

$$-\frac{\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f\sqrt{e^2-4df}}+\frac{\sqrt{f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}f\sqrt{e^2-4df}}+\frac{\sqrt{c}\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{f}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(d + e*x + f*x^2),x]
```

```
[Out] (Sqrt[c]*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])])/f - (Sqrt[
c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt[e^2 - 4*d*f]
))]*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 -
 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f
)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f*Sqrt[e^2 - 4*d*f])
 + (Sqrt[c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e + Sqrt[e^2
 - 4*d*f]))]*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + S
qrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (
c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f*Sqrt[e^2
- 4*d*f])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1003

```
Int[Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2]/((d_) + (e_.)*(x_) + (f_.)*(x_)^
2), x_Symbol] :> Dist[c/f, Int[1/Sqrt[a + b*x + c*x^2], x], x] - Dist[1/f,
Int[(c*d - a*f + (c*e - b*f)*x)/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),
x], x] /; FreeQ[{a, b, c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 -
4*d*f, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx = -\frac{\int \frac{cd-af+(ce-bf)x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{f} + \frac{c\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f}$$

$$= \frac{(2c)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f} - \frac{\left(2f(cd-af)-(ce-bf)\left(e-\sqrt{e^2\right.\right.}{}$$

$$= \frac{\sqrt{c}\,\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{f} + \frac{\left(2\left(2f(cd-af)-(ce-bf)\left(e-\sqrt{e^2-4d\right.\right.\right.}{}$$

$$= \frac{\sqrt{c}\,\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\,\sqrt{a+bx+cx^2}}\right)}{f} - \frac{\sqrt{c\left(e^2-2df-e\sqrt{e^2-4df}\right)+f\left(2af-\right.}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.02, size = 396, normalized size = 0.92

$$\frac{-\sqrt{c}\log\left(f\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)+\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}d\#1+2a\sqrt{c}e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}e\#1^3+f\#1^4\&,\frac{bcd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-ace\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+2a\sqrt{c}f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+ce\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-bf\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}d-a\sqrt{c}e-4cd\#1-be\#1+2af\#1+3\sqrt{c}e\#1^2-2f\#1^3}\&\right]}{f}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(d + e*x + f*x^2),x]
```

```
[Out] (-(Sqrt[c]*Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])]) + RootSum[
b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 +
b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*c*d*Log[-(Sqrt[c]*
x) + Sqrt[a + b*x + c*x^2] - #1] - a*c*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1] - 2*c^(3/2)*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#
1 + 2*a*Sqrt[c]*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + c*e*L
og[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - b*f*Log[-(Sqrt[c]*x) +
 Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1
- b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1546 vs. 2(376) = 752.
time = 0.00, size = 1547, normalized size = 3.59

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1547 |

---

Verification of antiderivative is not currently implemented for this CAS.

[In] int((c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)

[Out] -1/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + bx + cx^2}}{d + ex + fx^2}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(sqrt(a + b*x + c*x**2)/(d + e*x + f*x**2), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument Value
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2 + b\,x + a}}{f\,x^2 + e\,x + d}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(d + e*x + f*x^2),x)
```

```
[Out] int((a + b*x + c*x^2)^(1/2)/(d + e*x + f*x^2), x)
```

# 3.112 $\int \frac{\sqrt{a+bx+cx^2}}{x\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=523

$$-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}+\frac{\left(cd\left(e-\sqrt{e^2-4df}\right)-f\left(2bd-a\left(e+\sqrt{e^2-4df}\right)\right)\right)\tanh^{-1}\left(\right.}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{c\left(e^2-2df-e\sqrt{\right.}}}$$

```
[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))*a^(1/2)/d+1/2*arctanh(1
/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1
/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2
)^(1/2))^(1/2))*(c*d*(e-(-4*d*f+e^2)^(1/2))-f*(2*b*d-a*(e+(-4*d*f+e^2)^(1/2
))))/d*2^(1/2)/(-4*d*f+e^2)^(1/2)/(f*(2*a*f-b*(e-(-4*d*f+e^2)^(1/2)))+c*(e^
2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)-1/2*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^
2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(
c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(-f*(2*b*
d-a*(e-(-4*d*f+e^2)^(1/2)))+c*d*(e+(-4*d*f+e^2)^(1/2)))/d*2^(1/2)/(-4*d*f+e
^2)^(1/2)/(c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))+f*(2*a*f-b*(e+(-4*d*f+e^2)^(1
/2))))^(1/2)
```

**Rubi** [A]

time = 2.14, antiderivative size = 521, normalized size of antiderivative = 1.00, number of steps used = 17, number of rules used = 9, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.300$, Rules used = {6860, 748, 857, 635, 212, 738, 1033, 1090, 1046}

$$-\frac{\left(-af\left(\sqrt{e^2-4df}+e\right)+2bdf-cd\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\left(-af\left(e-\sqrt{e^2-4df}\right)+2bdf-cd\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(x*(d + e*x + f*x^2)),x]
```

```
[Out] -((Sqrt[a]*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/d) - ((2
*b*d*f - c*d*(e - Sqrt[e^2 - 4*d*f]) - a*f*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh
[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x
)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2
- 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[c*(e^2
 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt[e^2 - 4*d*f]))]) +
 ((2*b*d*f - a*f*(e - Sqrt[e^2 - 4*d*f]) - c*d*(e + Sqrt[e^2 - 4*d*f]))*Arc
Tanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]
))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[
e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[c*
(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e + Sqrt[e^2 - 4*d*f]))
])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1033

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d
+ e*x + f*x^2)^(q + 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), I
nt[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[h*p*(b*d - a*e) + a*(
h*e - 2*g*f)*(p + q + 1) + (2*h*p*(c*d - a*f) + b*(h*e - 2*g*f)*(p + q + 1)
)*x + (h*p*(c*e - b*f) + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /; Fr
eeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d
```

```
*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{x\left(d+ex+fx^2\right)}\,dx=\int \left(\frac{\sqrt{a+bx+cx^2}}{dx}+\frac{(-e-fx)\sqrt{a+bx+cx^2}}{d\left(d+ex+fx^2\right)}\right)dx$$

$$=\frac{\int \frac{\sqrt{a+bx+cx^2}}{x}\,dx}{d}+\frac{\int \frac{(-e-fx)\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx}{d}$$

$$=-\frac{\int \frac{-2a-bx}{x\sqrt{a+bx+cx^2}}\,dx}{2d}-\frac{\int \frac{-\frac{1}{2}(bd-2ae)f-\frac{1}{2}f(2cd-be-2af)x+\frac{1}{2}bf^2x^2}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{df}$$

$$=\frac{a\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{d}-\frac{\int \frac{-\frac{1}{2}bdf^2-\frac{1}{2}(bd-2ae)f^2+\left(-\frac{1}{2}bef^2-\frac{1}{2}f^2(2cd-be-2af)\right)x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{df^2}$$

$$=-\frac{(2a)\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx,x,\frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{d}-\frac{\left(2bdf-af\left(e-\sqrt{e^2-4df}\right)-c\right.}{}$$

$$=-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}+\frac{\left(2\left(2bdf-af\left(e-\sqrt{e^2-4df}\right)-cd\left(e\right.\right.\right.}{}$$

$$=-\frac{\sqrt{a}\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d}-\frac{\left(2bdf-cd\left(e-\sqrt{e^2-4df}\right)-af\left(e+\right.\right.}{\sqrt{2}\,c}$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.54, size = 467, normalized size = 0.89

$$\frac{2\sqrt{a}\tanh^{-1}\left(\frac{\sqrt{c}x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)-\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}d\#1+2a\sqrt{c}e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}e\#1^3+f\#1^4\&,\frac{b^2d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-acd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-abe\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)+a^2f\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)-2b\sqrt{c}d\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+2a\sqrt{c}e\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1+cd\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-af\log\left(-\sqrt{c}x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}d-a\sqrt{c}e-4cd\#1-be\#1+2af\#1+3\sqrt{c}e\#1^2-2f\#1^3}\&\right]}{d}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(x*(d + e*x + f*x^2)),x]
```

```
[Out] (2*Sqrt[a]*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]] - RootSum[b
^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b
*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b^2*d*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1] - a*c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c
*x^2] - #1] - a*b*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*f*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d*Log[-(Sqrt[c
]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*e*Log[-(Sqrt[c]*x) + Sq
```

```
rt[a + b*x + c*x^2] - #1]*#1 + c*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
 - #1]*#1^2 - a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b
*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2
- 2*f*#1^3) & ])/d
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 1690 vs. $2(460) = 920$.
time = 0.18, size = 1691, normalized size = 3.23

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1691 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/x/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2
*c*d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2/f*(-
c*(-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1/2)+(
(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x
+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4*d*f+
e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/
((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^
2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d
*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+
c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+2*f/(-e+(-4*d*f+e^2)^
(1/2))/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(
c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-
4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(
1/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^(1/2)+b*f
-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e+(-4*d*f+
e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2
*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-
4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c
*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2
)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e
```

```
^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(
1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)
-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(
-e+(-4*d*f+e^2)^(1/2)))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2)
)*((c*x^2+b*x+a)^(1/2)+1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^
(1/2)-a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] integrate(sqrt(c*x^2 + b*x + a)/((f*x^2 + x*e + d)*x), x)
```

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a + bx + cx^2}}{x\,(d + ex + fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/x/(f*x**2+e*x+d),x)
```

```
[Out] Integral(sqrt(a + b*x + c*x**2)/(x*(d + e*x + f*x**2)), x)
```

**Giac** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Warning, integration of abs or sign a
ssumes constant sign by intervals (correct if the argument is real):Check [
abs(sa
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+b\,x+a}}{x\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(x*(d + e*x + f*x^2)),x)
[Out] int((a + b*x + c*x^2)^(1/2)/(x*(d + e*x + f*x^2)), x)
```

# 3.113 $\int \frac{\sqrt{a+bx+cx^2}}{x^2(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=736

$$-\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d}+\frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d^2}+\frac{\sqrt{c}\,\tanh^{-1}\left(\right.}{}$$

```
[Out] -1/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d/a^(1/2)+e*arcta
nh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))*a^(1/2)/d^2-1/2*b*e*arctanh(1
/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/d^2/c^(1/2)-1/2*(-b*e+2*c*d)*arct
anh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/d^2/c^(1/2)+arctanh(1/2*(2*c
*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))*c^(1/2)/d-(c*x^2+b*x+a)^(1/2)/d/x-1/2*f*
arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/
2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-
4*d*f+e^2)^(1/2))^(1/2))*(2*c*d^2-b*d*(e+(-4*d*f+e^2)^(1/2))+a*(e^2-2*d*f+e
*(-4*d*f+e^2)^(1/2)))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(f*(2*a*f-b*(e-(-4*d*f
+e^2)^(1/2)))+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)+1/2*f*arctanh(1/4*(
4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/
(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1
/2))^(1/2))*(2*c*d^2-b*d*(e-(-4*d*f+e^2)^(1/2))+a*(e^2-2*d*f-e*(-4*d*f+e^2)
^(1/2)))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))
+f*(2*a*f-b*(e+(-4*d*f+e^2)^(1/2))))^(1/2)
```

**Rubi** [A]
time = 2.11, antiderivative size = 736, normalized size of antiderivative = 1.00, number of steps used = 23, number of rules used = 10, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.333$, Rules used = {6860, 746, 857, 635, 212, 738, 748, 1033, 1090, 1046}

$$-\frac{f\left(a\left(e\sqrt{e^2-4df}-2df+e^2\right)-bd\left(\sqrt{e^2-4df}+e\right)+2cd^2\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(a\left(-e\sqrt{e^2-4df}-2df+e^2\right)-bd\left(e-\sqrt{e^2-4df}\right)+2cd^2\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{d^2}-\frac{be\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,d^2}-\frac{(2cd-be)\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{c}\,d^2}-\frac{\sqrt{a+bx+cx^2}}{dx}-\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d}+\frac{\sqrt{c}\,\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{d}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[a + b*x + c*x^2]/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/(d*x)) - (b*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a +
 b*x + c*x^2])])/(2*Sqrt[a]*d) + (Sqrt[a]*e*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*
Sqrt[a + b*x + c*x^2])])/d^2 + (Sqrt[c]*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt
[a + b*x + c*x^2])])/d - (b*e*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x +
 c*x^2])])/(2*Sqrt[c]*d^2) - ((2*c*d - b*e)*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*
Sqrt[a + b*x + c*x^2])])/(2*Sqrt[c]*d^2) - (f*(2*c*d^2 - b*d*(e + Sqrt[e^2
- 4*d*f]) + a*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]))*ArcTanh[(4*a*f - b*(e -
Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt
```

```
[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a
+ b*x + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*(e^2 - 2*d*f - e*Sq
rt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt[e^2 - 4*d*f]))]) + (f*(2*c*d^2 -
b*d*(e - Sqrt[e^2 - 4*d*f]) + a*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]))*ArcTan
h[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*
x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2
 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*(
e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e + Sqrt[e^2 - 4*d*f]))]
)
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 746

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 1))), x] - Di
st[p/(e*(m + 1)), Int[(d + e*x)^(m + 1)*(b + 2*c*x)*(a + b*x + c*x^2)^(p -
1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*
d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && GtQ[p, 0] && (IntegerQ[p]
 || LtQ[m, -1]) && NeQ[m, -1] &&  !ILtQ[m + 2*p + 1, 0] && IntQuadraticQ[a,
 b, c, d, e, m, p, x]
```

Rule 748

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*((a + b*x + c*x^2)^p/(e*(m + 2*p + 1))), x
] - Dist[p/(e*(m + 2*p + 1)), Int[(d + e*x)^m*Simp[b*d - 2*a*e + (2*c*d - b
*e)*x, x]*(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e
, 0] && GtQ[p, 0] && NeQ[m + 2*p + 1, 0] && ( !RationalQ[m] || LtQ[m, 1]) &
```

```
&  !ILtQ[m + 2*p, 0] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 857

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Dist[g/e, Int[(d + e*x)^(m + 1)*(a + b*x +
c*x^2)^p, x], x] + Dist[(e*f - d*g)/e, Int[(d + e*x)^m*(a + b*x + c*x^2)^p,
 x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x] && NeQ[b^2 - 4*a*c, 0] &&
NeQ[c*d^2 - b*d*e + a*e^2, 0] &&  !IGtQ[m, 0]
```

Rule 1033

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[h*(a + b*x + c*x^2)^p*((d
+ e*x + f*x^2)^(q + 1)/(2*f*(p + q + 1))), x] - Dist[1/(2*f*(p + q + 1)), I
nt[(a + b*x + c*x^2)^(p - 1)*(d + e*x + f*x^2)^q*Simp[h*p*(b*d - a*e) + a*(
h*e - 2*g*f)*(p + q + 1) + (2*h*p*(c*d - a*f) + b*(h*e - 2*g*f)*(p + q + 1)
)*x + (h*p*(c*e - b*f) + c*(h*e - 2*g*f)*(p + q + 1))*x^2, x], x], x] /; Fr
eeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d
*f, 0] && GtQ[p, 0] && NeQ[p + q + 1, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1090

```
Int[((A_.) + (B_.)*(x_) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)
*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqr
t[d + e*x + f*x^2], x], x] + Dist[1/c, Int[(A*c - a*C + (B*c - b*C)*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A
, B, C}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{\sqrt{a+bx+cx^2}}{x^2\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{\sqrt{a+bx+cx^2}}{dx^2} - \frac{e\sqrt{a+bx+cx^2}}{d^2x} + \frac{\left(e^2-df+efx\right)\sqrt{a+bx+cx^2}}{d^2\left(d+ex+fx^2\right)}\right) d$$

$$= \frac{\int \frac{\left(e^2-df+efx\right)\sqrt{a+bx+cx^2}}{d+ex+fx^2}\,dx}{d^2} + \frac{\int \frac{\sqrt{a+bx+cx^2}}{x^2}\,dx}{d} - \frac{e\int \frac{\sqrt{a+bx+cx^2}}{x}\,d}{d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx} + \frac{\int \frac{b+2cx}{x\sqrt{a+bx+cx^2}}\,dx}{2d} + \frac{e\int \frac{-2a-bx}{x\sqrt{a+bx+cx^2}}\,dx}{2d^2} - \frac{\int \frac{\frac{1}{2}f(bd}{}}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx} + \frac{b\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{2d} + \frac{c\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{d} - \frac{(ae)\int}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx} - \frac{b\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx, x, \frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{d} + \frac{(2c)\text{Subst}\left(\int \frac{}{4c}\right.}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx} - \frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d} + \frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{}{2\sqrt{a}\sqrt{}}\right.}{d^2}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{dx} - \frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2\sqrt{a}\,d} + \frac{\sqrt{a}\,e\tanh^{-1}\left(\frac{}{2\sqrt{a}\sqrt{}}\right.}{d^2}$$

**<u>Mathematica</u>** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.74, size = 582, normalized size = 0.79

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[a + b*x + c*x^2]/(x^2*(d + e*x + f*x^2)),x]
```

```
[Out] (-((d*Sqrt[a + x*(b + c*x)])/x) + ((-(b*d) + 2*a*e)*ArcTanh[(-(Sqrt[c]*x) +
 Sqrt[a + x*(b + c*x)])/Sqrt[a]])/Sqrt[a] + RootSum[b^2*d - a*b*e + a^2*f -
 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 -
 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(b*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1]) + b^2*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a
*b*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*e*f*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1] + 2*c^(3/2)*d^2*Log[-(Sqrt[c]*x) + Sqrt
```

```
[a + b*x + c*x^2] - #1]*#1 - 2*b*Sqrt[c]*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1]*#1 - 2*a*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #
1]*#1 + b*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*e*f*L
og[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt
[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/d^
2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1905 vs. $2(633) = 1266$.
time = 0.19, size = 1906, normalized size = 2.59

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1906 |
| risch | Expression too large to display | 4093 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((c*x^2+b*x+a)^(1/2)/x^2/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x+1/2*(e+(-4*d*
f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*
e*f-2*c*d*f+c*e^2)/f^2)^(1/2)+1/2/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*ln((1/2
/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)+c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))/c^(1
/2)+((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*
e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^
2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(-b*f*(-4
*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(
1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^
(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1
/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c
*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+4*f^2/(-e+(-4*d*
f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*(1/2*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))
)^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+
2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^
2)/f^2)^(1/2)+1/2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*ln((1/2*(c*(-4*d*f+e^2)^
(1/2)+b*f-c*e)/f+c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))/c^(1/2)+((x-1/2/f*(-e
+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/c^(1/2)-1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2*2^(1/2)/((b*f*(-4*d*f+e
```

```
^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln
(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^
2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1
/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*
c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d
*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e
^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e
^2)^(1/2))*(-1/a/x*(c*x^2+b*x+a)^(3/2)+1/2*b/a*((c*x^2+b*x+a)^(1/2)+1/2*b*l
n((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b*x+2*a^
(1/2)*(c*x^2+b*x+a)^(1/2))/x))+2*c/a*(1/4*(2*c*x+b)/c*(c*x^2+b*x+a)^(1/2)+1
/8*(4*a*c-b^2)/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))))-16*f^2
*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2*((c*x^2+b*x+a)^(1/2)+
1/2*b*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-a^(1/2)*ln((2*a+b
*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(sqrt(c*x^2 + b*x + a)/((f*x^2 + x*e + d)*x^2), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{a+bx+cx^2}}{x^2\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x**2+b*x+a)**(1/2)/x**2/(f*x**2+e*x+d),x)
[Out] Integral(sqrt(a + b*x + c*x**2)/(x**2*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((c*x^2+b*x+a)^(1/2)/x^2/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{\sqrt{c\,x^2+b\,x+a}}{x^2\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((a + b*x + c*x^2)^(1/2)/(x^2*(d + e*x + f*x^2)),x)
```

```
[Out] int((a + b*x + c*x^2)^(1/2)/(x^2*(d + e*x + f*x^2)), x)
```

# 3.114 $\int \frac{x^3}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=545

$$\frac{\sqrt{a+bx+cx^2}}{cf}-\frac{e\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2}-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2c^{3/2}f}-\frac{\left(2def-(e^2-df\right.}{}$$

```
[Out] -1/2*b*arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/c^(3/2)/f-e*arcta
nh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f^2/c^(1/2)+(c*x^2+b*x+a)^(1/
2)/c/f-1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d
*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-
b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*e*f-(-d*f+e^2)*(e-(-4*d*f+e^2)^(1/
2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)
*(-4*d*f+e^2)^(1/2))^(1/2)+1/2*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+
2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*
d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*e*f-(-d*f+e^2)
*(e+(-4*d*f+e^2)^(1/2)))/f^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*
f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 2.30, antiderivative size = 545, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 6, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.200$, Rules used = {6860, 635, 212, 654, 1046, 738}

$$-\frac{b\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{2c^{3/2}f}-\frac{\left(2def-(e^2-df)\left(e-\sqrt{e^2-4df}\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}+\frac{\left(2def-(e^2-df)\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f^2\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}-\frac{e\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2}+\frac{\sqrt{a+bx+cx^2}}{cf}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] Sqrt[a + b*x + c*x^2]/(c*f) - (e*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*
x + c*x^2])])/(Sqrt[c]*f^2) - (b*ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*
x + c*x^2])])/(2*c^(3/2)*f) - ((2*d*e*f - (e^2 - d*f)*(e - Sqrt[e^2 - 4*d*f
]))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 -
 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f
)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f
]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
+ ((2*d*e*f - (e^2 - d*f)*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(4*a*f - b*(e +
Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt
[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a
+ b*x + c*x^2])])/(Sqrt[2]*f^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e
*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^3}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(-\frac{e}{f^2\sqrt{a+bx+cx^2}} + \frac{x}{f\sqrt{a+bx+cx^2}} + \frac{de+(e^2 -}{f^2\sqrt{a+bx+cx^2}}\right.$$

$$= \frac{\int \frac{de+(e^2-df)x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{f^2} - \frac{e\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f^2} + \frac{\int \frac{}{\sqrt{a-}}}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{cf} - \frac{(2e)\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f^2} - \frac{b}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{cf} - \frac{e\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2} - \frac{b\text{Subst}\left(\int\right.}{}$$

$$= \frac{\sqrt{a+bx+cx^2}}{cf} - \frac{e\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f^2} - \frac{b\tanh^{-1}\left(\right.}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.66, size = 423, normalized size = 0.78

$$\frac{\frac{2f\sqrt{a+x(b+cx)}}{c} + \frac{(2ce+bf)\log\left(c\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)}{c^{3/2}} - 2\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{bde\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-ae^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+adf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,de\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+e^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}\,d-a\sqrt{c}\,e-4cd\#1-be\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]}{2f^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((2*f*Sqrt[a + x*(b + c*x)])/c + ((2*c*e + b*f)*Log[c*(b + 2*c*x - 2*Sqrt[c
]*Sqrt[a + x*(b + c*x)])])/c^(3/2) - 2*RootSum[b^2*d - a*b*e + a^2*f - 4*b*
Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sq
rt[c]*e*#1^3 + f*#1^4 & , (b*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1] - a*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*d*f*Log[-(S
qrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*d*e*Log[-(Sqrt[c]*x) +
Sqrt[a + b*x + c*x^2] - #1]*#1 + e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^
2] - #1]*#1^2 - d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2
*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^
2 - 2*f*#1^3) & ])/(2*f^2)
```

**Maple** [A]
time = 0.19, size = 930, normalized size = 1.71

| method | result |
|---|---|
| default | $\frac{\frac{\sqrt{c\,x^{2}+b x +a}}{c}-\frac{b \ln\left(\frac{\frac{b}{2}+c x}{\sqrt{c}}+\sqrt{c\,x^{2}+b x +a}\right)}{2 c^{\frac{3}{2}}}}{f}-\frac{e \ln\left(\frac{\frac{b}{2}+c x}{\sqrt{c}}+\sqrt{c\,x^{2}+b x +a}\right)}{f^{2} \sqrt{c}}-\frac{\left(e^{3}-3 d e f+e^{2} \sqrt{-4 d f}\right.}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(1/c*(c*x^2+b*x+a)^(1/2)-1/2*b/c^(3/2)*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*
x+a)^(1/2)))-e/f^2*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-1/2*
(e^3-3*d*e*f+e^2*(-4*d*f+e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2))/f^3/(-4*d*f+e^2
)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*
e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)
*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*
d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-1/2*(-
d*f*(-4*d*f+e^2)^(1/2)+e^2*(-4*d*f+e^2)^(1/2)+3*d*e*f-e^3)/f^3/(-4*d*f+e^2)
^(1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/
2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^3}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**3/(sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: TypeError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:Error: Bad Argument Type
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{\sqrt{c\,x^2+b\,x+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^3/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.115 $\int \frac{x^2}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=463

$$\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}-\frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2(\ldots}{2\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(c\ldots}}\right.}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{ce^2-2cdf-bef+2af^2-(ce-\ldots}}$$

```
[Out] arctanh(1/2*(2*c*x+b)/c^(1/2)/(c*x^2+b*x+a)^(1/2))/f/c^(1/2)-1/2*arctanh(1/
4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/
2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)
^(1/2))^(1/2))*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f*2^(1/2)/(-4*d*f+e^2)^(1/2
)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*arc
tanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2)))
)*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d
*f+e^2)^(1/2))^(1/2))*(2*d*f-e*(e+(-4*d*f+e^2)^(1/2)))/f*2^(1/2)/(-4*d*f+e^
2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]
time = 2.09, antiderivative size = 463, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = $\{1091, 635, 212, 1046, 738\}$

$$-\frac{\left(-e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}-\frac{\left(2df-e\left(\sqrt{e^2-4df}+e\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,f\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}+\frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ArcTanh[(b + 2*c*x)/(2*Sqrt[c]*Sqrt[a + b*x + c*x^2])]/(Sqrt[c]*f) - ((e^2
- 2*d*f - e*Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) +
 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b
*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(S
qrt[2]*f*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e -
b*f)*Sqrt[e^2 - 4*d*f]]) - ((2*d*f - e*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(4*
a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2
*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*
d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*f*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*
c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1091

```
Int[((A_.) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C - b*C*x)/((a + b*x + c*x^2)*Sqrt[d
 + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A, C}, x] && NeQ[b^2 -
 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rubi steps

$$\int \frac{x^2}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \frac{\int \frac{1}{\sqrt{a+bx+cx^2}}\,dx}{f} + \frac{\int \frac{-d-ex}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{f}$$

$$= \frac{2\text{Subst}\left(\int \frac{1}{4c-x^2}\,dx, x, \frac{b+2cx}{\sqrt{a+bx+cx^2}}\right)}{f} + \frac{\left(e^2-2df-e\sqrt{e^2-4df}\right.}{}$$

$$= \frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f} - \frac{\left(2\left(e^2-2df-e\sqrt{e^2-4df}\right)\right)\text{S}}{}$$

$$= \frac{\tanh^{-1}\left(\frac{b+2cx}{2\sqrt{c}\sqrt{a+bx+cx^2}}\right)}{\sqrt{c}\,f} - \frac{\left(e^2-2df-e\sqrt{e^2-4df}\right)\tanh^{-1}}{\sqrt{2}\,f\sqrt{e^2-}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.45, size = 318, normalized size = 0.69

$$\frac{-\frac{\log\left(f\left(b+2cx-2\sqrt{c}\sqrt{a+x(b+cx)}\right)\right)}{\sqrt{c}} + \text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&, \frac{bd\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-ae\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,d\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+e\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}\,d-a\sqrt{c}\,e-4cd\#1-be\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]}{f}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] (-(Log[f*(b + 2*c*x - 2*Sqrt[c]*Sqrt[a + x*(b + c*x)])]/Sqrt[c]) + RootSum[
b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 +
b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*d*Log[-(Sqrt[c]*x)
 + Sqrt[a + b*x + c*x^2] - #1] - a*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^
2] - #1] - 2*Sqrt[c]*d*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 +
e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*S
qrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])
/f
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 843 vs. 2(407) = 814.
time = 0.17, size = 844, normalized size = 1.82

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{\ln\left(\frac{\frac{b}{2}+cx}{\sqrt{c}}+\sqrt{c\,x^2+bx+a}\right)}{f\sqrt{c}} - \frac{\left(-e\sqrt{-4df+e^2}+2df-e^2\right)\sqrt{2}\,\ln\left(\frac{-bf\sqrt{-4df+e^2}+\sqrt{-4df+e^2}\,ce+2a}{f^2}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*ln((1/2*b+c*x)/c^(1/2)+(c*x^2+b*x+a)^(1/2))/c^(1/2)-1/2*(-e*(-4*d*f+e^2
)^(1/2)+2*d*f-e^2)/f^2/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)
+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(
-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1
/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(
1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+
e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(
1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2
*(e+(-4*d*f+e^2)^(1/2))/f))-1/2*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d*
f+e^2)^(1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f
*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e
+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(
-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima** [F(-2)]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

$$\text{Timed out}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**2/(sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

$$\text{Exception raised: TypeError}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Exception raised: TypeError >> An error occurred running a Giac command:INP
UT:sage2:=int(sage0,sageVARx):;OUTPUT:sym2poly/r2sym(const gen & e,const in
dex_m & i,const vecteur & l) Error: Bad Argument Value
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{\sqrt{c\,x^2+b\,x+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^2/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.116 $\int \frac{x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=402

$$\frac{\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)x}{2\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{e^2-4df}}\sqrt{a+bx+cx^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{e^2-4df}}}-\cdots$$

```
[Out] 1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)
^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e
)*(-4*d*f+e^2)^(1/2))^(1/2))*(e-(-4*d*f+e^2)^(1/2))*2^(1/2)/(-4*d*f+e^2)^(1
/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*a
rctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2)
)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4
*d*f+e^2)^(1/2))^(1/2))*(e+(-4*d*f+e^2)^(1/2))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(
c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.61, antiderivative size = 402, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.107$, Rules used = {1046, 738, 212}

$$\frac{\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{\left(\sqrt{e^2-4df}+e\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}$$

Antiderivative was successfully verified.

```
[In] Int[x/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((e - Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*
f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f +
 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]
*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqr
t[e^2 - 4*d*f]]) - ((e + Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e + Sqrt[e^
2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2
- 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x +
 c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^
2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = -\left(\left(-1-\frac{e}{\sqrt{e^2-4df}}\right)\int \frac{1}{\left(e+\sqrt{e^2-4df}+2fx\right)\sqrt{a+bx+c\cdots}}\cdots\right.$$

$$= -\left(\left(2\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\right)\text{Subst}\left(\int \frac{\cdots}{16af^2-8bf\left(e-\sqrt{e^2-4d\cdots}\right.}\cdots\right.\right.$$

$$= -\frac{\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\cdots}{2\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(\cdots}}\cdots\right.}{\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(ce-b\cdots}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.38, size = 204, normalized size = 0.51

$$\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-a\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-2b\sqrt{c}\,d+a\sqrt{c}\,e+4cd\#1+be\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[x/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d
*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(a*Log[-(Sq
```

```
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]) + Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
 c*x^2] - #1]*#1^2)/(-2*b*Sqrt[c]*d + a*Sqrt[c]*e + 4*c*d*#1 + b*e*#1 - 2*a
*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ]
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 793 vs. $2(355) = 710$.

time = 0.16, size = 794, normalized size = 1.98

| method | result |
|---|---|
| default | $-\frac{\left(e+\sqrt{-4df+e^{2}}\right)\sqrt{2}\,\ln\left(\frac{-bf\sqrt{-4df+e^{2}}+\sqrt{-4df+e^{2}}\,ce+2a\,f^{2}-bef-2cdf+c\,e^{2}}{f^{2}}+\frac{\left(-c\sqrt{-4df+e^{2}}+bf\right.}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*2^(1/2)/((-b*f*(-4*d*f+e^2
)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln((
(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2
)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+
1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-
2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(
-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f
+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))
/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-1/2*(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)
^(1/2)/f*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*
e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-
1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f
+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4
*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d
*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*
e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima** [**F(-2)**]

time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 11131 vs. $2(351) = 702$.
time = 5.12, size = 11131, normalized size = 27.69

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt(2)*sqrt(-(2*c*d^2 - 2*a*d*f - b*d*e + a*e^2 + (4*c^2*d^3*f + 4*a^
2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*d^
2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(
b^2*d^2 - 2*a*b*d*e + a^2*e^2)/(4*c^4*d^5*f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*
a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a
^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2
*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f + (a^2*b^2 + 2*a^3*c)*f^2)*e^4 +
2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*
d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a
*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2 + 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*
d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^3*f^2 + (a*b^3 - a^2*b*c)*d^2*f^3
)*e)))/(4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*
c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2
*f + a*b*d*f^2)*e))*log(-(4*b*c*d^3*x + 2*b^2*d^3 + sqrt(2)*(4*b^2*d^3*f -
8*a*b*d^2*f*e + 2*a*b*d*e^3 - a^2*e^4 - (b^2*d^2 - 4*a^2*d*f)*e^2 + (8*c^3*
d^5*f - 8*a^3*d^2*f^4 + 8*(b^2*c - 3*a*c^2)*d^4*f^2 - 8*(a*b^2 - 3*a^2*c)*d
^3*f^3 - a^2*c*e^6 + (2*a*b*c*d + a^2*b*f)*e^5 - (a^3*f^2 + (b^2*c + 3*a*c^
2)*d^2 + 2*(a*b^2 - 4*a^2*c)*d*f)*e^4 + (3*b*c^2*d^3 - 5*a^2*b*d*f^2 + (b^3
 - 10*a*b*c)*d^2*f)*e^3 - 2*(c^3*d^4 - 3*a^3*d*f^3 - (b^2*c + 9*a*c^2)*d^3*
f - (5*a*b^2 - 11*a^2*c)*d^2*f^2)*e^2 - 4*(3*b*c^2*d^4*f - a^2*b*d^2*f^3 +
(b^3 - 2*a*b*c)*d^3*f^2)*e)*sqrt(-(b^2*d^2 - 2*a*b*d*e + a^2*e^2)/(4*c^4*d^
5*f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*
a^2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2
*d + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*
f + (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*
b*c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2
*c^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^
2 + 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*
d^3*f^2 + (a*b^3 - a^2*b*c)*d^2*f^3)*e)))*sqrt(c*x^2 + b*x + a)*sqrt(-(2*c*
d^2 - 2*a*d*f - b*d*e + a*e^2 + (4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c
```

```
)*d^2*f^2 - a*c*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a
*c)*d*f)*e^2 - 4*(b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(b^2*d^2 - 2*a*b*d*e + a^
2*e^2)/(4*c^4*d^5*f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4
- 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*
e^6 + 2*(a*b*c^2*d + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c
 - 2*a^2*c^2)*d*f + (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3
 + (b^3*c - 5*a*b*c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 +
a^4*f^4 - 2*(b^2*c^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2
*f^2 - 2*(a^2*b^2 + 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b
^3*c - a*b*c^2)*d^3*f^2 + (a*b^3 - a^2*b*c)*d^2*f^3)*e)))/(4*c^2*d^3*f + 4*
a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*
d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2*f + a*b*d*f^2)*e)) + (a
*b*d*x + 2*a^2*d)*e^2 - (4*a*b*d^2 + (b^2 + 4*a*c)*d^2*x)*e - (8*a*c^2*d^4*
f + 8*a^3*d^2*f^3 + 8*(a*b^2 - 2*a^2*c)*d^3*f^2 + 4*(b*c^2*d^4*f + a^2*b*d^
2*f^3 + (b^3 - 2*a*b*c)*d^3*f^2)*x - (a*b*c*d*x + 2*a^2*c*d)*e^4 + (2*a*b*c
*d^2 + 2*a^2*b*d*f + (b^2*c*d^2 + a*b^2*d*f)*x)*e^3 - (2*a*c^2*d^3 + 2*a^3*
d*f^2 + 2*(a*b^2 - 6*a^2*c)*d^2*f + (b*c^2*d^3 + a^2*b*d*f^2 + (b^3 - 6*a*b
*c)*d^2*f)*x)*e^2 - 4*(2*a*b*c*d^3*f + 2*a^2*b*d^2*f^2 + (b^2*c*d^3*f + a*b
^2*d^2*f^2)*x)*e)*sqrt(-(b^2*d^2 - 2*a*b*d*e + a^2*e^2)/(4*c^4*d^5*f + 4*a^
4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d
^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d + a^2*b
*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f + (a^2*b
^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*c^2)*d^2
*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c^2 + 6*a
*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2 + 6*a^3*
c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^3*f^2 +
(a*b^3 - a^2*b*c)*d^2*f^3)*e)))/x) + 1/4*sqrt(2)*sqrt(-(2*c*d^2 - 2*a*d*f -
 b*d*e + a*e^2 + (4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c
*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 -
4*(b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(b^2*d^2 - 2*a*b*d*e + a^2*e^2)/(4*c^4*d
^5*f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6
*a^2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^
2*d + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d
*f + (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a
*b*c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^
2*c^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b
^2 + 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3...
```

**Sympy [F]**

time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x/(sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

$$\text{Timed out}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\sqrt{c\,x^2+b\,x+a}\,\,\left(f\,x^2+e\,x+d\right)} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.117 $\int \frac{1}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=374

$$-\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)x}{2\sqrt{2}\,\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{e^2-4df}}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{e^2-4df}\,\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{e^2-4df}}}+\frac{\sqrt{2}\,f\tanh^{-1}}{}$$

```
[Out] -f*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^
(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)
*(-4*d*f+e^2)^(1/2))^(1/2))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f
+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)+f*arctanh(1/4*(4*a*f-b*(e+(-4
*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^
(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*2^
(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^
2)^(1/2))^(1/2)
```

**Rubi** [A]
time = 0.19, antiderivative size = 374, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 3, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.111$, Rules used = {997, 738, 212}

$$\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\,\sqrt{a+bx+cx^2}\,\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{e^2-4df}\,\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}-\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\,\sqrt{a+bx+cx^2}\,\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{e^2-4df}\,\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -((Sqrt[2]*f*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - S
qrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (
c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[e^2 - 4*d*f]*S
qrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]])) +
(Sqrt[2]*f*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqr
t[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*
e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[e^2 - 4*d*f]*Sqr
t[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 997

```
Int[1/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*(
x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dist[2*(c/q), Int[1/((
b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x], x] - Dist[2*(c/q), Int[1/((b + q
 + 2*c*x)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b, c, d, e, f}, x] &
& NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e - b*f, 0] && PosQ[b
^2 - 4*a*c]
```

Rubi steps

$$\int \frac{1}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \frac{(2f)\int \frac{1}{\left(e-\sqrt{e^2-4df}+2fx\right)\sqrt{a+bx+cx^2}}\,dx}{\sqrt{e^2-4df}} - \frac{(2f)\int \frac{1}{\left(e+\sqrt{e^2-\ldots\right.}}}{\ldots}$$

$$= -\frac{(4f)\mathrm{Subst}\left(\int \frac{1}{16af^2-8bf\left(e-\sqrt{e^2-4df}\right)+4c\left(e-\sqrt{e^2-4df}\right)^2-x^2}\,dx,\ldots\right.}{\sqrt{e^2-4df}\ldots}$$

$$= -\frac{\sqrt{2}\,f\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\left(bf-c\left(e-\sqrt{e^2-\ldots\right.\right.}}{2\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{e^2-\ldots}}}\right.}{\sqrt{e^2-4df}\sqrt{ce^2-2cdf-bef+2af^2-(ce-bf)\sqrt{\ldots}}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.01, size = 211, normalized size = 0.56

$$-\mathrm{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{b\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1}{2b\sqrt{c}\,d-a\sqrt{c}\,e-4cd\#1-be\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*
d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*Log[-(Sqr
```

```
t[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*Log[-(Sqrt[c]*x) + Sqrt[a
 + b*x + c*x^2] - #1]*#1)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1
+ 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ]
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 760 vs. $2(330) = 660$.
time = 0.00, size = 761, normalized size = 2.03

| method | result |
|---|---|
| default | $\sqrt{2}\ \ln\left(\frac{-bf\sqrt{-4df+e^2}+\sqrt{-4df+e^2}\,ce+2a f^2-bef-2cdf+c e^2}{f^2}+\frac{\left(-c\sqrt{-4df+e^2}+bf-ce\right)\left(x+\frac{e+\sqrt{-4df+e}}{2f}\right.}{f}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c
*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4
*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(
1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e
^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4
*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f))-1/(-4*d*f+e^2)^(1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2
)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(
1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f
+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*
(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f
*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)
^(1/2))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [B] Leaf count of result is larger than twice the leaf count of optimal. 11127 vs. $2(326) = 652$.
time = 4.97, size = 11127, normalized size = 29.75

Too large to display

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt(2)*sqrt((2*c*d*f - 2*a*f^2 + b*f*e - c*e^2 + (4*c^2*d^3*f + 4*a^2*
d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2
+ a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(b^
2*f^2 - 2*b*c*f*e + c^2*e^2)/(4*c^4*d^5*f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*
c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3
*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a
*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f + (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*
(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*
f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b
^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2 + 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^
4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^3*f^2 + (a*b^3 - a^2*b*c)*d^2*f^3)*
e)))/(4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*c*
d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2*f
 + a*b*d*f^2)*e))*log(-(4*b*c*d*f^2*x + 2*b^2*d*f^2 + sqrt(2)*(4*b*c*d^2*f^
2 - 4*a*b*d*f^3 + (c^2*d - a*c*f)*e^3 - (b*c*d*f - a*b*f^2)*e^2 - 4*(c^2*d^
2*f - a*c*d*f^2)*e - (8*b*c^2*d^4*f^2 + 8*a^2*b*d^2*f^4 + 8*(b^3 - 2*a*b*c)
*d^3*f^3 + (a*c^2*d + a^2*c*f)*e^5 - (b*c^2*d^2 + 4*a*b*c*d*f + a^2*b*f^2)*
e^4 + (c^3*d^3 + a^3*f^3 + (3*b^2*c - 5*a*c^2)*d^2*f + (3*a*b^2 - 5*a^2*c)*
d*f^2)*e^3 + 2*(b*c^2*d^3*f + a^2*b*d*f^3 - (b^3 - 10*a*b*c)*d^2*f^2)*e^2 -
 4*(c^3*d^4*f + a^3*d*f^4 + (3*b^2*c - a*c^2)*d^3*f^2 + (3*a*b^2 - a^2*c)*d
^2*f^3)*e)*sqrt(-(b^2*f^2 - 2*b*c*f*e + c^2*e^2)/(4*c^4*d^5*f + 4*a^4*d*f^5
 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3*f^3
+ 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d + a^2*b*c*f)*e
^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f + (a^2*b^2 + 2*
a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*c^2)*d^2*f + (a
*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c^2 + 6*a*c^3)*d
^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2 + 6*a^3*c)*d*f^
3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^3*f^2 + (a*b^3
- a^2*b*c)*d^2*f^3)*e)))*sqrt(c*x^2 + b*x + a)*sqrt((2*c*d*f - 2*a*f^2 + b*
f*e - c*e^2 + (4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^
```

```
4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(
b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(b^2*f^2 - 2*b*c*f*e + c^2*e^2)/(4*c^4*d^5*
f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^
2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d
 + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f
+ (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*
c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c
^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2
+ 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^
3*f^2 + (a*b^3 - a^2*b*c)*d^2*f^3)*e)))/(4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2
 - 2*a*c)*d^2*f^2 - a*c*e^4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b
^2 - 6*a*c)*d*f)*e^2 - 4*(b*c*d^2*f + a*b*d*f^2)*e)) + (b*c*f*x + 2*a*c*f)*
e^2 - (2*b*c*d*f + 2*a*b*f^2 + (4*c^2*d*f + b^2*f^2)*x)*e - (8*a*c^2*d^3*f^
2 + 8*a^3*d*f^4 + 8*(a*b^2 - 2*a^2*c)*d^2*f^3 + 4*(b*c^2*d^3*f^2 + a^2*b*d*
f^4 + (b^3 - 2*a*b*c)*d^2*f^3)*x - (a*b*c*f*x + 2*a^2*c*f)*e^4 + (2*a*b*c*d
*f + 2*a^2*b*f^2 + (b^2*c*d*f + a*b^2*f^2)*x)*e^3 - (2*a*c^2*d^2*f + 2*a^3*
f^3 + 2*(a*b^2 - 6*a^2*c)*d*f^2 + (b*c^2*d^2*f + a^2*b*f^3 + (b^3 - 6*a*b*c
)*d*f^2)*x)*e^2 - 4*(2*a*b*c*d^2*f^2 + 2*a^2*b*d*f^3 + (b^2*c*d^2*f^2 + a*b
^2*d*f^3)*x)*e)*sqrt(-(b^2*f^2 - 2*b*c*f*e + c^2*e^2)/(4*c^4*d^5*f + 4*a^4*
d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^2*c^2)*d^3
*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d + a^2*b*c
*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f + (a^2*b^2
 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*c^2)*d^2*f
 + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c^2 + 6*a*c
^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2 + 6*a^3*c)
*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*d*f^4 + (b^3*c - a*b*c^2)*d^3*f^2 + (a
*b^3 - a^2*b*c)*d^2*f^3)*e)))/x) - 1/4*sqrt(2)*sqrt((2*c*d*f - 2*a*f^2 + b*
f*e - c*e^2 + (4*c^2*d^3*f + 4*a^2*d*f^3 + 4*(b^2 - 2*a*c)*d^2*f^2 - a*c*e^
4 + (b*c*d + a*b*f)*e^3 - (c^2*d^2 + a^2*f^2 + (b^2 - 6*a*c)*d*f)*e^2 - 4*(
b*c*d^2*f + a*b*d*f^2)*e)*sqrt(-(b^2*f^2 - 2*b*c*f*e + c^2*e^2)/(4*c^4*d^5*
f + 4*a^4*d*f^5 + 8*(b^2*c^2 - 2*a*c^3)*d^4*f^2 + 4*(b^4 - 4*a*b^2*c + 6*a^
2*c^2)*d^3*f^3 + 8*(a^2*b^2 - 2*a^3*c)*d^2*f^4 - a^2*c^2*e^6 + 2*(a*b*c^2*d
 + a^2*b*c*f)*e^5 - ((b^2*c^2 + 2*a*c^3)*d^2 + 4*(a*b^2*c - 2*a^2*c^2)*d*f
+ (a^2*b^2 + 2*a^3*c)*f^2)*e^4 + 2*(b*c^3*d^3 + a^3*b*f^3 + (b^3*c - 5*a*b*
c^2)*d^2*f + (a*b^3 - 5*a^2*b*c)*d*f^2)*e^3 - (c^4*d^4 + a^4*f^4 - 2*(b^2*c
^2 + 6*a*c^3)*d^3*f + (b^4 - 20*a*b^2*c + 22*a^2*c^2)*d^2*f^2 - 2*(a^2*b^2
+ 6*a^3*c)*d*f^3)*e^2 - 8*(b*c^3*d^4*f + a^3*b*...
```

**Sympy** [**F**]

time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/(sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{\sqrt{c\,x^2+b\,x+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/((a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.118 $\int \frac{1}{x\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=451

$$-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d}+\frac{f\left(e+\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\left(bf-\right.}{2\sqrt{2}\sqrt{ce^2-2cdf-bef+2af^2-(ce-}\right.}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{ce^2-2cdf-bef+2af^2-(ce-}}$$

```
[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d/a^(1/2)+1/2*f*arctanh
(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^
(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e
^2)^(1/2))^(1/2))*(e+(-4*d*f+e^2)^(1/2))/d*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^
2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*f*arctanh(
1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(
1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^
2)^(1/2))^(1/2))*(e-(-4*d*f+e^2)^(1/2))/d*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2
-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]
time = 1.62, antiderivative size = 451, normalized size of antiderivative = 1.00, number of steps used = 9, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {6860, 738, 212, 1046}

$$\frac{f\left(\sqrt{e^2-4df}+e\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{f\left(e-\sqrt{e^2-4df}\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -(ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])]/(Sqrt[a]*d)) + (f*
(e + Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f
 - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f +
2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*
d*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sq
rt[e^2 - 4*d*f]]) - (f*(e - Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e + Sqrt
[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e
^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*
x + c*x^2])])/(Sqrt[2]*d*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2
*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
```

```
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx\sqrt{a+bx+cx^2}} + \frac{-e-fx}{d\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\right)dx$$

$$= \frac{\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{d} + \frac{\int \frac{-e-fx}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{d}$$

$$= -\frac{2\text{Subst}\left(\int \frac{1}{4a-x^2}\,dx, x, \frac{2a+bx}{\sqrt{a+bx+cx^2}}\right)}{d} - \frac{\left(f\left(1-\frac{e}{\sqrt{e^2-4df}}\right.\right.}{}$$

$$= -\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d} + \frac{\left(2f\left(1-\frac{e}{\sqrt{e^2-4df}}\right)\right)\text{Su}}{}$$

$$= -\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\,\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d} + \frac{f\left(1+\frac{e}{\sqrt{e^2-4df}}\right)\tanh^{-1}}{\sqrt{2}\,a}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.46, size = 319, normalized size = 0.71

$$\frac{\frac{2\tanh^{-1}\left(\frac{\sqrt{c}\,x-\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{\sqrt{a}}-\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{be\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-af\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,e\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-2b\sqrt{c}\,d+a\sqrt{c}\,e+4cd\#1+be\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{d}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((2*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]])/Sqrt[a] - RootSum
[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 +
 b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*e*Log[-(Sqrt[c]*x
) + Sqrt[a + b*x + c*x^2] - #1] - a*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1] - 2*Sqrt[c]*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 +
 f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(-2*b*Sqrt[c]*d + a
*Sqrt[c]*e + 4*c*d*#1 + b*e*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*f*#1^3) &
])/d
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 858 vs. 2(396) = 792.
time = 0.17, size = 859, normalized size = 1.90

| method | result |
|---|---|
| default | $-\frac{2f\sqrt{2}\,\ln\left(\frac{\frac{-bf\sqrt{-4df+e^2}+\sqrt{-4df+e^2}\,ce+2af^2-bef-2cdf+ce^2}{f^2}+\frac{\left(-c\sqrt{-4df+e^2}+bf-ce\right)\left(x+\frac{e+\sqrt{-4df}}{2f}\right.}{f}}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e^2)^
(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-
b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/
f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/
2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*
c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4
*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e
^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(
```

```
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-2*f/(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(
1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-
2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*
e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f
+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-
2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))+4*f/(-e+(-4*d
*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))/a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+
b*x+a)^(1/2))/x)
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 + x*e + d)*x), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
[Out] Integral(1/(x*sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\sqrt{c\,x^2+b\,x+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.119 $\int \frac{1}{x^2\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=543

$$-\frac{\sqrt{a+bx+cx^2}}{adx}+\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d}+\frac{e\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d^2}-\frac{f\left(e^2-2df+e\sqrt{\cdots}\right.}{\sqrt{2}\cdots}$$

```
[Out] 1/2*b*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)/d+e*arctan
h(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d^2/a^(1/2)-(c*x^2+b*x+a)^(1/2
)/a/d/x+1/2*f*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4
*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2
+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/d^2
*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f
+e^2)^(1/2))^(1/2)-1/2*f*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2
)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*
e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(e^2-2*d*f+e*(-4*d*f+e^2)
^(1/2))/d^2*2^(1/2)/(-4*d*f+e^2)^(1/2)/(f*(2*a*f-b*(e-(-4*d*f+e^2)^(1/2)))+
c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))^(1/2)
```

**Rubi** [A]
time = 2.98, antiderivative size = 543, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$,
Rules used = {6860, 744, 738, 212, 1046}

$$\frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d}-\frac{f\left(e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(-e\sqrt{e^2-4df}-2df+e^2\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\,d^2\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}+\frac{e\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d^2}-\frac{\sqrt{a+bx+cx^2}}{adx}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -(Sqrt[a + b*x + c*x^2]/(a*d*x)) + (b*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a
 + b*x + c*x^2])])/(2*a^(3/2)*d) + (e*ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a
 + b*x + c*x^2])])/(Sqrt[a]*d^2) - (f*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f])*A
rcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*
f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqr
t[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqr
t[c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt[e^2 - 4*d*
f]))]) + (f*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f])*ArcTanh[(4*a*f - b*(e + Sqr
t[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*
e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b
*x + c*x^2])])/(Sqrt[2]*d^2*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f
 + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 744

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[(2*c*d - b*e)/(2*(c*d^2 - b*d*e + a*e^2)),
 Int[(d + e*x)^(m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e,
 m, p}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2
*c*d - b*e, 0] && EqQ[m + 2*p + 3, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx^2\sqrt{a+bx+cx^2}} - \frac{e}{d^2x\sqrt{a+bx+cx^2}} + \frac{e^2-d}{d^2\sqrt{a+bx+cx}}\right.$$

$$= \frac{\int \frac{e^2-df+efx}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{d^2} + \frac{\int \frac{1}{x^2\sqrt{a+bx+cx^2}}\,dx}{d} - \frac{e\int -}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} - \frac{b\int \frac{1}{x\sqrt{a+bx+cx^2}}\,dx}{2ad} + \frac{(2e)\text{Subst}\left(\int \frac{1}{4a-x}\right.}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} + \frac{e\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}\,d^2} + \frac{b\text{Subst}\left(\right.}{}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{adx} + \frac{b\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d} + \frac{e\tanh^{-1}}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.72, size = 423, normalized size = 0.78

$$\frac{-\frac{d\sqrt{a+x(b+cx)}}{ax}+\frac{(bd+2ae)\tanh^{-1}\left(\frac{-\sqrt{c}\,x+\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{3/2}}+\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{be^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-bdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-aef\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,e^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+2\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+ef\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-2b\sqrt{c}\,d+a\sqrt{c}\,e+4cd\#1+be\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{d^2}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] (-((d*Sqrt[a + x*(b + c*x)])/(a*x)) + ((b*d + 2*a*e)*ArcTanh[(-(Sqrt[c]*x)
+ Sqrt[a + x*(b + c*x)])/Sqrt[a]])/a^(3/2) + RootSum[b^2*d - a*b*e + a^2*f
- 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2
- 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1] - b*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*e*f*L
og[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*e^2*Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqr
t[a + b*x + c*x^2] - #1]*#1 + e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1]*#1^2)/(-2*b*Sqrt[c]*d + a*Sqrt[c]*e + 4*c*d*#1 + b*e*#1 - 2*a*f*#1 -
3*Sqrt[c]*e*#1^2 + 2*f*#1^3) & ])/d^2
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 954 vs. $2(474) = 948$.
time = 0.19, size = 955, normalized size = 1.76

| method | result |
|---|---|
| default | $\frac{4f^2\sqrt{2}\,\ln\left(\frac{\frac{-bf\sqrt{-4df+e^2}+\sqrt{-4df+e^2}\,ce+2a\,f^2-bef-2cdf+c\,e^2}{f^2}+\frac{\left(-c\sqrt{-4df+e^2}+bf-ce\right)\left(x+\frac{e+\sqrt{-4df\ldots}}{2f}\ldots\right.}{f}\ldots}{\ldots}\right.}{\ldots}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 4*f^2/(e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e^
2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(
((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^
2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f
-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*
(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*
f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)
)/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-4*f^2/(-e+(-4*d*f+e^2)^(1/2))^2/(-4*d*f
+e^2)^(1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(
1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*
(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+
(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-
4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))-4*f/(-
e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(-1/a/x*(c*x^2+b*x+a)^(1/2)+1/
2*b/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))+16*f^2*e/(-e+(-4
*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2/a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(
c*x^2+b*x+a)^(1/2))/x)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 + x*e + d)*x^2), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^2 \sqrt{a + bx + cx^2} \, (d + ex + fx^2)} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/(x**2*sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^2 \sqrt{c\,x^2 + b\,x + a} \, \left(f\,x^2 + e\,x + d\right)} \, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x^2*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.120 $\int \frac{1}{x^3\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=679

$$-\frac{\sqrt{a+bx+cx^2}}{2adx^2}+\frac{3b\sqrt{a+bx+cx^2}}{4a^2dx}+\frac{e\sqrt{a+bx+cx^2}}{ad^2x}-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{5/2}d}-\frac{be}{\cdots}$$

```
[Out] -1/8*(-4*a*c+3*b^2)*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(5
/2)/d-1/2*b*e*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)/d^
2-(-d*f+e^2)*arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/d^3/a^(1/2)
-1/2*(c*x^2+b*x+a)^(1/2)/a/d/x^2+3/4*b*(c*x^2+b*x+a)^(1/2)/a^2/d/x+e*(c*x^2
+b*x+a)^(1/2)/a/d^2/x+1/2*f*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(
1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f
-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*e^3-4*d*e*f-(-d*f+e
^2)*(e-(-4*d*f+e^2)^(1/2)))/d^3*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b
*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*f*arctanh(1/4*(4*a*f-
b*(e+(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2
+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(
1/2))*(2*e^3-4*d*e*f-(-d*f+e^2)*(e+(-4*d*f+e^2)^(1/2)))/d^3*2^(1/2)/(-4*d*f
+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/
2)
```

**Rubi** [A]
time = 8.39, antiderivative size = 679, normalized size of antiderivative = 1.00, number of steps used = 16, number of rules used = 7, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.233$, Rules used = {6860, 758, 820, 738, 212, 744, 1046}

$$-\frac{(3b^2-4ac)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{8a^{5/2}d}-\frac{be\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{2a^{3/2}d^2}+\frac{3b\sqrt{a+bx+cx^2}}{4a^2dx}-\frac{(e^2-df)\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{a+bx+cx^2}}\right)}{\sqrt{a}d^3}+\frac{f\left(-(e^2-df)\left(e-\sqrt{e^2-4df}\right)-4def+2e^3\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}d^3\sqrt{e^2-4df}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{f\left(-(e^2-df)\left(\sqrt{e^2-4df}+e\right)-4def+2e^3\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}d^3\sqrt{e^2-4df}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}+\frac{e\sqrt{a+bx+cx^2}}{ad^2x}-\frac{\sqrt{a+bx+cx^2}}{2adx^2}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^3*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] -1/2*Sqrt[a + b*x + c*x^2]/(a*d*x^2) + (3*b*Sqrt[a + b*x + c*x^2])/(4*a^2*d
*x) + (e*Sqrt[a + b*x + c*x^2])/(a*d^2*x) - ((3*b^2 - 4*a*c)*ArcTanh[(2*a +
 b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(8*a^(5/2)*d) - (b*e*ArcTanh[(2*a
 + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(2*a^(3/2)*d^2) - ((e^2 - d*f)*
ArcTanh[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])])/(Sqrt[a]*d^3) + (f*
(2*e^3 - 4*d*e*f - (e^2 - d*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(4*a*f - b*
(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]
*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sq
rt[a + b*x + c*x^2])])/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f
```

```
- b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]) - (f*(2*e^3 - 4*d*e*f -
 (e^2 - d*f)*(e + Sqrt[e^2 - 4*d*f]))*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*
d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*
d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2
])])/(Sqrt[2]*d^3*Sqrt[e^2 - 4*d*f]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2
+ (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 744

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[(2*c*d - b*e)/(2*(c*d^2 - b*d*e + a*e^2)),
 Int[(d + e*x)^(m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e,
 m, p}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2
*c*d - b*e, 0] && EqQ[m + 2*p + 3, 0]
```

Rule 758

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[1/((m + 1)*(c*d^2 - b*d*e + a*e^2)), Int[(
d + e*x)^(m + 1)*Simp[c*d*(m + 1) - b*e*(m + p + 2) - c*e*(m + 2*p + 3)*x,
x]*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, m, p}, x] && NeQ[b^
2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && Ne
Q[m, -1] && ((LtQ[m, -1] && IntQuadraticQ[a, b, c, d, e, m, p, x]) || (SumS
implerQ[m, 1] && IntegerQ[p]) || ILtQ[Simplify[m + 2*p + 3], 0])
```

Rule 820

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(e*f - d*g))*(d + e*x)^(m + 1)*((a +
 b*x + c*x^2)^(p + 1)/(2*(p + 1)*(c*d^2 - b*d*e + a*e^2))), x] - Dist[(b*(e
*f + d*g) - 2*(c*d*f + a*e*g))/(2*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(
m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x]
```

```
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && EqQ[Simplify[m
+ 2*p + 3], 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^3\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx = \int \left(\frac{1}{dx^3\sqrt{a+bx+cx^2}} - \frac{e}{d^2x^2\sqrt{a+bx+cx^2}} + \frac{e^2-df}{d^3x\sqrt{a+bx-\cdots}}\right.$$

$$= \frac{\int \frac{-e(e^2-2df)-f(e^2-df)x}{\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx}{d^3} + \frac{\int \frac{1}{x^3\sqrt{a+bx+cx^2}}\,dx}{d} - \frac{e\int \cdots}{\cdots}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{e\sqrt{a+bx+cx^2}}{ad^2x} - \frac{\int \frac{\frac{3b}{2}+cx}{x^2\sqrt{a+bx+cx^2}}\,dx}{2ad} \cdots$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} + \frac{e\sqrt{a+bx+cx^2}}{ad^2x} - \frac{(e\cdots}{\cdots}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} + \frac{e\sqrt{a+bx+cx^2}}{ad^2x} - \frac{be\cdots}{\cdots}$$

$$= -\frac{\sqrt{a+bx+cx^2}}{2adx^2} + \frac{3b\sqrt{a+bx+cx^2}}{4a^2dx} + \frac{e\sqrt{a+bx+cx^2}}{ad^2x} - \frac{(3\cdots}{\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.32, size = 547, normalized size = 0.81

$$\frac{\frac{d(-2ad+3bdx+4aex)\sqrt{a+x(b+cx)}}{a^2x^2}+\frac{(-3b^2d^2-4abde+4a(cd^2-2ae^2+2adf))\tanh^{-1}\left(\frac{-\sqrt{c}\,x+\sqrt{a+x(b+cx)}}{\sqrt{a}}\right)}{a^{5/2}}-4\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{be^3\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2bdef\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-ae^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+adf^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2\sqrt{c}\,e^3\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+4\sqrt{c}\,def\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+e^2f\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-df^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{-2b\sqrt{c}\,d+a\sqrt{c}\,e+4cd\#1+be\#1-2af\#1-3\sqrt{c}\,e\#1^2+2f\#1^3}\&\right]}{4d^3}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^3*Sqrt[a + b*x + c*x^2]*(d + e*x + f*x^2)),x]
```

```
[Out] ((d*(-2*a*d + 3*b*d*x + 4*a*e*x)*Sqrt[a + x*(b + c*x)])/(a^2*x^2) + ((-3*b^
2*d^2 - 4*a*b*d*e + 4*a*(c*d^2 - 2*a*e^2 + 2*a*d*f))*ArcTanh[(-(Sqrt[c]*x)
+ Sqrt[a + x*(b + c*x)])/Sqrt[a]])/a^(5/2) - 4*RootSum[b^2*d - a*b*e + a^2*
f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^
2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1] - 2*b*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a
*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*d*f^2*Log[-(Sqrt[
c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*Sqrt[c]*e^3*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1]*#1 + 4*Sqrt[c]*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1]*#1 + e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*
#1^2 - d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(-2*b*Sqr
t[c]*d + a*Sqrt[c]*e + 4*c*d*#1 + b*e*#1 - 2*a*f*#1 - 3*Sqrt[c]*e*#1^2 + 2*
f*#1^3) & ])/(4*d^3)
```

**Maple** [A]
time = 0.19, size = 1117, normalized size = 1.65

| method | result |
|---|---|
| default | $-\frac{8f^3\sqrt{2}\,\ln\left(\frac{\frac{-bf\sqrt{-4df+e^2}+\sqrt{-4df+e^2}\,ce+2af^2-bef-2cdf+ce^2}{f^2}+\frac{\left(-c\sqrt{-4df+e^2}+bf-ce\right)\left(x+\frac{e+\sqrt{-4df}}{2f}\right.}{f}\right.}{}$ |
| risch | Expression too large to display |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -8*f^3/(e+(-4*d*f+e^2)^(1/2))^3/(-4*d*f+e^2)^(1/2)*2^(1/2)/((-b*f*(-4*d*f+e
^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln
```

```
(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e
^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f
)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c
*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d
*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2
))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f
+e^2)^(1/2))*(-1/2/a/x^2*(c*x^2+b*x+a)^(1/2)-3/4*b/a*(-1/a/x*(c*x^2+b*x+a)^
(1/2)+1/2*b/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))+1/2*c/a^
(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))-8*f^3/(-e+(-4*d*f+e^2)
^(1/2))^3/(-4*d*f+e^2)^(1/2)*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/
2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^
(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*
f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)
*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/
f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2
)^(1/2))))-16*f^2*e/(-e+(-4*d*f+e^2)^(1/2))^2/(e+(-4*d*f+e^2)^(1/2))^2*(-1/
a/x*(c*x^2+b*x+a)^(1/2)+1/2*b/a^(3/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(
1/2))/x))-64*f^3*(d*f-e^2)/(-e+(-4*d*f+e^2)^(1/2))^3/(e+(-4*d*f+e^2)^(1/2))
^3/a^(1/2)*ln((2*a+b*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x)
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(1/(sqrt(c*x^2 + b*x + a)*(f*x^2 + x*e + d)*x^3), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^3\sqrt{a+bx+cx^2}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**3/(c*x**2+b*x+a)**(1/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/(x**3*sqrt(a + b*x + c*x**2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^3/(c*x^2+b*x+a)^(1/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x^3\,\sqrt{c\,x^2+b\,x+a}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^3*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x^3*(a + b*x + c*x^2)^(1/2)*(d + e*x + f*x^2)), x)
```

# 3.121 $\int \frac{x^3}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=779

$$\frac{2(2a+bx)}{(b^2-4ac)\,f\sqrt{a+bx+cx^2}}+\frac{2e(b+2cx)}{(b^2-4ac)\,f^2\sqrt{a+bx+cx^2}}+\frac{2(cde(bcd-2ace+abf)-(bde-ae^2+adf)\,(\ldots}{(b^2-4\ldots}$$

```
[Out] 2*(b*x+2*a)/(-4*a*c+b^2)/f/(c*x^2+b*x+a)^(1/2)+2*e*(2*c*x+b)/(-4*a*c+b^2)/f
^2/(c*x^2+b*x+a)^(1/2)+2*(c*d*e*(a*b*f-2*a*c*e+b*c*d)-(a*d*f-a*e^2+b*d*e)*(
2*c^2*d+b^2*f-c*(2*a*f+b*e))+c*((a*b*f-2*a*c*e+b*c*d)*(-d*f+e^2)-d*e*(2*c^2
*d+b^2*f-c*(2*a*f+b*e)))*x)/(-4*a*c+b^2)/f^2/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f
+c*e))/(c*x^2+b*x+a)^(1/2)+1/2*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2
)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*
d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*(-a*e+b*d)*f+(
c*d^2-b*d*e+a*(-d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)^2-(-a*e+b*d)*
(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c
*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2
))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2
*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(2*d*(-a*e+b*d)*
f+(c*d^2-b*d*e+a*(-d*f+e^2))*(e+(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)^2-(-a*e+b*
d)*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*
f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 9.36, antiderivative size = 779, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 7, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.233$, Rules used = {6860, 627, 650, 1030, 1046, 738, 212}

$$\frac{2(cx((e^2-df)(abf-2ace+bcd)-de(-c(2af+be)+b^2f+2c^2d))-(adf-ae^2+bde)(-c(2af+be)+b^2f+2c^2d)+cde(abf-2ace+bcd))}{f^2(b^2-4ac)\sqrt{a+bx+cx^2}\,((cd-af)^2-(bd-ae)(ce-bf))}+\frac{2e(b+2cx)}{f^2(b^2-4ac)\sqrt{a+bx+cx^2}}+\frac{2(2a+bx)}{f(b^2-4ac)\sqrt{a+bx+cx^2}}+\frac{\left(\left(e-\sqrt{e^2-4df}\right)(a(e^2-df)-bde+cd^2)+2df(bd-ae)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2-(bd-ae)(ce-bf))\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}-\frac{\left(\left(\sqrt{e^2-4df}+e\right)(a(e^2-df)-bde+cd^2)+2df(bd-ae)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2-(bd-ae)(ce-bf))\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^3/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (2*(2*a + b*x))/((b^2 - 4*a*c)*f*Sqrt[a + b*x + c*x^2]) + (2*e*(b + 2*c*x))
/((b^2 - 4*a*c)*f^2*Sqrt[a + b*x + c*x^2]) + (2*(c*d*e*(b*c*d - 2*a*c*e + a
*b*f) - (b*d*e - a*e^2 + a*d*f)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) + c*((b
*c*d - 2*a*c*e + a*b*f)*(e^2 - d*f) - d*e*(2*c^2*d + b^2*f - c*(b*e + 2*a*f
)))*x))/((b^2 - 4*a*c)*f^2*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[a
 + b*x + c*x^2]) + ((2*d*(b*d - a*e)*f + (e - Sqrt[e^2 - 4*d*f])*(c*d^2 - b
*d*e + a*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f
- c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2
```

```
*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*S
qrt[e^2 - 4*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c
*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]) - ((2*d*(b*d - a*e
)*f + (e + Sqrt[e^2 - 4*d*f])*(c*d^2 - b*d*e + a*(e^2 - d*f)))*ArcTanh[(4*a
*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*
Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d
*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2 - (
b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f
)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 627

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-3/2), x_Symbol] :> Simp[-2*((b +
2*c*x)/((b^2 - 4*a*c)*Sqrt[a + b*x + c*x^2])), x] /; FreeQ[{a, b, c}, x] &&
 NeQ[b^2 - 4*a*c, 0]
```

Rule 650

```
Int[((d_.) + (e_.)*(x_))/((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(3/2), x_Symbo
l] :> Simp[-2*((b*d - 2*a*e + (2*c*d - b*e)*x)/((b^2 - 4*a*c)*Sqrt[a + b*x
+ c*x^2])), x] /; FreeQ[{a, b, c, d, e}, x] && NeQ[2*c*d - b*e, 0] && NeQ[b
^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
```

```
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^3}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx = \int \left(-\frac{e}{f^2(a+bx+cx^2)^{3/2}} + \frac{x}{f(a+bx+cx^2)^{3/2}} + \frac{de+}{f^2(a+bx+c}\right.$$

$$= \frac{\int \frac{de+(e^2-df)x}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx}{f^2} - \frac{e\int \frac{1}{(a+bx+cx^2)^{3/2}}\,dx}{f^2} + \frac{\int \frac{x}{(a+bx+cx^2)^{3/2}}\,d}{f}$$

$$= \frac{2(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2e(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} + \frac{2(c}{}$$

$$= \frac{2(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2e(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} + \frac{2(c}{}$$

$$= \frac{2(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2e(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} + \frac{2(c}{}$$

$$= \frac{2(2a+bx)}{(b^2-4ac)f\sqrt{a+bx+cx^2}} + \frac{2e(b+2cx)}{(b^2-4ac)f^2\sqrt{a+bx+cx^2}} + \frac{2(c}{}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 1.23, size = 690, normalized size = 0.89

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[x^3/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (2*(2*a^3*f + b^3*d*x + a*b*(b*d - 3*c*d*x - b*e*x) + a^2*(-2*c*d - b*e + 2
*c*e*x + b*f*x)) + (b^2 - 4*a*c)*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a*b*
e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2
*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (b^2*d^2*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1] + a*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] -
 #1] - 2*a*b*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*e^2*L
og[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*d*f*Log[-(Sqrt[c]*x) +
Sqrt[a + b*x + c*x^2] - #1] - 2*b*Sqrt[c]*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b
*x + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1]*#1 - c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 +
```

```
 b*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 - a*e^2*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + a*d*f*Log[-(Sqrt[c]*x) + Sqrt
[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e
*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/((b^2 - 4*a*c)*(c^2*d^2
- b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))*Sqrt[a + x*(b +
c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2082 vs. $2(726) = 1452$.
time = 0.16, size = 2083, normalized size = 2.67

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2083 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/f*(-1/c/(c*x^2+b*x+a)^(1/2)-b/c*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)
)-2*e/f^2*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)+1/2*(e^3-3*d*e*f+e^2*(-
4*d*f+e^2)^(1/2)-d*f*(-4*d*f+e^2)^(1/2))/f^3/(-4*d*f+e^2)^(1/2)*(2/(-b*f*(-
4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+
1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*f*(-c*(-4*d*f+e^2)^(1/2)+b
*f-c*e)/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d
*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+
b*f-c*e))/(2*c*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2-1/f^2*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2)/((x+1/2*(e+(-
4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*
d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*
f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/
2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f
*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2
+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2
^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d
*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*
f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)
^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1
/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/2*(-d*f*(-4*d*f+e^2)^(1/2)+e^2*(-4*d*f+e^2
)^(1/2)+3*d*e*f-e^3)/f^3/(-4*d*f+e^2)^(1/2)*(2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*
d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e+(-4*d*f+e
^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^
(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*
c*d*f+c*e^2)/f^2)^(1/2)-2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*f/(b*f*(-4*d*f+e^2
```

```
)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(x-1/2/f*(
-e+(-4*d*f+e^2)^(1/2)))+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f)/(2*c*(b*f*(-4*d*f
+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-(c*(-4*
d*f+e^2)^(1/2)+b*f-c*e)^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-
4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*
d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/
2)-2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c
*e^2)*f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*
f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-
4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*
d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^3}{(a+bx+cx^2)^{\frac{3}{2}}\,(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**3/((a + b*x + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^3}{\left(c\,x^2+b\,x+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^3/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.122 $\int \frac{x^2}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=609

$$-\frac{2(a(bcd-2ace+abf)+c(b^2d-abe-2a(cd-af))\,x)}{(b^2-4ac)\,((cd-af)^2-(bd-ae)(ce-bf))\,\sqrt{a+bx+cx^2}}-\frac{f\left(2d(cd-af)-(bd-ae)\left(e-\sqrt{e^2-4df}\right.\right.}{\sqrt{2}\,\sqrt{e^2-4df}\,((cd-af)^2}$$

```
[Out] -2*(a*(a*b*f-2*a*c*e+b*c*d)+c*(b^2*d-a*b*e-2*a*(-a*f+c*d))*x)/(-4*a*c+b^2)/
((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))/(c*x^2+b*x+a)^(1/2)-1/2*f*arctanh(1/4*
(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2)
/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(
1/2))^(1/2))*(2*d*(-a*f+c*d)-(-a*e+b*d)*(e-(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)
^2-(-a*e+b*d)*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2
*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)+1/2*f*arctanh(1/4*(4*a*f-b*(e+(
-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a
)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*
(2*d*(-a*f+c*d)-(-a*e+b*d)*(e+(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)^2-(-a*e+b*d)
*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+
c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 3.77, antiderivative size = 609, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.133$, Rules used = {1075, 1046, 738, 212}

$$-\frac{2(cx(-abe-2a(cd-af)+b^2d)+a(abf-2ace+bcd))}{(b^2-4ac)\sqrt{a+bx+cx^2}\,((cd-af)^2-(bd-ae)(ce-bf))}-\frac{f\left(2d(cd-af)-\left(e-\sqrt{e^2-4df}\right)(bd-ae)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2-(bd-ae)(ce-bf))\sqrt{2af^2-\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}+\frac{f\left(2d(cd-af)-\left(\sqrt{e^2-4df}+e\right)(bd-ae)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\,((cd-af)^2-(bd-ae)(ce-bf))\sqrt{2af^2+\sqrt{e^2-4df}\,(ce-bf)-bef-2cdf+ce^2}}$$

Antiderivative was successfully verified.

```
[In] Int[x^2/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-2*(a*(b*c*d - 2*a*c*e + a*b*f) + c*(b^2*d - a*b*e - 2*a*(c*d - a*f))*x))/
((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[a + b*x + c*x
^2]) - (f*(2*d*(c*d - a*f) - (b*d - a*e)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[(
4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/
(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 -
4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2
- (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e -
b*f)*Sqrt[e^2 - 4*d*f]]) + (f*(2*d*(c*d - a*f) - (b*d - a*e)*(e + Sqrt[e^2
- 4*d*f]))*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sqr
t[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*
e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*
```

d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 1075

```
Int[((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((A_.) + (C_.)*(x_)^2)*((d_) +
(e_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1
)*((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c
*e - b*f))*(p + 1)))*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b)*(2*c^2*
d + b^2*f - c*(b*e + 2*a*f)) + c*(A*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) + C
*(b^2*d - a*b*e - 2*a*(c*d - a*f)))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d -
a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(
d + e*x + f*x^2)^q*Simp[(-2*A*c - 2*a*C)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
- b*f))*(p + 1) + (b^2*(C*d + A*f) - b*((Plus[A])*c*e + a*C*e) + 2*(A*c*(c*
d - a*f) - a*(c*C*d - a*C*f)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((A*c - a*C)
*(2*a*c*e - b*(c*d + a*f)) + (A*b)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p
+ q + 2) - (2*f*((A*c - a*C)*(2*a*c*e - b*(c*d + a*f)) + (A*b)*(2*c^2*d + b
^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (b^2*(C*d + A*f) - b*(A*c*e + a*C*e)
 + 2*(A*c*(c*d - a*f) - a*(c*C*d - a*C*f)))*(b*f*(p + 1) - c*e*(2*p + q + 4
)))*x - c*f*(b^2*(C*d + A*f) - b*(A*c*e + a*C*e) + 2*(A*c*(c*d - a*f) - a*(
c*C*d - a*C*f)))*(2*p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, e, f
, A, C, q}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1]
&& NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f), 0] &&  !( !IntegerQ[p] && I
LtQ[q, -1]) &&  !IGtQ[q, 0]
```

Rubi steps

$$\int \frac{x^2}{\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx = -\frac{2(a(bcd-2ace+abf)+c(b^2d-abe-2a(cd-af))\,x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} -$$

$$= -\frac{2(a(bcd-2ace+abf)+c(b^2d-abe-2a(cd-af))\,x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} +$$

$$= -\frac{2(a(bcd-2ace+abf)+c(b^2d-abe-2a(cd-af))\,x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} -$$

$$= -\frac{2(a(bcd-2ace+abf)+c(b^2d-abe-2a(cd-af))\,x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} -$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.93, size = 534, normalized size = 0.88

$$\frac{-2(b^2cdx+ac(bd-2cdx-bex)+a^2(-2ce+bf+2cfx))-(b^2-4ac)\sqrt{a+x(b+cx)}\,\text{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{bcd^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2abdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+a^2ef\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-2c^{3/2}d^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+2a\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1+bdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2-aef\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}\,d-a\sqrt{c}\,e-4cd\#1-be\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]}{(b^2-4ac)\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)\right)\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-2*(b^2*c*d*x + a*c*(b*d - 2*c*d*x - b*e*x) + a^2*(-2*c*e + b*f + 2*c*f*x)
) - (b^2 - 4*a*c)*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a*b*e + a^2*f - 4*b
*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*S
qrt[c]*e*#1^3 + f*#1^4 & , (b*c*d^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1] - 2*a*b*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*e*
f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*c^(3/2)*d^2*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*a*Sqrt[c]*d*f*Log[-(Sqrt[c]*x)
+ Sqrt[a + b*x + c*x^2] - #1]*#1 + b*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x +
c*x^2] - #1]*#1^2 - a*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1
^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]
*e*#1^2 - 2*f*#1^3) & ])/((b^2 - 4*a*c)*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b
*e + a^2*f) + a*c*(e^2 - 2*d*f))*Sqrt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1991 vs. 2(560) = 1120.
time = 0.18, size = 1992, normalized size = 3.27

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1992 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2/f*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)+1/2*(-e*(-4*d*f+e^2)^(1/2)+2*
d*f-e^2)/f^2/(-4*d*f+e^2)^(1/2)*(2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^
2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/
2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e
^2)/f^2)^(1/2)-2*f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/(-b*f*(-4*d*f+e^2)^(1/2)
+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e))/(2*c*(-b*f*(-4*d*f+e^2
)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-1/f^2*(-c*(
-4*d*f+e^2)^(1/2)+b*f-c*e)^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c
*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4
*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1
/2)-2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)*f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^
2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-
c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)
+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(
e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+
(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1
/2*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2))/f^2/(-4*d*f+e^2)^(1/2)*(2/(b*f*(-4*d*f+
e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x-1/2/
f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-
e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2
*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*f/(
b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*
(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f)/(2
*c*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e
^2)/f^2-(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(1
/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2))
)+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+
c*e^2)/f^2)^(1/2)-2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-
b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-(
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/2
)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e^
```

```
2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*
(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2
)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(1
/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2}{(a+bx+cx^2)^{\frac{3}{2}}(d+ex+fx^2)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x**2/((a + b*x + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]

time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x^2}{\left(c\,x^2+b\,x+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(x^2/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.123 $\int \frac{x}{\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=609

$$\frac{2(a(2c^2d-bce+b^2f-2acf)+c(bcd-2ace+abf)x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}+\frac{f\left(2d(ce-bf)-(cd-af)\left(e-\sqrt{e^2-4df}\right)\right.}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2-\right.}$$

```
[Out] 2*(a*(-2*a*c*f+b^2*f-b*c*e+2*c^2*d)+c*(a*b*f-2*a*c*e+b*c*d)*x)/(-4*a*c+b^2)
/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))/(c*x^2+b*x+a)^(1/2)+1/2*f*arctanh(1/4
*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^2)^(1/2)))*2^(1/2
)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^
(1/2))^(1/2))*(2*d*(-b*f+c*e)-(-a*f+c*d)*(e-(-4*d*f+e^2)^(1/2)))/((-a*f+c*d
)^2-(-a*e+b*d)*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+
2*a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*f*arctanh(1/4*(4*a*f-b*(e+
(-4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+
a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))
*(2*d*(-b*f+c*e)-(-a*f+c*d)*(e+(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)^2-(-a*e+b*d
)*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f
+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)
```

**Rubi** [A]

time = 3.56, antiderivative size = 609, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {1030, 1046, 738, 212}

$$\frac{2(a(-2acf+b^2f-bce+2c^2d)+cx(abf-2ace+bcd))}{(b^2-4ac)\sqrt{a+bx+cx^2}\left((cd-af)^2-(bd-ae)(ce-bf)\right)}+\frac{f\left(2d(ce-bf)-\left(e-\sqrt{e^2-4df}\right)(cd-af)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}-\frac{f\left(2d(ce-bf)-\left(\sqrt{e^2-4df}+e\right)(cd-af)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}$$

Antiderivative was successfully verified.

```
[In] Int[x/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (2*(a*(2*c^2*d - b*c*e + b^2*f - 2*a*c*f) + c*(b*c*d - 2*a*c*e + a*b*f)*x))
/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[a + b*x + c*
x^2]) + (f*(2*d*(c*e - b*f) - (c*d - a*f)*(e - Sqrt[e^2 - 4*d*f]))*ArcTanh[
(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)
/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 -
 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2
 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 - (c*e -
 b*f)*Sqrt[e^2 - 4*d*f]]) - (f*(2*d*(c*e - b*f) - (c*d - a*f)*(e + Sqrt[e^2
 - 4*d*f]))*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e + Sq
rt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2 + (c
*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^2 - 4
```

```
*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2 - 2*c*d*f - b*e*
f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{x}{(a+bx+cx^2)^{3/2}\,(d+ex+fx^2)}\,dx = \frac{2(a(2c^2d-bce+b^2f-2acf)+c(bcd-2ace+abf)x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} - \frac{\cdots}{(b^2 \cdots}$$

$$= \frac{2(a(2c^2d-bce+b^2f-2acf)+c(bcd-2ace+abf)x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} - \frac{\left(f \cdots\right.}{\cdots}$$

$$= \frac{2(a(2c^2d-bce+b^2f-2acf)+c(bcd-2ace+abf)x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} + \frac{\left(2 \cdots\right.}{\cdots}$$

$$= \frac{2(a(2c^2d-bce+b^2f-2acf)+c(bcd-2ace+abf)x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}} + \frac{f\left( \cdots\right.}{\cdots}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.93, size = 570, normalized size = 0.94

$$\frac{-4a^2cf+2bc^2dx+2a(b^2f+2c^2(d-ex)+bc(-e+fx))-(b^2-4ac)\sqrt{a+x(b+cx)}\,\mathrm{RootSum}\left[b^2d-abe+a^2f-4b\sqrt{c}\,d\#1+2a\sqrt{c}\,e\#1+4cd\#1^2+be\#1^2-2af\#1^2-2\sqrt{c}\,e\#1^3+f\#1^4\&,\frac{-bcde\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+b^2df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+acdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)-a^2f^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)+2c^{3/2}de\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-2b\sqrt{c}\,df\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1-cdf\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2+af^2\log\left(-\sqrt{c}\,x+\sqrt{a+bx+cx^2}-\#1\right)\#1^2}{2b\sqrt{c}\,d-a\sqrt{c}\,e-4cd\#1-be\#1+2af\#1+3\sqrt{c}\,e\#1^2-2f\#1^3}\&\right]}{(b^2-4ac)\left(c^2d^2-bcde+f\left(b^2d-abe+a^2f\right)+ac\left(e^2-2df\right)\right)\sqrt{a+x(b+cx)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-4*a^2*c*f + 2*b*c^2*d*x + 2*a*(b^2*f + 2*c^2*(d - e*x) + b*c*(-e + f*x))
- (b^2 - 4*a*c)*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a*b*e + a^2*f - 4*b*S
qrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqr
t[c]*e*#1^3 + f*#1^4 & , (-(b*c*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1]) + b^2*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*c*d*f
*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a^2*f^2*Log[-(Sqrt[c]*x)
+ Sqrt[a + b*x + c*x^2] - #1] + 2*c^(3/2)*d*e*Log[-(Sqrt[c]*x) + Sqrt[a + b
*x + c*x^2] - #1]*#1 - 2*b*Sqrt[c]*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*
x^2] - #1]*#1 - c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 +
 a*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d
- a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3)
 & ])/((b^2 - 4*a*c)*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(
e^2 - 2*d*f))*Sqrt[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1938 vs. 2(560) = 1120.
time = 0.16, size = 1939, normalized size = 3.18

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1939 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(2/(-b*f*(-4*d*f+e^2)^(1/2)
+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f
+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b
*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/(-b*f*(-
4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(
x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e))/(2*c*(
-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)
/f^2-1/f^2*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2)
)/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*
f+c*e^2)/f^2)^(1/2)-2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f
^2-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^
(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1
/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f
+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-
4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(
1/2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f
-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^
2)^(1/2))/f)))+1/2*(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)/f*(2/(b*f*(-4
*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e
)*f/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*
e^2)*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/
f)/(2*c*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*
f+c*e^2)/f^2-(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^
2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c
*d*f+c*e^2)/f^2)^(1/2)-2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1
/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)
^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d
*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2
```

```
)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)
/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)
^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^
2)^(1/2)))))
```

**Maxima** [**F(-2)**]
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x}{\left(a + bx + cx^{2}\right)^{\frac{3}{2}} \left(d + ex + fx^{2}\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(x/((a + b*x + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{x}{\left(c\,x^2+b\,x+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
[Out] int(x/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.124 $\int \frac{1}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx$

**Optimal**. Leaf size=666

$$\frac{2(b^2ce-2ac^2e-b^3f-bc(cd-3af)-c(2c^2d-bce+b^2f-2acf)\,x)}{(b^2-4ac)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}-\frac{f\left(c\left(e^2-2df+e\sqrt{e^2-4df}\right)+f\left(\cdots\right.\right.}{\sqrt{2}\,\sqrt{e^2-4df}\,\left((c\cdots\right.}$$

[Out] 2*(b^2*c*e-2*a*c^2*e-b^3*f-b*c*(-3*a*f+c*d)-c*(-2*a*c*f+b^2*f-b*c*e+2*c^2*d
)*x)/(-4*a*c+b^2)/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))/(c*x^2+b*x+a)^(1/2)-
1/2*f*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4*d*f+e^
2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-(-b*f+c
*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))+f*(2*a*f
-b*(e+(-4*d*f+e^2)^(1/2))))/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))*2^(1/2)/(-
4*d*f+e^2)^(1/2)/(f*(2*a*f-b*(e-(-4*d*f+e^2)^(1/2)))+c*(e^2-2*d*f-e*(-4*d*f
+e^2)^(1/2)))^(1/2)+1/2*f*arctanh(1/4*(4*a*f-b*(e+(-4*d*f+e^2)^(1/2))+2*x*(
b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b
*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(f*(2*a*f-b*(e-(-4*d*f+e
^2)^(1/2)))+c*(e^2-2*d*f-e*(-4*d*f+e^2)^(1/2)))/((-a*f+c*d)^2-(-a*e+b*d)*(-
b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*(e^2-2*d*f+e*(-4*d*f+e^2)^(1/2))+f*
(2*a*f-b*(e+(-4*d*f+e^2)^(1/2))))^(1/2)

**Rubi** [A]

time = 0.94, antiderivative size = 666, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 4, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.148$, Rules used = {988, 1046, 738, 212}

$$\frac{2(-cx(-2acf+b^2f-bce+2c^2d)-bc(cd-3af)-2ac^2e+b^3(-f)+b^2ce)}{(b^2-4ac)\sqrt{a+bx+cx^2}\left((cd-af)^2-(bd-ae)(ce-bf)\right)}-\frac{f\left(f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)-b\left(e-\sqrt{e^2-4df}\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2-\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)}}+\frac{f\left(f\left(2af-b\left(e-\sqrt{e^2-4df}\right)\right)+c\left(-e\sqrt{e^2-4df}-2df+e^2\right)\right)\tanh^{-1}\left(\frac{4af+2x\left(bf-c\left(\sqrt{e^2-4df}+e\right)\right)-b\left(\sqrt{e^2-4df}+e\right)}{2\sqrt{2}\sqrt{a+bx+cx^2}\sqrt{2af^2+\sqrt{e^2-4df}(ce-bf)-bef-2cdf+ce^2}}\right)}{\sqrt{2}\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{f\left(2af-b\left(\sqrt{e^2-4df}+e\right)\right)+c\left(e\sqrt{e^2-4df}-2df+e^2\right)}}$$

Antiderivative was successfully verified.

[In] Int[1/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]

[Out] (2*(b^2*c*e - 2*a*c^2*e - b^3*f - b*c*(c*d - 3*a*f) - c*(2*c^2*d - b*c*e +
b^2*f - 2*a*c*f)*x))/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f
))*Sqrt[a + b*x + c*x^2]) - (f*(c*(e^2 - 2*d*f + e*Sqrt[e^2 - 4*d*f]) + f*(
2*a*f - b*(e + Sqrt[e^2 - 4*d*f])))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*
f]) + 2*(b*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*
f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])
])/(Sqrt[2]*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqr
t[c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt[e^2 - 4*d*
f]))]) + (f*(c*(e^2 - 2*d*f - e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e - Sqrt
[e^2 - 4*d*f])))*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(b*f - c*(e

```
 + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f + 2*a*f^2
 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2]*Sqrt[e^
2 - 4*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*(e^2 - 2*d*f +
e*Sqrt[e^2 - 4*d*f]) + f*(2*a*f - b*(e + Sqrt[e^2 - 4*d*f]))])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 988

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.) + (e_.)*(x_) + (f_.)*(x
_)^2)^(q_), x_Symbol] :> Simp[(2*a*c^2*e - b^2*c*e + b^3*f + b*c*(c*d - 3*a
*f) + c*(2*c^2*d + b^2*f - c*(b*e + 2*a*f))*x)*(a + b*x + c*x^2)^(p + 1)*((
d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e -
 b*f))*(p + 1))), x] - Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(
c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*x^2)^q*Sim
p[2*c*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1) - (2*c^2*d + b^2*f
- c*(b*e + 2*a*f))*(a*f*(p + 1) - c*d*(p + 2)) - e*(b^2*c*e - 2*a*c^2*e - b
^3*f - b*c*(c*d - 3*a*f))*(p + q + 2) + (2*f*(2*a*c^2*e - b^2*c*e + b^3*f +
 b*c*(c*d - 3*a*f))*(p + q + 2) - (2*c^2*d + b^2*f - c*(b*e + 2*a*f))*(b*f*
(p + 1) - c*e*(2*p + q + 4)))*x + c*f*(2*c^2*d + b^2*f - c*(b*e + 2*a*f))*(
2*p + 2*q + 5)*x^2, x], x], x] /; FreeQ[{a, b, c, d, e, f, q}, x] && NeQ[b^
2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (
b*d - a*e)*(c*e - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1]) &&  !IGtQ[q,
 0]
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rubi steps

$$\int \frac{1}{\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx = \frac{2(b^2ce-2ac^2e-b^3f-bc(cd-3af)-c(2c^2d-bce+b^2f-2acf}{\left(b^2-4ac\right)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}$$

$$= \frac{2(b^2ce-2ac^2e-b^3f-bc(cd-3af)-c(2c^2d-bce+b^2f-2acf}{\left(b^2-4ac\right)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}$$

$$= \frac{2(b^2ce-2ac^2e-b^3f-bc(cd-3af)-c(2c^2d-bce+b^2f-2acf}{\left(b^2-4ac\right)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}$$

$$= \frac{2(b^2ce-2ac^2e-b^3f-bc(cd-3af)-c(2c^2d-bce+b^2f-2acf}{\left(b^2-4ac\right)\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx+cx^2}}$$

**Mathematica** [C] Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 0.07, size = 692, normalized size = 1.04

[illegible]

Antiderivative was successfully verified.

```
[In] Integrate[1/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-2*(b^3*f + b^2*c*(-e + f*x) + b*c*(-3*a*f + c*(d - e*x)) + 2*c^2*(c*d*x +
 a*(e - f*x))) + (b^2 - 4*a*c)*Sqrt[a + x*(b + c*x)]*RootSum[b^2*d - a*b*e
+ a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a
*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(b*c*e^2*Log[-(Sqrt[c]*x) + Sqrt[
a + b*x + c*x^2] - #1]) + b*c*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1] + b^2*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a*c*e*f*Lo
g[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a*b*f^2*Log[-(Sqrt[c]*x) +
 Sqrt[a + b*x + c*x^2] - #1] + 2*c^(3/2)*e^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*
x + c*x^2] - #1]*#1 - 2*c^(3/2)*d*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2
] - #1]*#1 - 2*b*Sqrt[c]*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]
*#1 + 2*a*Sqrt[c]*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c
*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + b*f^2*Log[-(Sqrt
[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4
*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ])/((b^2 - 4*a
*c)*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))*Sqr
t[a + x*(b + c*x)])
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 1905 vs. $2(609) = 1218$.
time = 0.00, size = 1906, normalized size = 2.86

| method | result | size |
|---|---|---|
| default | Expression too large to display | 1906 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] -1/(-4*d*f+e^2)^(1/2)*(2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*
a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-
c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-
4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(
1/2)-2*f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(x+1/2*(e+(-4*d*f+e^2)^(1/2
))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e))/(2*c*(-b*f*(-4*d*f+e^2)^(1/2)+(-
4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-1/f^2*(-c*(-4*d*f+e^2
)^(1/2)+b*f-c*e)^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e
^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^
(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(-b*
f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^
2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*
c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/
2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e
^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4*(x+1/2*(e+(-4*d*f+
e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^
2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*
f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)))+1/(-4*d*f+e
^2)^(1/2)*(2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2
*c*d*f+c*e^2)*f^2/((x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1
/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2
)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*(c*(-4*d
*f+e^2)^(1/2)+b*f-c*e)*f/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a
*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+(c*(-4*d*f
+e^2)^(1/2)+b*f-c*e)/f)/(2*c*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e
+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2-(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2/f^2)/((x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*
c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d
*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((b*f*(-4*d*f+e^
2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(
((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2
```

```
)/f^2+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/
2*2^(1/2)*((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c
*d*f+c*e^2)/f^2)^(1/2)*(4*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*
f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^
2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x
-1/2/f*(-e+(-4*d*f+e^2)^(1/2))))
```

**Maxima [F(-2)]**
time = 0.00, size = 0, normalized size = 0.00

Exception raised: ValueError

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
```

```
[Out] Exception raised: ValueError >> Computation failed since Maxima requested a
dditional constraints; using the 'assume' command before evaluation *may* h
elp (example of legal syntax is 'assume(4*d*f-%e^2>0)', see `assume?` for m
ore det
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\left(a + bx + cx^{2}\right)^{\frac{3}{2}} \left(d + ex + fx^{2}\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/((a + b*x + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] Timed out
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{\left(c\,x^2+b\,x+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/((a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.125 $\int \frac{1}{x\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx$

**Optimal**. Leaf size=816

$$\frac{2\left(b^2-2ac+bcx\right)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}}+\frac{2\left(ce(2ace-b(cd+af))+(be-af)\left(2c^2d+b^2f-c(be+2af)\right)+c\left(2c^2de+\right.\right.}{\left(b^2-4ac\right)d\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{a+bx}}$$

```
[Out] -arctanh(1/2*(b*x+2*a)/a^(1/2)/(c*x^2+b*x+a)^(1/2))/a^(3/2)/d+2*(b*c*x-2*a*
c+b^2)/a/(-4*a*c+b^2)/d/(c*x^2+b*x+a)^(1/2)+2*(c*e*(2*a*c*e-b*(a*f+c*d))+(-
a*f+b*e)*(2*c^2*d+b^2*f-c*(2*a*f+b*e))+c*(2*c^2*d*e+b*f*(-a*f+b*e)-b*c*(d*f
+e^2))*x)/(-4*a*c+b^2)/d/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))/(c*x^2+b*x+a)
^(1/2)+1/2*f*arctanh(1/4*(4*a*f+2*x*(b*f-c*(e-(-4*d*f+e^2)^(1/2)))-b*(e-(-4
*d*f+e^2)^(1/2)))*2^(1/2)/(c*x^2+b*x+a)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2-
(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(-2*f*(-a*e*f-b*d*f+b*e^2)+2*c*(-2*d*
e*f+e^3)+(f*(-a*f+b*e)-c*(-d*f+e^2))*(e-(-4*d*f+e^2)^(1/2)))/d/((-a*f+c*d)^
2-(-a*e+b*d)*(-b*f+c*e))*2^(1/2)/(-4*d*f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*
a*f^2-(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2)-1/2*f*arctanh(1/4*(4*a*f-b*(e+(-
4*d*f+e^2)^(1/2))+2*x*(b*f-c*(e+(-4*d*f+e^2)^(1/2))))*2^(1/2)/(c*x^2+b*x+a)
^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1/2))*(
-2*f*(-a*e*f-b*d*f+b*e^2)+2*c*(-2*d*e*f+e^3)+(f*(-a*f+b*e)-c*(-d*f+e^2))*(e
+(-4*d*f+e^2)^(1/2)))/d/((-a*f+c*d)^2-(-a*e+b*d)*(-b*f+c*e))*2^(1/2)/(-4*d*
f+e^2)^(1/2)/(c*e^2-2*c*d*f-b*e*f+2*a*f^2+(-b*f+c*e)*(-4*d*f+e^2)^(1/2))^(1
/2)
```

**Rubi** [A]

time = 14.81, antiderivative size = 814, normalized size of antiderivative = 1.00, number of steps used = 12, number of rules used = 7, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.233$, Rules used = {6860, 754, 12, 738, 212, 1030, 1046}

$$\frac{2\left(b^2+cxb-2ac\right)}{a\left(b^2-4ac\right)d\sqrt{cx^2+bx+a}}-\frac{\tanh^{-1}\left(\frac{2a+bx}{2\sqrt{a}\sqrt{cx^2+bx+a}}\right)}{a^{3/2}d}-\frac{f\left(2f\left(be^2-afe-bdf\right)-2c\left(e^3-2def\right)-\left(e-\sqrt{e^2-4df}\right)\left(f(be-af)-c\left(e^2-df\right)\right)\right)\tanh^{-1}\left(\frac{4af-b\left(e-\sqrt{e^2-4df}\right)+2\left(bf-c\left(e-\sqrt{e^2-4df}\right)\right)x}{2\sqrt{2}\sqrt{ce^2-bfe+2af^2-2cdf-(ce-bf)\sqrt{e^2-4df}}\sqrt{cx^2+bx+a}}\right)}{\sqrt{2}d\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{ce^2-bfe+2af^2-2cdf-(ce-bf)\sqrt{e^2-4df}}}+\frac{f\left(2f\left(be^2-afe-bdf\right)-2c\left(e^3-2def\right)-\left(e+\sqrt{e^2-4df}\right)\left(f(be-af)-c\left(e^2-df\right)\right)\right)\tanh^{-1}\left(\frac{4af-b\left(e+\sqrt{e^2-4df}\right)+2\left(bf-c\left(e+\sqrt{e^2-4df}\right)\right)x}{2\sqrt{2}\sqrt{ce^2-bfe+2af^2-2cdf+(ce-bf)\sqrt{e^2-4df}}\sqrt{cx^2+bx+a}}\right)}{\sqrt{2}d\sqrt{e^2-4df}\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{ce^2-bfe+2af^2-2cdf+(ce-bf)\sqrt{e^2-4df}}}+\frac{2\left(ce(2ace-b(cd+af))+(be-af)\left(fb^2+2c^2d-c(be+2af)\right)+c\left(2dec^2-b\left(e^2+df\right)c+bf(be-af)\right)x\right)}{\left(b^2-4ac\right)d\left((cd-af)^2-(bd-ae)(ce-bf)\right)\sqrt{cx^2+bx+a}}$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*(a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (2*(b^2 - 2*a*c + b*c*x))/(a*(b^2 - 4*a*c)*d*Sqrt[a + b*x + c*x^2]) + (2*(c
*e*(2*a*c*e - b*(c*d + a*f)) + (b*e - a*f)*(2*c^2*d + b^2*f - c*(b*e + 2*a*
f)) + c*(2*c^2*d*e + b*f*(b*e - a*f) - b*c*(e^2 + d*f))*x))/((b^2 - 4*a*c)*
d*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[a + b*x + c*x^2]) - ArcTan
h[(2*a + b*x)/(2*Sqrt[a]*Sqrt[a + b*x + c*x^2])]/(a^(3/2)*d) - (f*(2*f*(b*e
^2 - b*d*f - a*e*f) - 2*c*(e^3 - 2*d*e*f) - (e - Sqrt[e^2 - 4*d*f])*(f*(b*e
 - a*f) - c*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e - Sqrt[e^2 - 4*d*f]) + 2*(b
*f - c*(e - Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f
```

```
+ 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[2
]*d*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2
- 2*c*d*f - b*e*f + 2*a*f^2 - (c*e - b*f)*Sqrt[e^2 - 4*d*f]]) + (f*(2*f*(b*
e^2 - b*d*f - a*e*f) - 2*c*(e^3 - 2*d*e*f) - (e + Sqrt[e^2 - 4*d*f])*(f*(b*
e - a*f) - c*(e^2 - d*f)))*ArcTanh[(4*a*f - b*(e + Sqrt[e^2 - 4*d*f]) + 2*(
b*f - c*(e + Sqrt[e^2 - 4*d*f]))*x)/(2*Sqrt[2]*Sqrt[c*e^2 - 2*c*d*f - b*e*f
 + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]]*Sqrt[a + b*x + c*x^2])])/(Sqrt[
2]*d*Sqrt[e^2 - 4*d*f]*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*Sqrt[c*e^2
 - 2*c*d*f - b*e*f + 2*a*f^2 + (c*e - b*f)*Sqrt[e^2 - 4*d*f]])
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 754

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*(b*c*d - b^2*e + 2*a*c*e + c*(2*c*d - b*e)
*x)*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^
2))), x] + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d +
 e*x)^m*Simp[b*c*d*e*(2*p - m + 2) + b^2*e^2*(m + p + 2) - 2*c^2*d^2*(2*p +
 3) - 2*a*c*e^2*(m + 2*p + 3) - c*e*(2*c*d - b*e)*(m + 2*p + 4)*x, x]*(a +
b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4
*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p,
-1] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 1030

```
Int[((g_.) + (h_.)*(x_))*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_) + (e
_.)*(x_) + (f_.)*(x_)^2)^(q_), x_Symbol] :> Simp[(a + b*x + c*x^2)^(p + 1)*
((d + e*x + f*x^2)^(q + 1)/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f))*(p + 1)))*(g*c*(2*a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d +
b^2*f - c*(b*e + 2*a*f)) + c*(g*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)) - h*(b*
```

```
c*d - 2*a*c*e + a*b*f))*x), x] + Dist[1/((b^2 - 4*a*c)*((c*d - a*f)^2 - (b*
d - a*e)*(c*e - b*f))*(p + 1)), Int[(a + b*x + c*x^2)^(p + 1)*(d + e*x + f*
x^2)^q*Simp[(b*h - 2*g*c)*((c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f))*(p + 1)
 + (b^2*(g*f) - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*((-h)*c*
e)))*(a*f*(p + 1) - c*d*(p + 2)) - e*((g*c)*(2*a*c*e - b*(c*d + a*f)) + (g*
b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(p + q + 2) - (2*f*((g*c)*(2*
a*c*e - b*(c*d + a*f)) + (g*b - a*h)*(2*c^2*d + b^2*f - c*(b*e + 2*a*f)))*(
p + q + 2) - (b^2*g*f - b*(h*c*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) - a*
((-h)*c*e)))*(b*f*(p + 1) - c*e*(2*p + q + 4)))*x - c*f*(b^2*(g*f) - b*(h*c
*d + g*c*e + a*h*f) + 2*(g*c*(c*d - a*f) + a*h*c*e))*(2*p + 2*q + 5)*x^2, x
], x], x] /; FreeQ[{a, b, c, d, e, f, g, h, q}, x] && NeQ[b^2 - 4*a*c, 0] &
& NeQ[e^2 - 4*d*f, 0] && LtQ[p, -1] && NeQ[(c*d - a*f)^2 - (b*d - a*e)*(c*e
 - b*f), 0] &&  !( !IntegerQ[p] && ILtQ[q, -1])
```

Rule 1046

```
Int[((g_.) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (
e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> With[{q = Rt[b^2 - 4*a*c, 2]}, Dis
t[(2*c*g - h*(b - q))/q, Int[1/((b - q + 2*c*x)*Sqrt[d + e*x + f*x^2]), x],
 x] - Dist[(2*c*g - h*(b + q))/q, Int[1/((b + q + 2*c*x)*Sqrt[d + e*x + f*x
^2]), x], x]] /; FreeQ[{a, b, c, d, e, f, g, h}, x] && NeQ[b^2 - 4*a*c, 0]
&& NeQ[e^2 - 4*d*f, 0] && PosQ[b^2 - 4*a*c]
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\,dx = \int \left(\frac{1}{dx\left(a+bx+cx^2\right)^{3/2}} + \frac{-e-fx}{d\left(a+bx+cx^2\right)^{3/2}\left(d+ex+fx^2\right)}\right.$$

$$= \frac{\int \frac{1}{x(a+bx+cx^2)^{3/2}}\,dx}{d} + \frac{\int \frac{-e-fx}{(a+bx+cx^2)^{3/2}(d+ex+fx^2)}\,dx}{d}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} + \frac{2(ce(2ace-b(cd+af))+(be}{(b^2-4ac}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} + \frac{2(ce(2ace-b(cd+af))+(be}{(b^2-4ac}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} + \frac{2(ce(2ace-b(cd+af))+(be}{(b^2-4ac}$$

$$= \frac{2(b^2-2ac+bcx)}{a\left(b^2-4ac\right)d\sqrt{a+bx+cx^2}} + \frac{2(ce(2ace-b(cd+af))+(be}{(b^2-4ac}$$

**Mathematica [C]** Result contains higher order function than in optimal. Order 9 vs. order 3 in optimal.
time = 4.65, size = 1025, normalized size = 1.26

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*(a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x]
```

```
[Out] (-2*(b^4*f + 2*a*c^2*(-(c*d) + a*f + c*e*x) + b^3*c*(-e + f*x) + b^2*c*(-4*
a*f + c*(d - e*x)) + b*c^2*(c*d*x + 3*a*(e - f*x))))/(a*(-b^2 + 4*a*c)*(c^2
*d^2 - b*c*d*e + f*(b^2*d - a*b*e + a^2*f) + a*c*(e^2 - 2*d*f))*Sqrt[a + x*
(b + c*x)]) + (2*ArcTanh[(Sqrt[c]*x - Sqrt[a + x*(b + c*x)])/Sqrt[a]])/(a^(
3/2)*d) - RootSum[b^2*d - a*b*e + a^2*f - 4*b*Sqrt[c]*d*#1 + 2*a*Sqrt[c]*e*
#1 + 4*c*d*#1^2 + b*e*#1^2 - 2*a*f*#1^2 - 2*Sqrt[c]*e*#1^3 + f*#1^4 & , (-(
b*c*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]) + 2*b*c*d*e*f*Log[-
(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + b^2*e^2*f*Log[-(Sqrt[c]*x) + Sq
rt[a + b*x + c*x^2] - #1] + a*c*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x
^2] - #1] - b^2*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - a*c*
```

```
d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] - 2*a*b*e*f^2*Log[-(Sq
rt[c]*x) + Sqrt[a + b*x + c*x^2] - #1] + a^2*f^3*Log[-(Sqrt[c]*x) + Sqrt[a
+ b*x + c*x^2] - #1] + 2*c^(3/2)*e^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^
2] - #1]*#1 - 4*c^(3/2)*d*e*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1
]*#1 - 2*b*Sqrt[c]*e^2*f*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1
+ 2*b*Sqrt[c]*d*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 + 2*a
*Sqrt[c]*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1 - c*e^2*f*
Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + c*d*f^2*Log[-(Sqrt[c]
*x) + Sqrt[a + b*x + c*x^2] - #1]*#1^2 + b*e*f^2*Log[-(Sqrt[c]*x) + Sqrt[a
+ b*x + c*x^2] - #1]*#1^2 - a*f^3*Log[-(Sqrt[c]*x) + Sqrt[a + b*x + c*x^2]
- #1]*#1^2)/(2*b*Sqrt[c]*d - a*Sqrt[c]*e - 4*c*d*#1 - b*e*#1 + 2*a*f*#1 + 3
*Sqrt[c]*e*#1^2 - 2*f*#1^3) & ]/(d*(c^2*d^2 - b*c*d*e + f*(b^2*d - a*b*e +
a^2*f) + a*c*(e^2 - 2*d*f)))
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 2058 vs. $2(753) = 1506$.
time = 0.17, size = 2059, normalized size = 2.52

| method | result | size |
|---|---|---|
| default | Expression too large to display | 2059 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x,method=_RETURNVERBOSE)
```

```
[Out] 2*f/(e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(2/(-b*f*(-4*d*f+e^2)^(1/2)+(
-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x+1/2*(e+(-4*d*f+e
^2)^(1/2))/f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2
)^(1/2))/f)+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e
*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/(-b*f*(-4*
d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*(2*c*(x+
1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e))/(2*c*(-b
*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f
^2-1/f^2*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2)/((x+1/2*(e+(-4*d*f+e^2)^(1/2))/
f)^2*c+1/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)
+1/2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+
c*e^2)/f^2)^(1/2)-2/(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1
/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((-b*f*(-4*d*f+e^2)^(1/2
)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+1/f*(-c*(-4*d*f+e
^2)^(1/2)+b*f-c*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+1/2*2^(1/2)*((-b*f*(-4*
d*f+e^2)^(1/2)+(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/
2)*(4*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)^2*c+4/f*(-c*(-4*d*f+e^2)^(1/2)+b*f-c
*e)*(x+1/2*(e+(-4*d*f+e^2)^(1/2))/f)+2*(-b*f*(-4*d*f+e^2)^(1/2)+(-4*d*f+e^2
)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x+1/2*(e+(-4*d*f+e^2)
^(1/2))/f)))+2*f/(-e+(-4*d*f+e^2)^(1/2))/(-4*d*f+e^2)^(1/2)*(2/(b*f*(-4*d*f
```

```
+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)*f^2/((x-1/2
/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(
-e+(-4*d*f+e^2)^(1/2)))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+
2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)-2*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)*f/
(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)
*(2*c*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f)/(
2*c*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*
e^2)/f^2-(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)^2/f^2)/((x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))^2*c+(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)
))+1/2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f
+c*e^2)/f^2)^(1/2)-2/(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2
-b*e*f-2*c*d*f+c*e^2)*f^2*2^(1/2)/((b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*ln(((b*f*(-4*d*f+e^2)^(1/2)-
(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2+(c*(-4*d*f+e^2)^(1/
2)+b*f-c*e)/f*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+1/2*2^(1/2)*((b*f*(-4*d*f+e
^2)^(1/2)-(-4*d*f+e^2)^(1/2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2)*(4
*(x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))^2*c+4*(c*(-4*d*f+e^2)^(1/2)+b*f-c*e)/f*(
x-1/2/f*(-e+(-4*d*f+e^2)^(1/2)))+2*(b*f*(-4*d*f+e^2)^(1/2)-(-4*d*f+e^2)^(1/
2)*c*e+2*a*f^2-b*e*f-2*c*d*f+c*e^2)/f^2)^(1/2))/(x-1/2/f*(-e+(-4*d*f+e^2)^(
1/2)))))-4*f/(-e+(-4*d*f+e^2)^(1/2))/(e+(-4*d*f+e^2)^(1/2))*(1/a/(c*x^2+b*x
+a)^(1/2)-b/a*(2*c*x+b)/(4*a*c-b^2)/(c*x^2+b*x+a)^(1/2)-1/a^(3/2)*ln((2*a+b
*x+2*a^(1/2)*(c*x^2+b*x+a)^(1/2))/x))
```

**Maxima** [**F**]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="maxima")
[Out] integrate(1/((c*x^2 + b*x + a)^(3/2)*(f*x^2 + x*e + d)*x), x)
```

**Fricas** [**F(-1)**] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="fricas")
[Out] Timed out
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x\left(a + bx + cx^{2}\right)^{\frac{3}{2}}\left(d + ex + fx^{2}\right)}\, dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x**2+b*x+a)**(3/2)/(f*x**2+e*x+d),x)
```

```
[Out] Integral(1/(x*(a + b*x + c*x**2)**(3/2)*(d + e*x + f*x**2)), x)
```

**Giac [F]**
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(c*x^2+b*x+a)^(3/2)/(f*x^2+e*x+d),x, algorithm="giac")
```

```
[Out] sage0*x
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{1}{x\left(c\,x^2+b\,x+a\right)^{3/2}\,\left(f\,x^2+e\,x+d\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)),x)
```

```
[Out] int(1/(x*(a + b*x + c*x^2)^(3/2)*(d + e*x + f*x^2)), x)
```

# 3.126 $\int \frac{x^4}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=140

$$\frac{5}{2}\sqrt{-3-4x-x^2}-\frac{1}{4}x\sqrt{-3-4x-x^2}+\frac{11}{2}\sin^{-1}(2+x)+\frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{\tan^{-1}\left(\frac{1+\frac{}{\sqrt{-3}}}{}\right)}{2\sqrt{2}}$$

```
[Out] 11/2*arcsin(2+x)-5/4*arctanh(x/(-x^2-4*x-3)^(1/2))+1/4*arctan(1/2*(1+(-3-x)
/(-x^2-4*x-3)^(1/2))*2^(1/2))*2^(1/2)-1/4*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^
(1/2))*2^(1/2))*2^(1/2)+5/2*(-x^2-4*x-3)^(1/2)-1/4*x*(-x^2-4*x-3)^(1/2)
```

**Rubi** [A]
time = 0.31, antiderivative size = 140, normalized size of antiderivative = 1.00, number of steps used = 24, number of rules used = 14, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.467$, Rules used = {6860, 633, 222, 654, 756, 1042, 1000, 12, 1040, 1175, 632, 210, 1041, 212}

$$\frac{11}{2}\text{ArcSin}(x+2)+\frac{\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{1}{4}\sqrt{-x^2-4x-3}\,x+\frac{5}{2}\sqrt{-x^2-4x-3}-\frac{5}{4}\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[x^4/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] (5*Sqrt[-3 - 4*x - x^2])/2 - (x*Sqrt[-3 - 4*x - x^2])/4 + (11*ArcSin[2 + x]
)/2 + ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/(2*Sqrt[2]) - ArcT
an[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/(2*Sqrt[2]) - (5*ArcTanh[x/S
qrt[-3 - 4*x - x^2]])/4
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 222

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSin[Rt[-b, 2]*(x/Sqrt
[a])]/Rt[-b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && NegQ[b]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 633

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Dist[1/(2*c*(-4*
(c/(b^2 - 4*a*c)))^p), Subst[Int[Simp[1 - x^2/(b^2 - 4*a*c), x]^p, x], x, b
 + 2*c*x], x] /; FreeQ[{a, b, c, p}, x] && GtQ[4*a - b^2/c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 756

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m - 1)*((a + b*x + c*x^2)^(p + 1)/(c*(m + 2*p +
 1))), x] + Dist[1/(c*(m + 2*p + 1)), Int[(d + e*x)^(m - 2)*Simp[c*d^2*(m +
 2*p + 1) - e*(a*e*(m - 1) + b*d*(p + 1)) + e*(2*c*d - b*e)*(m + p)*x, x]*(
a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, m, p}, x] && NeQ[b^2 -
4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && If[Rat
ionalQ[m], GtQ[m, 1], SumSimplerQ[m, -2]] && NeQ[m + 2*p + 1, 0] && IntQuad
raticQ[a, b, c, d, e, m, p, x]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
 && NeQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^4}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = \int \left(\frac{5}{4\sqrt{-3-4x-x^2}} - \frac{x}{\sqrt{-3-4x-x^2}} + \frac{x^2}{2\sqrt{-3-4x-x^2}} -$$

$$= -\left(\frac{1}{4}\int \frac{15+8x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx\right) + \frac{1}{2}\int \frac{x^2}{\sqrt{-3-4x-}$$

$$= \sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} - \frac{1}{4}\int \frac{3+6x}{\sqrt{-3-4x-x^2}}\,dx$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{5}{4}\sin^{-1}(2+x) + \frac{1}{8}\int$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{13}{4}\sin^{-1}(2+x) - \tanh$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{11}{2}\sin^{-1}(2+x) - \frac{5}{4}\tan$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{11}{2}\sin^{-1}(2+x) - \frac{5}{4}\tan$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{11}{2}\sin^{-1}(2+x) - \frac{5}{4}\tan$$

$$= \frac{5}{2}\sqrt{-3-4x-x^2} - \frac{1}{4}x\sqrt{-3-4x-x^2} + \frac{11}{2}\sin^{-1}(2+x) + \frac{\tan^{-}}{}$$

**Mathematica** [A]
time = 0.24, size = 99, normalized size = 0.71

$$\frac{1}{4}\left(-\left((-10+x)\sqrt{-3-4x-x^2}\right) - \sqrt{2}\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right) - 44\tan^{-1}\left(\frac{\sqrt{-3-4x-x^2}}{3+x}\right) - 5\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[x^4/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] (-((-10 + x)*Sqrt[-3 - 4*x - x^2]) - Sqrt[2]*ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt
[-3 - 4*x - x^2])] - 44*ArcTan[Sqrt[-3 - 4*x - x^2]/(3 + x)] - 5*ArcTanh[x/
Sqrt[-3 - 4*x - x^2]])/4
```

**Maple [A]**
time = 0.45, size = 159, normalized size = 1.14

| method | result |
|---|---|
| risch | $\frac{(x-10)(x^2+4x+3)}{4\sqrt{-x^2-4x-3}}+\frac{11\arcsin(x+2)}{2}+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\,\sqrt{2}}{6}\right)\right.}{24\sqrt{\frac{\frac{x^2}{(-\frac{3}{2}-x)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ |
| default | $-\frac{x\sqrt{-x^2-4x-3}}{4}+\frac{5\sqrt{-x^2-4x-3}}{2}+\frac{11\arcsin(x+2)}{2}+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\right.\right.}{24\sqrt{}}$ |
| trager | $\left(-\frac{x}{4}+\frac{5}{2}\right)\sqrt{-x^2-4x-3}+\frac{11\,\mathrm{RootOf}\left(\_Z^2+1\right)\ln\left(-x\,\mathrm{RootOf}\left(\_Z^2+1\right)-2\,\mathrm{RootOf}\left(\_Z^2+1\right)+\sqrt{-x^2-4x}\right.}{2}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/4*x*(-x^2-4*x-3)^(1/2)+5/2*(-x^2-4*x-3)^(1/2)+11/2*arcsin(x+2)+1/24*3^(1
/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*arctan(1/6*(3*x^2/(-3/2-x)
^2-12)^(1/2)*2^(1/2))+5*arctanh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)^(1/2)))/
((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2-x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(x^4/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.36, size = 178, normalized size = 1.27

$$-\frac{1}{4}\sqrt{-x^2-4x-3}\,(x-10)+\frac{1}{8}\sqrt{2}\arctan\left(\frac{\sqrt{2}x+3\sqrt{2}\sqrt{-x^2-4x-3}}{2(2x+3)}\right)+\frac{1}{8}\sqrt{2}\arctan\left(-\frac{\sqrt{2}x-3\sqrt{2}\sqrt{-x^2-4x-3}}{2(2x+3)}\right)-\frac{11}{2}\arctan\left(\frac{\sqrt{-x^2-4x-3}\,(x+2)}{x^2+4x+3}\right)+\frac{5}{16}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)-\frac{5}{16}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt(-x^2 - 4*x - 3)*(x - 10) + 1/8*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*
sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x + 3)) + 1/8*sqrt(2)*arctan(-1/2*(sqrt(2)
*x - 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x + 3)) - 11/2*arctan(sqrt(-x^2 - 4
*x - 3)*(x + 2)/(x^2 + 4*x + 3)) + 5/16*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*
x + 3)/x^2) - 5/16*log((2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^4}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**4/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(x**4/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [A]
time = 5.90, size = 188, normalized size = 1.34

$$-\frac{1}{4}\sqrt{-x^2-4x-3}\,(x-10)+\frac{1}{4}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\frac{1}{4}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)+\frac{11}{2}\arcsin(x+2)-\frac{5}{8}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)+\frac{5}{8}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^4/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] -1/4*sqrt(-x^2 - 4*x - 3)*(x - 10) + 1/4*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqr
t(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)) + 1/4*sqrt(2)*arctan(1/2*sqrt(2)*((sqr
t(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)) + 11/2*arcsin(x + 2) - 5/8*log(2*(sqrt
(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 +
1) + 5/8*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt(-x^2 - 4*x - 3) -
 1)^2/(x + 2)^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x^4}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^4/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(x^4/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

# 3.127 $\int \frac{x^3}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=115

$$-\frac{1}{2}\sqrt{-3-4x-x^2}-2\sin^{-1}(2+x)+\frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{2\sqrt{2}}+\tanh^{-1}$$

```
[Out] -2*arcsin(2+x)+arctanh(x/(-x^2-4*x-3)^(1/2))+1/4*arctan(1/2*(1+(-3-x)/(-x^2
-4*x-3)^(1/2))*2^(1/2))*2^(1/2)-1/4*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(1/2))
*2^(1/2))*2^(1/2)-1/2*(-x^2-4*x-3)^(1/2)
```

**Rubi** [A]
time = 0.26, antiderivative size = 115, normalized size of antiderivative = 1.00, number of steps used = 20, number of rules used = 13, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.433$, Rules used = {6860, 633, 222, 654, 1042, 1000, 12, 1040, 1175, 632, 210, 1041, 212}

$$-2\text{ArcSin}(x+2)+\frac{\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)}{2\sqrt{2}}-\frac{1}{2}\sqrt{-x^2-4x-3}+\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[x^3/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] -1/2*Sqrt[-3 - 4*x - x^2] - 2*ArcSin[2 + x] + ArcTan[(1 - (3 + x)/Sqrt[-3 -
 4*x - x^2])/Sqrt[2]]/(2*Sqrt[2]) - ArcTan[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2
])/Sqrt[2]]/(2*Sqrt[2]) + ArcTanh[x/Sqrt[-3 - 4*x - x^2]]
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 222

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSin[Rt[-b, 2]*(x/Sqrt
[a])]/Rt[-b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && NegQ[b]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 633

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Dist[1/(2*c*(-4*
(c/(b^2 - 4*a*c)))^p), Subst[Int[Simp[1 - x^2/(b^2 - 4*a*c), x]^p, x], x, b
 + 2*c*x], x] /; FreeQ[{a, b, c, p}, x] && GtQ[4*a - b^2/c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
```

```
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
 && NeQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{x^3}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = \int \left(-\frac{1}{\sqrt{-3-4x-x^2}} + \frac{x}{2\sqrt{-3-4x-x^2}} + \frac{6+5x}{2\sqrt{-3-4x-x^2}\,(\ldots}\right.$$

$$= \frac{1}{2}\int \frac{x}{\sqrt{-3-4x-x^2}}\,dx + \frac{1}{2}\int \frac{6+5x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,d\ldots$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} + \frac{1}{2}\text{Subst}\left(\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx, x, -4-2x\right) - \frac{5}{8}\ldots$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - \sin^{-1}(2+x) + \frac{1}{8}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+4\ldots}$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - 2\sin^{-1}(2+x) + \frac{5}{4}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\ldots\right.$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - 2\sin^{-1}(2+x) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\ldots$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - 2\sin^{-1}(2+x) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\ldots$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - 2\sin^{-1}(2+x) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\ldots$$

$$= -\frac{1}{2}\sqrt{-3-4x-x^2} - 2\sin^{-1}(2+x) + \frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}\ldots}{\sqrt{2}}\right.}{2\sqrt{2}}\ldots$$

**Mathematica** [A]
time = 0.20, size = 94, normalized size = 0.82

$$-\frac{1}{2}\sqrt{-3-4x-x^2} - \frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right)}{2\sqrt{2}} + 4\tan^{-1}\left(\frac{\sqrt{-3-4x-x^2}}{3+x}\right) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[x^3/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] -1/2*Sqrt[-3 - 4*x - x^2] - ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])
]/(2*Sqrt[2]) + 4*ArcTan[Sqrt[-3 - 4*x - x^2]/(3 + x)] + ArcTanh[x/Sqrt[-3
- 4*x - x^2]]
```

**Maple** [A]
time = 0.41, size = 144, normalized size = 1.25

| method | result |
|---|---|
| default | $-\frac{\sqrt{-x^2-4x-3}}{2}-2\arcsin(x+2)+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\left(\sqrt{2}\arctan\left(\frac{\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\,\sqrt{}}{6}\right.\right.}{24\sqrt{\frac{\frac{x^2}{(-\frac{3}{2}-x)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\left(1+\right.}$ |
| risch | $\frac{x^2+4x+3}{2\sqrt{-x^2-4x-3}}-2\arcsin(x+2)+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\left(\sqrt{2}\arctan\left(\frac{\sqrt{\frac{3x^2}{(-\frac{3}{2}-x)^2}-12}\,\sqrt{}}{6}\right.\right.}{24\sqrt{\frac{\frac{x^2}{(-\frac{3}{2}-x)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\left(1+\right.}$ |
| trager | $-\frac{\sqrt{-x^2-4x-3}}{2}+\frac{3\,\mathrm{RootOf}\left(24\_Z^2-16\_Z+3\right)\ln\left(-\frac{-72\,\mathrm{RootOf}\left(24\_Z^2-16\_Z+3\right)^2x+72\,\mathrm{RootOf}\left(24\_Z^2-16\_Z+\right.}{12\,\mathrm{RootOf}\left(24\_\right.}\right.}{2}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] -1/2*(-x^2-4*x-3)^(1/2)-2*arcsin(x+2)+1/24*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^
2-12)^(1/2)*(2^(1/2)*arctan(1/6*(3*x^2/(-3/2-x)^2-12)^(1/2)*2^(1/2))-4*arct
anh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)^(1/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3
/2-x))^2)^(1/2)/(1+x/(-3/2-x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(x^3/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.45, size = 175, normalized size = 1.52

$$\frac{1}{8}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)+\frac{1}{8}\sqrt{2}\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)-\frac{1}{2}\sqrt{-x^2-4x-3}+2\arctan\left(\frac{\sqrt{-x^2-4x-3}\,(x+2)}{x^2+4x+3}\right)-\frac{1}{4}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)+\frac{1}{4}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/8*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x +
3)) + 1/8*sqrt(2)*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/
(2*x + 3)) - 1/2*sqrt(-x^2 - 4*x - 3) + 2*arctan(sqrt(-x^2 - 4*x - 3)*(x +
2)/(x^2 + 4*x + 3)) - 1/4*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) +
1/4*log((2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^3}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**3/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(x**3/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [A]
time = 3.58, size = 185, normalized size = 1.61

$$\frac{1}{4}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\frac{1}{4}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)-\frac{1}{2}\sqrt{-x^2-4x-3}-2\arcsin(x+2)+\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)-\frac{1}{2}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^3/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] 1/4*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
+ 1/4*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
- 1/2*sqrt(-x^2 - 4*x - 3) - 2*arcsin(x + 2) + 1/2*log(2*(sqrt(-x^2 - 4*x -
 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 1) - 1/2*log(
2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)
^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x^3}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^3/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(x^3/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

# 3.128 $\int \frac{x^2}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=98

$$\frac{1}{2}\sin^{-1}(2+x)-\frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{\sqrt{2}}+\frac{\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{\sqrt{2}}-\frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

```
[Out] 1/2*arcsin(2+x)-1/2*arctanh(x/(-x^2-4*x-3)^(1/2))-1/2*arctan(1/2*(1+(-3-x)/
(-x^2-4*x-3)^(1/2))*2^(1/2))*2^(1/2)+1/2*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(
1/2))*2^(1/2))*2^(1/2)
```

**Rubi** [A]
time = 0.12, antiderivative size = 98, normalized size of antiderivative = 1.00, number of steps used = 16, number of rules used = 12, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {1091, 633, 222, 1042, 1000, 12, 1040, 1175, 632, 210, 1041, 212}

$$\frac{1}{2}\text{ArcSin}(x+2)-\frac{\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)}{\sqrt{2}}+\frac{\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)}{\sqrt{2}}-\frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[x^2/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] ArcSin[2 + x]/2 - ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/Sqrt[2
] + ArcTan[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/Sqrt[2] - ArcTanh[x/
Sqrt[-3 - 4*x - x^2]]/2
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 222

```
Int[1/Sqrt[(a_) + (b_.)*(x_)^2], x_Symbol] :> Simp[ArcSin[Rt[-b, 2]*(x/Sqrt
[a])]/Rt[-b, 2], x] /; FreeQ[{a, b}, x] && GtQ[a, 0] && NegQ[b]
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 633

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Dist[1/(2*c*(-4*
(c/(b^2 - 4*a*c)))^p), Subst[Int[Simp[1 - x^2/(b^2 - 4*a*c), x]^p, x], x, b
 + 2*c*x], x] /; FreeQ[{a, b, c, p}, x] && GtQ[4*a - b^2/c, 0]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
```

```
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
 && NeQ[2*h*d - g*e, 0]
```

Rule 1091

```
Int[((A_.) + (C_.)*(x_)^2)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) +
 (e_.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[C/c, Int[1/Sqrt[d + e*x + f
*x^2], x], x] + Dist[1/c, Int[(A*c - a*C - b*C*x)/((a + b*x + c*x^2)*Sqrt[d
 + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, A, C}, x] && NeQ[b^2 -
 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rubi steps

$$\int \frac{x^2}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{-3-4x-x^2}}\,dx + \frac{1}{2}\int \frac{-3-4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$$
$$= -\left(\frac{1}{4}\text{Subst}\left(\int \frac{1}{\sqrt{1-\frac{x^2}{4}}}\,dx, x, -4-2x\right)\right) + \frac{1}{2}\int \frac{-}{\sqrt{-3-4x-}}$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \frac{1}{4}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx + \frac{1}{4}\int -$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + \frac{3}{2}\text{Subst}\left(\int \frac{1}{3-3x}\right.$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) - 8\text{Subst}\left(\int \frac{}{-4-}\right.$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + \frac{1}{3}\text{Subst}\left(\int \frac{}{\frac{1}{3}-}\right.$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) - \frac{2}{3}\text{Subst}\left(\int \frac{1}{-\frac{8}{9}-}\right.$$
$$= \frac{1}{2}\sin^{-1}(2+x) - \frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{\sqrt{2}} + \frac{\tan^{-1}\left(\frac{1+\frac{}{\sqrt{-3}}}{}\right.}{\sqrt{2}}$$

**Mathematica** [A]
time = 0.17, size = 77, normalized size = 0.79

$$\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right)}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{-3-4x-x^2}}{3+x}\right) - \frac{1}{2}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[x^2/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])]/Sqrt[2] - ArcTan[Sqrt[-3 -
 4*x - x^2]/(3 + x)] - ArcTanh[x/Sqrt[-3 - 4*x - x^2]]/2
```

**Maple** [A]
time = 0.32, size = 130, normalized size = 1.33

| method | result |
|---|---|

| | |
|---|---|
| default | $\frac{\arcsin(x+2)}{2}-\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}}{6}\right)-\operatorname{arctanh}\left(\frac{3x}{\left(-\frac{3}{2}-x\right)\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right.}}}\right.\right.}{12\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ |
| trager | $\frac{\operatorname{RootOf}\left(\_Z^2+1\right)\ln\left(-x\operatorname{RootOf}\left(\_Z^2+1\right)-2\operatorname{RootOf}\left(\_Z^2+1\right)+\sqrt{-x^2-4x-3}\right)}{2}+\frac{\ln\left(-\frac{16\operatorname{RootOf}\left(16\_Z^2-8\_Z\right.}{}\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/2*arcsin(x+2)-1/12*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*a
rctan(1/6*(3*x^2/(-3/2-x)^2-12)^(1/2)*2^(1/2))-arctanh(3*x/(-3/2-x)/(3*x^2/
(-3/2-x)^2-12)^(1/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3
/2-x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(x^2/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.45, size = 161, normalized size = 1.64

$$-\frac{1}{4}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)-\frac{1}{4}\sqrt{2}\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2(2x+3)}\right)-\frac{1}{2}\arctan\left(\frac{\sqrt{-x^2-4x-3}\,(x+2)}{x^2+4x+3}\right)+\frac{1}{8}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)-\frac{1}{8}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/4*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x +
 3)) - 1/4*sqrt(2)*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))
/(2*x + 3)) - 1/2*arctan(sqrt(-x^2 - 4*x - 3)*(x + 2)/(x^2 + 4*x + 3)) + 1/
8*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) - 1/8*log((2*sqrt(-x^2 - 4
*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x^2}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x**2/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(x**2/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 171 vs. 2(82) = 164.
time = 3.98, size = 171, normalized size = 1.74

$$-\frac{1}{2}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)-\frac{1}{2}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)+\frac{1}{2}\arcsin\left(x+2\right)-\frac{1}{4}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)+\frac{1}{4}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] -1/2*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
 - 1/2*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
 + 1/2*arcsin(x + 2) - 1/4*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sq
rt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 1) + 1/4*log(2*(sqrt(-x^2 - 4*x - 3)
- 1)/(x + 2) + (sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 3)
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x^2}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x^2/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(x^2/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

## 3.129 $\int \frac{x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=68

$$-\frac{\tan^{-1}\left(\frac{1-\frac{3\sqrt{-1-x}}{\sqrt{3+x}}}{\sqrt{2}}\right)}{\sqrt{2}}+\frac{\tan^{-1}\left(\frac{1+\frac{3\sqrt{-1-x}}{\sqrt{3+x}}}{\sqrt{2}}\right)}{\sqrt{2}}$$

```
[Out] -1/2*arctan(1/2*(1-3*(-1-x)^(1/2)/(3+x)^(1/2))*2^(1/2))*2^(1/2)+1/2*arctan(
1/2*(1+3*(-1-x)^(1/2)/(3+x)^(1/2))*2^(1/2))*2^(1/2)
```

**Rubi** [A]
time = 0.04, antiderivative size = 69, normalized size of antiderivative = 1.01, number of steps used = 6, number of rules used = 4, integrand size = 28, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {1040, 1175, 632, 210}

$$\frac{\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)}{\sqrt{2}}-\frac{\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)}{\sqrt{2}}$$

Antiderivative was successfully verified.

```
[In] Int[x/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/Sqrt[2] - ArcTan[(1 + (3
 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]/Sqrt[2]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
```

```
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rubi steps

$$\int \frac{x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = 8\text{Subst}\left(\int \frac{1+3x^2}{-4-8x^2-36x^4}\,dx, x, \frac{1+\frac{x}{3}}{\sqrt{-3-4x-x^2}}\right)$$
$$= -\left(\frac{1}{3}\text{Subst}\left(\int \frac{1}{\frac{1}{3}-\frac{2x}{3}+x^2}\,dx, x, \frac{1+\frac{x}{3}}{\sqrt{-3-4x-x^2}}\right)\right) - \frac{1}{3}\text{Subst}\Bigg($$
$$= \frac{2}{3}\text{Subst}\left(\int \frac{1}{-\frac{8}{9}-x^2}\,dx, x, \frac{2}{3}\left(-1+\frac{3+x}{\sqrt{-3-4x-x^2}}\right)\right) + \frac{2}{3}\text{Subst}$$
$$= \frac{\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{\sqrt{2}} - \frac{\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)}{\sqrt{2}}$$

**Mathematica [A]**
time = 0.12, size = 33, normalized size = 0.49

$$-\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right)}{\sqrt{2}}$$

Antiderivative was successfully verified.

```
[In] Integrate[x/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
[Out] -(ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])]/Sqrt[2])
```

**Maple [A]**
time = 0.14, size = 92, normalized size = 1.35

| method | result |
|---|---|

| | |
|---|---|
| trager | $\frac{\mathrm{RootOf}\left(\_Z^2+2\right)\ln\left(\frac{6\,\mathrm{RootOf}\left(\_Z^2+2\right)x^2+8x\sqrt{-x^2-4x-3}+20\,\mathrm{RootOf}\left(\_Z^2+2\right)x+12\sqrt{-x^2-4x-3}+15\,\mathrm{RootOf}}{2x^2+4x+3}\right)}{4}$ |
| default | $\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}}{6}\right)}{12\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/(-x^2-4*x-3)^(1/2)/(2*x^2+4*x+3),x,method=_RETURNVERBOSE)
```

```
[Out] 1/12*3^(1/2)*4^(1/2)/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2
-x))*(3*x^2/(-3/2-x)^2-12)^(1/2)*2^(1/2)*arctan(1/6*(3*x^2/(-3/2-x)^2-12)^(
1/2)*2^(1/2))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(x/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.47, size = 50, normalized size = 0.74

$$\frac{1}{4}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,(6\,x^2+20\,x+15)\sqrt{-x^2-4\,x-3}}{4\,(2\,x^3+11\,x^2+18\,x+9)}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/4*sqrt(2)*arctan(1/4*sqrt(2)*(6*x^2 + 20*x + 15)*sqrt(-x^2 - 4*x - 3)/(2*
x^3 + 11*x^2 + 18*x + 9))
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{x}{\sqrt{-(x+1)\,(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(x/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac [A]**
time = 3.62, size = 68, normalized size = 1.00

$$\frac{1}{2}\,\sqrt{2}\,\arctan\left(\frac{1}{2}\,\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4\,x-3}-1\right)}{x+2}+1\right)\right)+\frac{1}{2}\,\sqrt{2}\,\arctan\left(\frac{1}{2}\,\sqrt{2}\left(\frac{\sqrt{-x^2-4\,x-3}-1}{x+2}+1\right)\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] 1/2*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
+ 1/2*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{x}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(x/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

# 3.130 $\int \frac{1}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=95

$$-\frac{1}{3}\sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)+\frac{1}{3}\sqrt{2}\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)+\frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-}}\right.$$

```
[Out] 1/3*arctanh(x/(-x^2-4*x-3)^(1/2))-1/3*arctan(1/2*(1+(-3-x)/(-x^2-4*x-3)^(1/
2))*2^(1/2))*2^(1/2)+1/3*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(1/2))*2^(1/2))*2
^(1/2)
```

**Rubi** [A]
time = 0.07, antiderivative size = 95, normalized size of antiderivative = 1.00, number of steps used = 10, number of rules used = 8, integrand size = 27, $\frac{\text{number of rules}}{\text{integrand size}} = 0.296$, Rules used = {1000, 12, 1040, 1175, 632, 210, 1041, 212}

$$-\frac{1}{3}\sqrt{2}\,\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)+\frac{1}{3}\sqrt{2}\,\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)+\frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[1/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] -1/3*(Sqrt[2]*ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]) + (Sqrt[2
]*ArcTan[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]])/3 + ArcTanh[x/Sqrt[-3
 - 4*x - x^2]]/3
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rubi steps

$$\int \frac{1}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = -\left(\frac{1}{6}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx\right) + \frac{1}{6}\int -\frac{}{\sqrt{-3-4x}}$$

$$= -\left(\frac{2}{3}\int \frac{x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx\right) + \text{Subst}\left(\int \frac{1}{3-3x}\right.$$

$$= \frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) - \frac{16}{3}\text{Subst}\left(\int \frac{1+3x^2}{-4-8x^2-36x^4}\,dx\right.$$

$$= \frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) + \frac{2}{9}\text{Subst}\left(\int \frac{1}{\frac{1}{3}-\frac{2x}{3}+x^2}\,dx, x, -\frac{}{\sqrt{}}\right.$$

$$= \frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right) - \frac{4}{9}\text{Subst}\left(\int \frac{1}{-\frac{8}{9}-x^2}\,dx, x, \frac{2}{3}\left(-\right.\right.$$

$$= -\frac{1}{3}\sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right) + \frac{1}{3}\sqrt{2}\tan^{-1}\left(\frac{1+\frac{}{\sqrt{-}}}{}\right.$$

**Mathematica [A]**
time = 0.16, size = 54, normalized size = 0.57

$$\frac{1}{3}\left(\sqrt{2}\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right) + \tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
[Out] (Sqrt[2]*ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])] + ArcTanh[x/Sqrt[
-3 - 4*x - x^2]])/3
```

**Maple [A]**
time = 0.32, size = 121, normalized size = 1.27

| method | result |
|---|---|
| default | $-\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}}{6}\right)+\text{arctanh}\left(\frac{3x}{\left(-\frac{3}{2}-x\right)\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}}\right)\right)}{18\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ |

| | |
|---|---|
| trager | $\mathrm{RootOf}\left(12\_Z^2+4\_Z+1\right)\ln\left(-\frac{12\,\mathrm{RootOf}\left(12\_Z^2+4\_Z+1\right)^2 x-4\,\mathrm{RootOf}\left(12\_Z^2+4\_Z+1\right)x+2\sqrt{-x^2-4x}}{2\,\mathrm{RootOf}\left(12\_Z^2+4\_Z+1\right)x+x+1}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(-x^2-4*x-3)^(1/2)/(2*x^2+4*x+3),x,method=_RETURNVERBOSE)
```

```
[Out] -1/18*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*arctan(1/6*(3*x^
2/(-3/2-x)^2-12)^(1/2)*2^(1/2))+arctanh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)^
(1/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2-x))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(1/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)), x)
```

**Fricas [A]**
time = 0.40, size = 132, normalized size = 1.39

$$-\frac{1}{6}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2\,(2x+3)}\right)-\frac{1}{6}\sqrt{2}\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2\,(2x+3)}\right)-\frac{1}{12}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)+\frac{1}{12}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/6*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/(2*x +
 3)) - 1/6*sqrt(2)*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))
/(2*x + 3)) - 1/12*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) + 1/12*lo
g((2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{\sqrt{-(x+1)(x+3)}\,(2x^2+4x+3)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(1/(sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 165 vs. 2(76) = 152.
time = 3.46, size = 165, normalized size = 1.74

$$-\frac{1}{3}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)-\frac{1}{3}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)+\frac{1}{6}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)-\frac{1}{6}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] -1/3*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
 - 1/3*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
 + 1/6*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) -
 1)^2/(x + 2)^2 + 1) - 1/6*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt
(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{1}{\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(1/((- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

# 3.131 $\int \frac{1}{x\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=130

$$-\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}}+\frac{1}{9}\sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)-\frac{1}{9}\sqrt{2}\tan^{-1}\left(\frac{1+\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right.$$

```
[Out] -4/9*arctanh(x/(-x^2-4*x-3)^(1/2))+1/9*arctan(1/2*(1+(-3-x)/(-x^2-4*x-3)^(1
/2))*2^(1/2))*2^(1/2)-1/9*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(1/2))*2^(1/2))*
2^(1/2)-1/9*arctan(1/3*(3+2*x)*3^(1/2)/(-x^2-4*x-3)^(1/2))*3^(1/2)
```

**Rubi** [A]

time = 0.26, antiderivative size = 130, normalized size of antiderivative = 1.00, number of steps used = 17, number of rules used = 11, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.367$, Rules used = {6860, 738, 210, 1042, 1000, 12, 1040, 1175, 632, 1041, 212}

$$-\frac{\text{ArcTan}\left(\frac{2x+3}{\sqrt{3}\sqrt{-x^2-4x-3}}\right)}{3\sqrt{3}}+\frac{1}{9}\sqrt{2}\,\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)-\frac{1}{9}\sqrt{2}\,\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)-\frac{4}{9}\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[1/(x*Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] -1/3*ArcTan[(3 + 2*x)/(Sqrt[3]*Sqrt[-3 - 4*x - x^2])]/Sqrt[3] + (Sqrt[2]*Ar
cTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]])/9 - (Sqrt[2]*ArcTan[(1 +
(3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]])/9 - (4*ArcTanh[x/Sqrt[-3 - 4*x - x^
2]])/9
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
 EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

```
&& NeQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = \int \left(\frac{1}{3x\sqrt{-3-4x-x^2}} - \frac{2(2+x)}{3\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\right) d$$

$$= \frac{1}{3}\int \frac{1}{x\sqrt{-3-4x-x^2}}\,dx - \frac{2}{3}\int \frac{2+x}{\sqrt{-3-4x-x^2}\,(3+4x+2x$$

$$= \frac{1}{6}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx - \frac{1}{3}\int \frac{1}{\sqrt{-3-4x-x^2}}$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{1}{18}\int \frac{-6-4x}{\sqrt{-3-4x-x^2}\,(3+}$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} - \frac{1}{3}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right.$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} - \frac{4}{9}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right.$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} - \frac{4}{9}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right.$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} - \frac{4}{9}\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right.$$

$$= -\frac{\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\,\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{1}{9}\sqrt{2}\,\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4}}}{\sqrt{2}}\right.$$

**Mathematica** [A]
time = 0.18, size = 90, normalized size = 0.69

$$\frac{1}{9}\left(-\sqrt{2}\,\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\,\sqrt{-3-4x-x^2}}\right)+2\sqrt{3}\,\tan^{-1}\left(\frac{\sqrt{3}\,\sqrt{-3-4x-x^2}}{3+x}\right)-4\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x*Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] (-(Sqrt[2]*ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])]) + 2*Sqrt[3]*Ar
cTan[(Sqrt[3]*Sqrt[-3 - 4*x - x^2])/(3 + x)] - 4*ArcTanh[x/Sqrt[-3 - 4*x -
x^2]])/9
```

**Maple [A]**
time = 0.37, size = 152, normalized size = 1.17

| method | result |
|---|---|
| default | $\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\sqrt{2}}{6}\right)+4\,\mathrm{arctanh}\left(\frac{3x}{\left(-\frac{3}{2}-x\right)\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}}\right)\right)}{54\sqrt{\frac{\frac{x^2}{\left(-\frac{3}{2}-x\right)^2}-4}{\left(1+\frac{x}{-\frac{3}{2}-x}\right)^2}}\,\left(1+\frac{x}{-\frac{3}{2}-x}\right)}$ - |
| trager | $\mathrm{RootOf}\left(18\_Z^2-8\_Z+1\right)\ln\left(-\frac{40500\,\mathrm{RootOf}\left(18\_Z^2-8\_Z+1\right)^2 x+4680\sqrt{-x^2-4x-3}\,\mathrm{RootOf}\left(18\_Z^2\right.}{}\right.$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/54*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*arctan(1/6*(3*x^2
/(-3/2-x)^2-12)^(1/2)*2^(1/2))+4*arctanh(3*x/(-3/2-x)/(3*x^2/(-3/2-x)^2-12)
^(1/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2-x))+1/9*3^(
1/2)*arctan(1/6*(-6-4*x)*3^(1/2)/(-x^2-4*x-3)^(1/2))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(1/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)*x), x)
```

**Fricas [A]**
time = 0.38, size = 170, normalized size = 1.31

$$\frac{1}{9}\sqrt{3}\arctan\left(\frac{\sqrt{3}\sqrt{-x^2-4x-3}\,(2x+3)}{3(x^2+4x+3)}\right)+\frac{1}{18}\sqrt{2}\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\sqrt{-x^2-4x-3}}{2(2x+3)}\right)+\frac{1}{18}\sqrt{2}\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\sqrt{-x^2-4x-3}}{2(2x+3)}\right)+\frac{1}{9}\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)-\frac{1}{9}\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] 1/9*sqrt(3)*arctan(1/3*sqrt(3)*sqrt(-x^2 - 4*x - 3)*(2*x + 3)/(x^2 + 4*x +
3)) + 1/18*sqrt(2)*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x - 3))/
(2*x + 3)) + 1/18*sqrt(2)*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^2 - 4*
```

```
x - 3))/(2*x + 3)) + 1/9*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^2) - 1
/9*log((2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x\sqrt{-\left(x+1\right)\left(x+3\right)}\,\left(2x^2+4x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(1/(x*sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [A]
time = 3.09, size = 199, normalized size = 1.53

$$\frac{1}{9}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\frac{2}{9}\sqrt{3}\arctan\left(\frac{1}{3}\sqrt{3}\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\frac{1}{9}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)-\frac{2}{9}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)+\frac{2}{9}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] 1/9*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
+ 2/9*sqrt(3)*arctan(1/3*sqrt(3)*(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)
) + 1/9*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1)
) - 2/9*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3)
- 1)^2/(x + 2)^2 + 1) + 2/9*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqr
t(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 3)
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{1}{x\,\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x*(- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(1/(x*(- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

# 3.132 $\int \frac{1}{x^2\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx$

**Optimal**. Leaf size=151

$$\frac{\sqrt{-3-4x-x^2}}{9x}+\frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}}+\frac{2}{27}\sqrt{2}\tan^{-1}\left(\frac{1-\frac{3+x}{\sqrt{-3-4x-x^2}}}{\sqrt{2}}\right)-\frac{2}{27}\sqrt{2}\tan^{-1}$$

```
[Out] 10/27*arctanh(x/(-x^2-4*x-3)^(1/2))+2/27*arctan(1/2*(1+(-3-x)/(-x^2-4*x-3)^
(1/2))*2^(1/2))*2^(1/2)-2/27*arctan(1/2*(1+(3+x)/(-x^2-4*x-3)^(1/2))*2^(1/2
))*2^(1/2)+2/9*arctan(1/3*(3+2*x)*3^(1/2)/(-x^2-4*x-3)^(1/2))*3^(1/2)+1/9*(
-x^2-4*x-3)^(1/2)/x
```

**Rubi** [A]

time = 0.28, antiderivative size = 151, normalized size of antiderivative = 1.00, number of steps used = 20, number of rules used = 12, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.400$, Rules used = {6860, 744, 738, 210, 1042, 1000, 12, 1040, 1175, 632, 1041, 212}

$$\frac{2\text{ArcTan}\left(\frac{2x+3}{\sqrt{3}\sqrt{-x^2-4x-3}}\right)}{3\sqrt{3}}+\frac{2}{27}\sqrt{2}\,\text{ArcTan}\left(\frac{1-\frac{x+3}{\sqrt{-x^2-4x-3}}}{\sqrt{2}}\right)-\frac{2}{27}\sqrt{2}\,\text{ArcTan}\left(\frac{\frac{x+3}{\sqrt{-x^2-4x-3}}+1}{\sqrt{2}}\right)+\frac{\sqrt{-x^2-4x-3}}{9x}+\frac{10}{27}\tanh^{-1}\left(\frac{x}{\sqrt{-x^2-4x-3}}\right)$$

Antiderivative was successfully verified.

```
[In] Int[1/(x^2*Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] Sqrt[-3 - 4*x - x^2]/(9*x) + (2*ArcTan[(3 + 2*x)/(Sqrt[3]*Sqrt[-3 - 4*x - x
^2])])/(3*Sqrt[3]) + (2*Sqrt[2]*ArcTan[(1 - (3 + x)/Sqrt[-3 - 4*x - x^2])/S
qrt[2]])/27 - (2*Sqrt[2]*ArcTan[(1 + (3 + x)/Sqrt[-3 - 4*x - x^2])/Sqrt[2]]
)/27 + (10*ArcTanh[x/Sqrt[-3 - 4*x - x^2]])/27
```

Rule 12

```
Int[(a_)*(u_), x_Symbol] :> Dist[a, Int[u, x], x] /; FreeQ[a, x] &&  !Match
Q[u, (b_)*(v_) /; FreeQ[b, x]]
```

Rule 210

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(-(Rt[-a, 2]*Rt[-b, 2])^(
-1))*ArcTan[Rt[-b, 2]*(x/Rt[-a, 2])], x] /; FreeQ[{a, b}, x] && PosQ[a/b] &
& (LtQ[a, 0] || LtQ[b, 0])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 632

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(-1), x_Symbol] :> Dist[-2, Subst[I
nt[1/Simp[b^2 - 4*a*c - x^2, x], x], x, b + 2*c*x], x] /; FreeQ[{a, b, c},
x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 744

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m + 1)*((a + b*x + c*x^2)^(p + 1)/((m + 1)*(c*d
^2 - b*d*e + a*e^2))), x] + Dist[(2*c*d - b*e)/(2*(c*d^2 - b*d*e + a*e^2)),
 Int[(d + e*x)^(m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e,
 m, p}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2
*c*d - b*e, 0] && EqQ[m + 2*p + 3, 0]
```

Rule 1000

```
Int[1/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_.) + (e_.)*(x_) + (f_.)*
(x_)^2]), x_Symbol] :> With[{q = Rt[(c*d - a*f)^2 - (b*d - a*e)*(c*e - b*f)
, 2]}, Dist[1/(2*q), Int[(c*d - a*f + q + (c*e - b*f)*x)/((a + b*x + c*x^2)
*Sqrt[d + e*x + f*x^2]), x], x] - Dist[1/(2*q), Int[(c*d - a*f - q + (c*e -
 b*f)*x)/((a + b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x]] /; FreeQ[{a, b,
 c, d, e, f}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && NeQ[c*e -
 b*f, 0] && NegQ[b^2 - 4*a*c]
```

Rule 1040

```
Int[(x_)/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_.)*(x_) + (f_.)
*(x_)^2]), x_Symbol] :> Dist[-2*e, Subst[Int[(1 - d*x^2)/(c*e - b*f - e*(2*
c*d - b*e + 2*a*f)*x^2 + d^2*(c*e - b*f)*x^4), x], x, (1 + (e + Sqrt[e^2 -
4*d*f])*(x/(2*d)))/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f},
x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
```

Rule 1041

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[g, Subst[Int[1/(a + (c*d - a*f)
*x^2), x], x, x/Sqrt[d + e*x + f*x^2]], x] /; FreeQ[{a, b, c, d, e, f, g, h
}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0] &&
```

```
EqQ[2*h*d - g*e, 0]
```

Rule 1042

```
Int[((g_) + (h_.)*(x_))/(((a_) + (b_.)*(x_) + (c_.)*(x_)^2)*Sqrt[(d_) + (e_
.)*(x_) + (f_.)*(x_)^2]), x_Symbol] :> Dist[-(2*h*d - g*e)/e, Int[1/((a + b
*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] + Dist[h/e, Int[(2*d + e*x)/((a
+ b*x + c*x^2)*Sqrt[d + e*x + f*x^2]), x], x] /; FreeQ[{a, b, c, d, e, f, g
, h}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[e^2 - 4*d*f, 0] && EqQ[b*d - a*e, 0]
 && NeQ[2*h*d - g*e, 0]
```

Rule 1175

```
Int[((d_) + (e_.)*(x_)^2)/((a_) + (b_.)*(x_)^2 + (c_.)*(x_)^4), x_Symbol] :
> With[{q = Rt[2*(d/e) - b/c, 2]}, Dist[e/(2*c), Int[1/Simp[d/e + q*x + x^2
, x], x], x] + Dist[e/(2*c), Int[1/Simp[d/e - q*x + x^2, x], x], x]] /; Fre
eQ[{a, b, c, d, e}, x] && NeQ[b^2 - 4*a*c, 0] && EqQ[c*d^2 - a*e^2, 0] && (
GtQ[2*(d/e) - b/c, 0] || ( !LtQ[2*(d/e) - b/c, 0] && EqQ[d - e*Rt[a/c, 2],
0]))
```

Rule 6860

```
Int[(u_)/((a_.) + (b_.)*(x_)^(n_.) + (c_.)*(x_)^(n2_.)), x_Symbol] :> With[
{v = RationalFunctionExpand[u/(a + b*x^n + c*x^(2*n)), x]}, Int[v, x] /; Su
mQ[v]] /; FreeQ[{a, b, c}, x] && EqQ[n2, 2*n] && IGtQ[n, 0]
```

Rubi steps

$$\int \frac{1}{x^2\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx = \int \left(\frac{1}{3x^2\sqrt{-3-4x-x^2}} - \frac{4}{9x\sqrt{-3-4x-x^2}} + \frac{2}{9\sqrt{-3-4x}}\right.$$
$$= \frac{2}{9}\int \frac{5+4x}{\sqrt{-3-4x-x^2}\,(3+4x+2x^2)}\,dx + \frac{1}{3}\int \frac{1}{x^2\sqrt{-3-4x-}}$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} - \frac{2}{9}\int \frac{1}{x\sqrt{-3-4x-x^2}}\,dx - \frac{2}{9}\int \frac{}{\sqrt{-3-4}}$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{4\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{9\sqrt{3}} + \frac{1}{27}\int$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{4}{9}\tanh^{-}$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{10}{27}\tanh$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{10}{27}\tanh$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{10}{27}\tanh$$
$$= \frac{\sqrt{-3-4x-x^2}}{9x} + \frac{2\tan^{-1}\left(\frac{3+2x}{\sqrt{3}\sqrt{-3-4x-x^2}}\right)}{3\sqrt{3}} + \frac{2}{27}\sqrt{2}$$

**Mathematica [A]**
time = 0.23, size = 113, normalized size = 0.75

$$\frac{-2\sqrt{2}\,x\tan^{-1}\left(\frac{3+2x}{\sqrt{2}\sqrt{-3-4x-x^2}}\right)+3\left(\sqrt{-3-4x-x^2}-4\sqrt{3}\,x\tan^{-1}\left(\frac{\sqrt{3}\sqrt{-3-4x-x^2}}{3+x}\right)\right)+10x\tanh^{-1}\left(\frac{x}{\sqrt{-3-4x-x^2}}\right)}{27x}$$

Antiderivative was successfully verified.

```
[In] Integrate[1/(x^2*Sqrt[-3 - 4*x - x^2]*(3 + 4*x + 2*x^2)),x]
```

```
[Out] (-2*Sqrt[2]*x*ArcTan[(3 + 2*x)/(Sqrt[2]*Sqrt[-3 - 4*x - x^2])] + 3*(Sqrt[-3
 - 4*x - x^2] - 4*Sqrt[3]*x*ArcTan[(Sqrt[3]*Sqrt[-3 - 4*x - x^2])/(3 + x)])
 + 10*x*ArcTanh[x/Sqrt[-3 - 4*x - x^2]])/(27*x)
```

**Maple [A]**
time = 0.39, size = 169, normalized size = 1.12

| method | result |
|---|---|
| default | $\frac{\sqrt{-x^2-4x-3}}{9x}-\frac{2\sqrt{3}\,\arctan\left(\frac{(-6-4x)\sqrt{3}}{6\sqrt{-x^2-4x-3}}\right)}{9}+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\frac{\sqrt{(-\cdots}}{\cdots}\right.\right.}{81\sqrt{\frac{\cdots}{\left(1+\cdots\right.}}}$ |
| risch | $-\frac{x^2+4x+3}{9x\sqrt{-x^2-4x-3}}-\frac{2\sqrt{3}\,\arctan\left(\frac{(-6-4x)\sqrt{3}}{6\sqrt{-x^2-4x-3}}\right)}{9}+\frac{\sqrt{3}\,\sqrt{4}\,\sqrt{\frac{3x^2}{\left(-\frac{3}{2}-x\right)^2}-12}\,\left(\sqrt{2}\,\arctan\left(\cdots\right.\right.}{81\sqrt{\cdots}}$ |
| trager | $\frac{\sqrt{-x^2-4x-3}}{9x}+\frac{16\,\mathrm{RootOf}\left(768\_Z^2+160\_Z+9\right)\ln\left(-\frac{288000\,\mathrm{RootOf}\left(768\_Z^2+160\_Z+9\right)^2x+12480\sqrt{-x^2-4x-\cdots}}{\cdots}\right.}{\cdots}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x,method=_RETURNVERBOSE)
```

```
[Out] 1/9*(-x^2-4*x-3)^(1/2)/x-2/9*3^(1/2)*arctan(1/6*(-6-4*x)*3^(1/2)/(-x^2-4*x-
3)^(1/2))+1/81*3^(1/2)*4^(1/2)*(3*x^2/(-3/2-x)^2-12)^(1/2)*(2^(1/2)*arctan(
1/6*(3*x^2/(-3/2-x)^2-12)^(1/2)*2^(1/2))-5*arctanh(3*x/(-3/2-x)/(3*x^2/(-3/
2-x)^2-12)^(1/2)))/((x^2/(-3/2-x)^2-4)/(1+x/(-3/2-x))^2)^(1/2)/(1+x/(-3/2-x
))
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="maxima")
```

```
[Out] integrate(1/((2*x^2 + 4*x + 3)*sqrt(-x^2 - 4*x - 3)*x^2), x)
```

**Fricas [A]**
time = 0.39, size = 194, normalized size = 1.28

$$-\frac{12\sqrt{3}\,x\arctan\left(\frac{\sqrt{3}\,\sqrt{-x^2-4x-3}\,(2x+3)}{3\,(x^2+4x+3)}\right)-2\sqrt{2}\,x\arctan\left(\frac{\sqrt{2}\,x+3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2\,(2x+3)}\right)-2\sqrt{2}\,x\arctan\left(-\frac{\sqrt{2}\,x-3\sqrt{2}\,\sqrt{-x^2-4x-3}}{2\,(2x+3)}\right)+5x\log\left(-\frac{2\sqrt{-x^2-4x-3}\,x+4x+3}{x^2}\right)-5x\log\left(\frac{2\sqrt{-x^2-4x-3}\,x-4x-3}{x^2}\right)-6\sqrt{-x^2-4x-3}}{54x}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="fricas")
```

```
[Out] -1/54*(12*sqrt(3)*x*arctan(1/3*sqrt(3)*sqrt(-x^2 - 4*x - 3)*(2*x + 3)/(x^2
+ 4*x + 3)) - 2*sqrt(2)*x*arctan(1/2*(sqrt(2)*x + 3*sqrt(2)*sqrt(-x^2 - 4*x
 - 3))/(2*x + 3)) - 2*sqrt(2)*x*arctan(-1/2*(sqrt(2)*x - 3*sqrt(2)*sqrt(-x^
2 - 4*x - 3))/(2*x + 3)) + 5*x*log(-(2*sqrt(-x^2 - 4*x - 3)*x + 4*x + 3)/x^
2) - 5*x*log((2*sqrt(-x^2 - 4*x - 3)*x - 4*x - 3)/x^2) - 6*sqrt(-x^2 - 4*x
- 3))/x
```

**Sympy** [**F**]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{1}{x^2\sqrt{-\left(x+1\right)\left(x+3\right)}\,\left(2x^2+4x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x**2/(2*x**2+4*x+3)/(-x**2-4*x-3)**(1/2),x)
```

```
[Out] Integral(1/(x**2*sqrt(-(x + 1)*(x + 3))*(2*x**2 + 4*x + 3)), x)
```

**Giac** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 269 vs. 2(121) = 242.
time = 3.93, size = 269, normalized size = 1.78

$$\frac{2}{27}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{3\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)-\frac{4}{9}\sqrt{3}\arctan\left(\frac{1}{3}\sqrt{3}\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+1\right)\right)+\frac{2}{27}\sqrt{2}\arctan\left(\frac{1}{2}\sqrt{2}\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+1\right)\right)-\frac{\frac{\sqrt{-x^2-4x-3}-1}{x+2}+2}{18\left(\frac{\sqrt{-x^2-4x-3}-1}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)}+\frac{5}{27}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{3\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+1\right)-\frac{5}{27}\log\left(\frac{2\left(\sqrt{-x^2-4x-3}-1\right)}{x+2}+\frac{\left(\sqrt{-x^2-4x-3}-1\right)^2}{(x+2)^2}+3\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(1/x^2/(2*x^2+4*x+3)/(-x^2-4*x-3)^(1/2),x, algorithm="giac")
```

```
[Out] 2/27*sqrt(2)*arctan(1/2*sqrt(2)*(3*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1))
 - 4/9*sqrt(3)*arctan(1/3*sqrt(3)*(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 1
)) + 2/27*sqrt(2)*arctan(1/2*sqrt(2)*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) +
1)) - 1/18*((sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + 2)/((sqrt(-x^2 - 4*x - 3)
- 1)/(x + 2) + (sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2 + 1) + 5/27*log(2*(sq
rt(-x^2 - 4*x - 3) - 1)/(x + 2) + 3*(sqrt(-x^2 - 4*x - 3) - 1)^2/(x + 2)^2
+ 1) - 5/27*log(2*(sqrt(-x^2 - 4*x - 3) - 1)/(x + 2) + (sqrt(-x^2 - 4*x - 3
) - 1)^2/(x + 2)^2 + 3)
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{1}{x^2\,\sqrt{-x^2-4\,x-3}\,\left(2\,x^2+4\,x+3\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(1/(x^2*(- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)),x)
```

```
[Out] int(1/(x^2*(- 4*x - x^2 - 3)^(1/2)*(4*x + 2*x^2 + 3)), x)
```

## 3.133 $\int (2+3x)^2 \left(30+31x-12x^2\right)^2 \sqrt{6+17x+12x^2}\,dx$

**Optimal**. Leaf size=149

$$\frac{125455(17+24x)\sqrt{6+17x+12x^2}}{150994944}-\frac{125455(17+24x)\left(6+17x+12x^2\right)^{3/2}}{4718592}+\frac{25091(17+24x)\left(6+17x+1\right.}{24576}$$

```
[Out] -125455/4718592*(17+24*x)*(12*x^2+17*x+6)^(3/2)+25091/24576*(17+24*x)*(12*x
^2+17*x+6)^(5/2)-873/1792*(12*x^2+17*x+6)^(7/2)-1/32*(10-3*x)*(12*x^2+17*x+
6)^(7/2)-125455/1811939328*arctanh(1/12*(17+24*x)*3^(1/2)/(12*x^2+17*x+6)^(
1/2))*3^(1/2)+125455/150994944*(17+24*x)*(12*x^2+17*x+6)^(1/2)
```

**Rubi** [A]
time = 0.06, antiderivative size = 149, normalized size of antiderivative = 1.00, number of steps used = 8, number of rules used = 6, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}} = 0.176$, Rules used = {1016, 756, 654, 626, 635, 212}

$$-\frac{1}{32}(10-3x)\left(12x^2+17x+6\right)^{7/2}-\frac{873\left(12x^2+17x+6\right)^{7/2}}{1792}+\frac{25091(24x+17)\left(12x^2+17x+6\right)^{5/2}}{24576}-\frac{125455(24x+17)\left(12x^2+17x+6\right)^{3/2}}{4718592}+\frac{125455(24x+17)\sqrt{12x^2+17x+6}}{150994944}-\frac{125455\tanh^{-1}\left(\frac{24x+17}{4\sqrt{3}\sqrt{12x^2+17x+6}}\right)}{603979776\sqrt{3}}$$

Antiderivative was successfully verified.

```
[In] Int[(2 + 3*x)^2*(30 + 31*x - 12*x^2)^2*Sqrt[6 + 17*x + 12*x^2],x]
```

```
[Out] (125455*(17 + 24*x)*Sqrt[6 + 17*x + 12*x^2])/150994944 - (125455*(17 + 24*x
)*(6 + 17*x + 12*x^2)^(3/2))/4718592 + (25091*(17 + 24*x)*(6 + 17*x + 12*x^
2)^(5/2))/24576 - (873*(6 + 17*x + 12*x^2)^(7/2))/1792 - ((10 - 3*x)*(6 + 1
7*x + 12*x^2)^(7/2))/32 - (125455*ArcTanh[(17 + 24*x)/(4*Sqrt[3]*Sqrt[6 + 1
7*x + 12*x^2])])/(603979776*Sqrt[3])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
```

```
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 756

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[e*(d + e*x)^(m - 1)*((a + b*x + c*x^2)^(p + 1)/(c*(m + 2*p +
 1))), x] + Dist[1/(c*(m + 2*p + 1)), Int[(d + e*x)^(m - 2)*Simp[c*d^2*(m +
 2*p + 1) - e*(a*e*(m - 1) + b*d*(p + 1)) + e*(2*c*d - b*e)*(m + p)*x, x]*(
a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, m, p}, x] && NeQ[b^2 -
4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && If[Rat
ionalQ[m], GtQ[m, 1], SumSimplerQ[m, -2]] && NeQ[m + 2*p + 1, 0] && IntQuad
raticQ[a, b, c, d, e, m, p, x]
```

Rule 1016

```
Int[((g_) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.
) + (e_.)*(x_) + (f_.)*(x_)^2)^(m_.), x_Symbol] :> Int[(d*(g/a) + f*h*(x/c)
)^m*(a + b*x + c*x^2)^(m + p), x] /; FreeQ[{a, b, c, d, e, f, g, h, p}, x]
&& EqQ[c*g^2 - b*g*h + a*h^2, 0] && EqQ[c^2*d*g^2 - a*c*e*g*h + a^2*f*h^2,
0] && IntegerQ[m]
```

Rubi steps

$$\int (2+3x)^2 \left(30+31x-12x^2\right)^2 \sqrt{6+17x+12x^2}\, dx = \int (10-3x)^2 \left(6+17x+12x^2\right)^{5/2}\, dx$$
$$= -\frac{1}{32}(10-3x)\left(6+17x+12x^2\right)^{7/2} + \frac{1}{96}\int \Big(11331$$
$$= -\frac{873(6+17x+12x^2)^{7/2}}{1792} - \frac{1}{32}(10-3x)\left(6+17x+\right.$$
$$= \frac{25091(17+24x)\left(6+17x+12x^2\right)^{5/2}}{24576} - \frac{873(6+17}{17}$$
$$= -\frac{125455(17+24x)\left(6+17x+12x^2\right)^{3/2}}{4718592} + \frac{25091(1}{}$$
$$= \frac{125455(17+24x)\sqrt{6+17x+12x^2}}{150994944} - \frac{125455(17+}{}$$
$$= \frac{125455(17+24x)\sqrt{6+17x+12x^2}}{150994944} - \frac{125455(17+}{}$$
$$= \frac{125455(17+24x)\sqrt{6+17x+12x^2}}{150994944} - \frac{125455(17+}{}$$

**Mathematica [A]**
time = 0.50, size = 89, normalized size = 0.60

$$\frac{6\sqrt{6+17x+12x^2}\,(474999091769+3132157281976x+7899203409792x^2+8974844476416x^3+3438453030912x^4-1190083166208x^5-732816211968x^6+171228266496x^7)-878185\sqrt{3}\,\tanh^{-1}\left(\frac{2\sqrt{2+\frac{17x}{3}+4x^2}}{3+4x}\right)}{6341787648}$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + 3*x)^2*(30 + 31*x - 12*x^2)^2*Sqrt[6 + 17*x + 12*x^2],x]
```

```
[Out] (6*Sqrt[6 + 17*x + 12*x^2]*(474999091769 + 3132157281976*x + 7899203409792*
x^2 + 8974844476416*x^3 + 3438453030912*x^4 - 1190083166208*x^5 - 732816211
968*x^6 + 171228266496*x^7) - 878185*Sqrt[3]*ArcTanh[(2*Sqrt[2 + (17*x)/3 +
 4*x^2])/(3 + 4*x)])/6341787648
```

**Maple [A]**
time = 0.14, size = 147, normalized size = 0.99

| method | result |
|---|---|
| risch | $\frac{(171228266496x^7-732816211968x^6-1190083166208x^5+3438453030912x^4+8974844476416x^3+7899203409792x^2+3132157281976x-}{1056964608}$ |

| | |
|---|---|
| trager | $\left(162x^7 - \frac{19413}{28}x^6 - \frac{504423}{448}x^5 + \frac{11659251}{3584}x^4 + \frac{139118993}{16384}x^3 + \frac{20570842213}{2752512}x^2 + \frac{391519660247}{132120576}x + \frac{47499909176}{1056964608}\right.$ |
| default | $\frac{129220757x(12x^2+17x+6)^{\frac{3}{2}}}{458752} - \frac{125455\ln\left(\frac{\left(\frac{17}{2}+12x\right)\sqrt{12}}{12}+\sqrt{12x^2+17x+6}\right)\sqrt{12}}{3623878656} + \frac{2473875847(12x^2+17x+6)^{\frac{3}{2}}}{33030144}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2+3*x)^2*(-12*x^2+31*x+30)^2*(12*x^2+17*x+6)^(1/2),x,method=_RETURNVER
BOSE)
```

```
[Out] 129220757/458752*x*(12*x^2+17*x+6)^(3/2)-125455/3623878656*ln(1/12*(17/2+12
*x)*12^(1/2)+(12*x^2+17*x+6)^(1/2))*12^(1/2)+2473875847/33030144*(12*x^2+17
*x+6)^(3/2)+27/2*x^5*(12*x^2+17*x+6)^(3/2)-8613/112*x^4*(12*x^2+17*x+6)^(3/
2)+125455/150994944*(17+24*x)*(12*x^2+17*x+6)^(1/2)+14991/1792*x^3*(12*x^2+
17*x+6)^(3/2)+4267751/14336*x^2*(12*x^2+17*x+6)^(3/2)
```

**Maxima [A]**
time = 0.51, size = 155, normalized size = 1.04

$$\frac{27}{2}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} x^{5} - \frac{8613}{112}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} x^{4} + \frac{14991}{1792}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} x^{3} + \frac{4267751}{14336}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} x^{2} + \frac{129220757}{458752}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} x + \frac{2473875847}{33030144}\,{\left(12\,x^{2} + 17\,x + 6\right)}^{\frac{3}{2}} + \frac{125455}{6291456}\,\sqrt{12\,x^{2} + 17\,x + 6}\,x - \frac{125455}{1811939328}\,\sqrt{3}\log\left(4\,\sqrt{3}\,\sqrt{12\,x^{2} + 17\,x + 6} + 24\,x + 17\right) + \frac{2132735}{150994944}\,\sqrt{12\,x^{2} + 17\,x + 6}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)^2*(-12*x^2+31*x+30)^2*(12*x^2+17*x+6)^(1/2),x, algorithm=
"maxima")
```

```
[Out] 27/2*(12*x^2 + 17*x + 6)^(3/2)*x^5 - 8613/112*(12*x^2 + 17*x + 6)^(3/2)*x^4
 + 14991/1792*(12*x^2 + 17*x + 6)^(3/2)*x^3 + 4267751/14336*(12*x^2 + 17*x
+ 6)^(3/2)*x^2 + 129220757/458752*(12*x^2 + 17*x + 6)^(3/2)*x + 2473875847/
33030144*(12*x^2 + 17*x + 6)^(3/2) + 125455/6291456*sqrt(12*x^2 + 17*x + 6)
*x - 125455/1811939328*sqrt(3)*log(4*sqrt(3)*sqrt(12*x^2 + 17*x + 6) + 24*x
 + 17) + 2132735/150994944*sqrt(12*x^2 + 17*x + 6)
```

**Fricas [A]**
time = 0.38, size = 88, normalized size = 0.59

$$\frac{1}{1056964608}\,{\left(171228266496\,x^{7} - 732816211968\,x^{6} - 1190083166208\,x^{5} + 3438453030912\,x^{4} + 8974844476416\,x^{3} + 7899203409792\,x^{2} + 3132157281976\,x + 474999091769\right)}\sqrt{12\,x^{2} + 17\,x + 6} + \frac{125455}{3623878656}\,\sqrt{3}\log\left(-8\,\sqrt{3}\,\sqrt{12\,x^{2} + 17\,x + 6}\,{\left(24\,x + 17\right)} + 1152\,x^{2} + 1632\,x + 577\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)^2*(-12*x^2+31*x+30)^2*(12*x^2+17*x+6)^(1/2),x, algorithm=
"fricas")
```

```
[Out] 1/1056964608*(171228266496*x^7 - 732816211968*x^6 - 1190083166208*x^5 + 343
8453030912*x^4 + 8974844476416*x^3 + 7899203409792*x^2 + 3132157281976*x +
474999091769)*sqrt(12*x^2 + 17*x + 6) + 125455/3623878656*sqrt(3)*log(-8*sq
rt(3)*sqrt(12*x^2 + 17*x + 6)*(24*x + 17) + 1152*x^2 + 1632*x + 577)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \sqrt{(3x+2)\,(4x+3)}\,(3x-10)^2\,(3x+2)^2\,(4x+3)^2\;dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)**2*(-12*x**2+31*x+30)**2*(12*x**2+17*x+6)**(1/2),x)
```

```
[Out] Integral(sqrt((3*x + 2)*(4*x + 3))*(3*x - 10)**2*(3*x + 2)**2*(4*x + 3)**2,
 x)
```

**Giac** [A]
time = 3.38, size = 85, normalized size = 0.57

$$\frac{1}{1056964608}\left(8\left(48\left(24\left(96\left(24\left(48\left(168\,x-719\right)x-56047\right)x+3886417\right)x+973832951\right)x+20570842213\right)x+391519660247\right)x+474999091769\right)\sqrt{12\,x^2+17\,x+6}+\frac{125455}{1811939328}\sqrt{3}\,\log\left(\left|-4\,\sqrt{3}\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)-17\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)^2*(-12*x^2+31*x+30)^2*(12*x^2+17*x+6)^(1/2),x, algorithm=
"giac")
```

```
[Out] 1/1056964608*(8*(48*(24*(96*(24*(48*(168*x - 719)*x - 56047)*x + 3886417)*x
 + 973832951)*x + 20570842213)*x + 391519660247)*x + 474999091769)*sqrt(12*
x^2 + 17*x + 6) + 125455/1811939328*sqrt(3)*log(abs(-4*sqrt(3)*(2*sqrt(3)*x
 - sqrt(12*x^2 + 17*x + 6)) - 17))
```

**Mupad** [B]
time = 5.08, size = 187, normalized size = 1.26

$$\frac{4267751\,x^2\,{\left(12\,x^2+17\,x+6\right)}^{3/2}}{14336}+\frac{14991\,x^3\,{\left(12\,x^2+17\,x+6\right)}^{3/2}}{1792}-\frac{8613\,x^4\,{\left(12\,x^2+17\,x+6\right)}^{3/2}}{112}+\frac{27\,x^5\,{\left(12\,x^2+17\,x+6\right)}^{3/2}}{2}-\frac{146030443\,\sqrt{12}\,\ln\left(\sqrt{12\,x^2+17\,x+6}+\frac{\sqrt{12}\,\left(12\,x+\frac{17}{2}\right)}{12}\right)}{88080384}+\frac{438091329\,\left(\frac{x}{2}+\frac{17}{48}\right)\,\sqrt{12\,x^2+17\,x+6}}{229376}+\frac{2473875847\,\sqrt{12\,x^2+17\,x+6}\,\left(1152\,x^2+408\,x-291\right)}{3170893824}+\frac{129220757\,x\,{\left(12\,x^2+17\,x+6\right)}^{3/2}}{458752}+\frac{42055889399\,\sqrt{12}\,\ln\left(2\,\sqrt{12\,x^2+17\,x+6}+\frac{\sqrt{12}\,\left(24\,x+17\right)}{12}\right)}{25367150592}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((3*x + 2)^2*(17*x + 12*x^2 + 6)^(1/2)*(31*x - 12*x^2 + 30)^2,x)
```

```
[Out] (4267751*x^2*(17*x + 12*x^2 + 6)^(3/2))/14336 + (14991*x^3*(17*x + 12*x^2 +
 6)^(3/2))/1792 - (8613*x^4*(17*x + 12*x^2 + 6)^(3/2))/112 + (27*x^5*(17*x
+ 12*x^2 + 6)^(3/2))/2 - (146030443*12^(1/2)*log((17*x + 12*x^2 + 6)^(1/2)
+ (12^(1/2)*(12*x + 17/2))/12))/88080384 + (438091329*(x/2 + 17/48)*(17*x +
 12*x^2 + 6)^(1/2))/229376 + (2473875847*(17*x + 12*x^2 + 6)^(1/2)*(408*x +
 1152*x^2 - 291))/3170893824 + (129220757*x*(17*x + 12*x^2 + 6)^(3/2))/4587
52 + (42055889399*12^(1/2)*log(2*(17*x + 12*x^2 + 6)^(1/2) + (12^(1/2)*(24*
x + 17))/12))/25367150592
```

# 3.134 $\int (2+3x)\left(30+31x-12x^2\right)\sqrt{6+17x+12x^2}\,dx$

**Optimal**. Leaf size=103

$$-\frac{97(17+24x)\sqrt{6+17x+12x^2}}{24576}+\frac{97}{768}(17+24x)\left(6+17x+12x^2\right)^{3/2}-\frac{1}{20}\left(6+17x+12x^2\right)^{5/2}+\frac{97\tanh^{-1}}{}$$

```
[Out] 97/768*(17+24*x)*(12*x^2+17*x+6)^(3/2)-1/20*(12*x^2+17*x+6)^(5/2)+97/294912
*arctanh(1/12*(17+24*x)*3^(1/2)/(12*x^2+17*x+6)^(1/2))*3^(1/2)-97/24576*(17
+24*x)*(12*x^2+17*x+6)^(1/2)
```

**Rubi** [A]

time = 0.03, antiderivative size = 103, normalized size of antiderivative = 1.00, number of steps used = 6, number of rules used = 5, integrand size = 30, $\frac{\text{number of rules}}{\text{integrand size}} = 0.167$, Rules used = {1016, 654, 626, 635, 212}

$$-\frac{1}{20}\left(12x^2+17x+6\right)^{5/2}+\frac{97}{768}(24x+17)\left(12x^2+17x+6\right)^{3/2}-\frac{97(24x+17)\sqrt{12x^2+17x+6}}{24576}+\frac{97\tanh^{-1}\left(\frac{24x+17}{4\sqrt{3}\sqrt{12x^2+17x+6}}\right)}{98304\sqrt{3}}$$

Antiderivative was successfully verified.

```
[In] Int[(2 + 3*x)*(30 + 31*x - 12*x^2)*Sqrt[6 + 17*x + 12*x^2],x]
```

```
[Out] (-97*(17 + 24*x)*Sqrt[6 + 17*x + 12*x^2])/24576 + (97*(17 + 24*x)*(6 + 17*x
 + 12*x^2)^(3/2))/768 - (6 + 17*x + 12*x^2)^(5/2)/20 + (97*ArcTanh[(17 + 24
*x)/(4*Sqrt[3]*Sqrt[6 + 17*x + 12*x^2])])/(98304*Sqrt[3])
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 626

```
Int[((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(b + 2*c*x)
*((a + b*x + c*x^2)^p/(2*c*(2*p + 1))), x] - Dist[p*((b^2 - 4*a*c)/(2*c*(2*
p + 1))), Int[(a + b*x + c*x^2)^(p - 1), x], x] /; FreeQ[{a, b, c}, x] && N
eQ[b^2 - 4*a*c, 0] && GtQ[p, 0] && IntegerQ[4*p]
```

Rule 635

```
Int[1/Sqrt[(a_) + (b_.)*(x_) + (c_.)*(x_)^2], x_Symbol] :> Dist[2, Subst[In
t[1/(4*c - x^2), x], x, (b + 2*c*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a,
 b, c}, x] && NeQ[b^2 - 4*a*c, 0]
```

Rule 654

```
Int[((d_.) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_Symbol
] :> Simp[e*((a + b*x + c*x^2)^(p + 1)/(2*c*(p + 1))), x] + Dist[(2*c*d - b
*e)/(2*c), Int[(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, p}, x]
&& NeQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 1016

```
Int[((g_) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.
) + (e_.)*(x_) + (f_.)*(x_)^2)^(m_.), x_Symbol] :> Int[(d*(g/a) + f*h*(x/c)
)^m*(a + b*x + c*x^2)^(m + p), x] /; FreeQ[{a, b, c, d, e, f, g, h, p}, x]
&& EqQ[c*g^2 - b*g*h + a*h^2, 0] && EqQ[c^2*d*g^2 - a*c*e*g*h + a^2*f*h^2,
0] && IntegerQ[m]
```

Rubi steps

$$\begin{aligned}\int (2+3x)\left(30+31x-12x^2\right)\sqrt{6+17x+12x^2}\,dx &= \int (10-3x)\left(6+17x+12x^2\right)^{3/2}\,dx\\
&= -\frac{1}{20}\left(6+17x+12x^2\right)^{5/2}+\frac{97}{8}\int \left(6+17x+12x^2\right)^{3/2}\\
&= \frac{97}{768}(17+24x)\left(6+17x+12x^2\right)^{3/2}-\frac{1}{20}\left(6+17x+1\right.\\
&= -\frac{97(17+24x)\sqrt{6+17x+12x^2}}{24576}+\frac{97}{768}(17+24x)\left(6\right.\\
&= -\frac{97(17+24x)\sqrt{6+17x+12x^2}}{24576}+\frac{97}{768}(17+24x)\left(6\right.\\
&= -\frac{97(17+24x)\sqrt{6+17x+12x^2}}{24576}+\frac{97}{768}(17+24x)\left(6\right.\end{aligned}$$

**Mathematica [A]**
time = 0.27, size = 74, normalized size = 0.72

$$\frac{6\sqrt{6+17x+12x^2}\left(1353611+5455144x+6837888x^2+1963008x^3-884736x^4\right)+485\sqrt{3}\tanh^{-1}\left(\frac{2\sqrt{2+\frac{17x}{3}+4x^2}}{3+4x}\right)}{737280}$$

Antiderivative was successfully verified.

```
[In] Integrate[(2 + 3*x)*(30 + 31*x - 12*x^2)*Sqrt[6 + 17*x + 12*x^2],x]
```

```
[Out] (6*Sqrt[6 + 17*x + 12*x^2]*(1353611 + 5455144*x + 6837888*x^2 + 1963008*x^3
 - 884736*x^4) + 485*Sqrt[3]*ArcTanh[(2*Sqrt[2 + (17*x)/3 + 4*x^2])/(3 + 4*
x)])/737280
```

**Maple [A]**
time = 0.09, size = 96, normalized size = 0.93

| method | result |
|---|---|
| risch | $-\frac{(884736x^4-1963008x^3-6837888x^2-5455144x-1353611)\sqrt{12x^2+17x+6}}{122880}+\frac{97\ln\left(\frac{(\frac{17}{2}+12x)\sqrt{12}}{12}+\sqrt{12x^2+1}\right.}{589824}$ |
| trager | $\left(-\frac{36}{5}x^4+\frac{639}{40}x^3+\frac{17807}{320}x^2+\frac{681893}{15360}x+\frac{1353611}{122880}\right)\sqrt{12x^2+17x+6}+\frac{97\,\mathrm{RootOf}\left(\_Z^2-3\right)\ln\left(24\,\mathrm{RootOf}\left(\right.\right.}{}$ |
| default | $-\frac{3x^2(12x^2+17x+6)^{\frac{3}{2}}}{5}+\frac{349x(12x^2+17x+6)^{\frac{3}{2}}}{160}+\frac{7093(12x^2+17x+6)^{\frac{3}{2}}}{3840}-\frac{97(17+24x)\sqrt{12x^2+17x+6}}{24576}+\frac{97\ln\left(\right.}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2+3*x)*(-12*x^2+31*x+30)*(12*x^2+17*x+6)^(1/2),x,method=_RETURNVERBOSE
)
```

```
[Out] -3/5*x^2*(12*x^2+17*x+6)^(3/2)+349/160*x*(12*x^2+17*x+6)^(3/2)+7093/3840*(1
2*x^2+17*x+6)^(3/2)-97/24576*(17+24*x)*(12*x^2+17*x+6)^(1/2)+97/589824*ln(1
/12*(17/2+12*x)*12^(1/2)+(12*x^2+17*x+6)^(1/2))*12^(1/2)
```

**Maxima [A]**
time = 0.49, size = 104, normalized size = 1.01

$$-\frac{3}{5}\,(12\,x^2+17\,x+6)^{\frac{3}{2}}x^2+\frac{349}{160}\,(12\,x^2+17\,x+6)^{\frac{3}{2}}x+\frac{7093}{3840}\,(12\,x^2+17\,x+6)^{\frac{3}{2}}-\frac{97}{1024}\,\sqrt{12\,x^2+17\,x+6}\,x+\frac{97}{294912}\,\sqrt{3}\,\log\left(4\,\sqrt{3}\,\sqrt{12\,x^2+17\,x+6}+24\,x+17\right)-\frac{1649}{24576}\,\sqrt{12\,x^2+17\,x+6}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)*(-12*x^2+31*x+30)*(12*x^2+17*x+6)^(1/2),x, algorithm="max
ima")
```

```
[Out] -3/5*(12*x^2 + 17*x + 6)^(3/2)*x^2 + 349/160*(12*x^2 + 17*x + 6)^(3/2)*x +
7093/3840*(12*x^2 + 17*x + 6)^(3/2) - 97/1024*sqrt(12*x^2 + 17*x + 6)*x + 9
7/294912*sqrt(3)*log(4*sqrt(3)*sqrt(12*x^2 + 17*x + 6) + 24*x + 17) - 1649/
24576*sqrt(12*x^2 + 17*x + 6)
```

**Fricas [A]**
time = 0.44, size = 73, normalized size = 0.71

$$-\frac{1}{122880}\,\left(884736\,x^4-1963008\,x^3-6837888\,x^2-5455144\,x-1353611\right)\sqrt{12\,x^2+17\,x+6}+\frac{97}{589824}\,\sqrt{3}\,\log\left(8\,\sqrt{3}\,\sqrt{12\,x^2+17\,x+6}\,(24\,x+17)+1152\,x^2+1632\,x+577\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)*(-12*x^2+31*x+30)*(12*x^2+17*x+6)^(1/2),x, algorithm="fri
cas")
```

```
[Out] -1/122880*(884736*x^4 - 1963008*x^3 - 6837888*x^2 - 5455144*x - 1353611)*sq
rt(12*x^2 + 17*x + 6) + 97/589824*sqrt(3)*log(8*sqrt(3)*sqrt(12*x^2 + 17*x
+ 6)*(24*x + 17) + 1152*x^2 + 1632*x + 577)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int\left(-152x\sqrt{12x^2+17x+6}\right)dx-\int\left(-69x^2\sqrt{12x^2+17x+6}\right)dx-\int 36x^3\sqrt{12x^2+17x+6}\,dx-\int\left(-60\sqrt{12x^2+17x+6}\right)dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)*(-12*x**2+31*x+30)*(12*x**2+17*x+6)**(1/2),x)
```

```
[Out] -Integral(-152*x*sqrt(12*x**2 + 17*x + 6), x) - Integral(-69*x**2*sqrt(12*x
**2 + 17*x + 6), x) - Integral(36*x**3*sqrt(12*x**2 + 17*x + 6), x) - Integ
ral(-60*sqrt(12*x**2 + 17*x + 6), x)
```

**Giac [A]**
time = 3.69, size = 70, normalized size = 0.68

$$-\frac{1}{122880}\left(8\left(48\left(72\left(32x-71\right)x-17807\right)x-681893\right)x-1353611\right)\sqrt{12x^2+17x+6}-\frac{97}{294912}\sqrt{3}\log\left(\left|-4\sqrt{3}\left(2\sqrt{3}x-\sqrt{12x^2+17x+6}\right)-17\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((2+3*x)*(-12*x^2+31*x+30)*(12*x^2+17*x+6)^(1/2),x, algorithm="gia
c")
```

```
[Out] -1/122880*(8*(48*(72*(32*x - 71)*x - 17807)*x - 681893)*x - 1353611)*sqrt(1
2*x^2 + 17*x + 6) - 97/294912*sqrt(3)*log(abs(-4*sqrt(3)*(2*sqrt(3)*x - sqr
t(12*x^2 + 17*x + 6)) - 17))
```

**Mupad [B]**
time = 4.69, size = 136, normalized size = 1.32

$$\frac{3753\left(\frac{x}{2}+\frac{17}{48}\right)\sqrt{12x^2+17x+6}}{80}-\frac{417\sqrt{12}\ln\left(\sqrt{12x^2+17x+6}+\frac{\sqrt{12}\left(12x+\frac{17}{2}\right)}{12}\right)}{10240}-\frac{3x^2\left(12x^2+17x+6\right)^{3/2}}{5}+\frac{7093\sqrt{12x^2+17x+6}\left(1152x^2+408x-291\right)}{368640}+\frac{349x\left(12x^2+17x+6\right)^{3/2}}{160}+\frac{120581\sqrt{12}\ln\left(2\sqrt{12x^2+17x+6}+\frac{\sqrt{12}\left(24x+17\right)}{12}\right)}{2949120}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((3*x + 2)*(17*x + 12*x^2 + 6)^(1/2)*(31*x - 12*x^2 + 30),x)
```

```
[Out] (3753*(x/2 + 17/48)*(17*x + 12*x^2 + 6)^(1/2))/80 - (417*12^(1/2)*log((17*x
 + 12*x^2 + 6)^(1/2) + (12^(1/2)*(12*x + 17/2))/12))/10240 - (3*x^2*(17*x +
 12*x^2 + 6)^(3/2))/5 + (7093*(17*x + 12*x^2 + 6)^(1/2)*(408*x + 1152*x^2 -
 291))/368640 + (349*x*(17*x + 12*x^2 + 6)^(3/2))/160 + (120581*12^(1/2)*lo
g(2*(17*x + 12*x^2 + 6)^(1/2) + (12^(1/2)*(24*x + 17))/12))/2949120
```

## 3.135 $\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)(30+31x-12x^2)}\,dx$

**Optimal**. Leaf size=28

$$\frac{1}{42}\tanh^{-1}\left(\frac{206+291x}{84\sqrt{6+17x+12x^2}}\right)$$

[Out] 1/42*arctanh(1/84*(206+291*x)/(12*x^2+17*x+6)^(1/2))

**Rubi** [A]
time = 0.03, antiderivative size = 28, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}}$ = 0.088, Rules used = {1016, 738, 212}

$$\frac{1}{42}\tanh^{-1}\left(\frac{291x+206}{84\sqrt{12x^2+17x+6}}\right)$$

Antiderivative was successfully verified.

[In] Int[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)*(30 + 31*x - 12*x^2)),x]

[Out] ArcTanh[(206 + 291*x)/(84*Sqrt[6 + 17*x + 12*x^2])]/42

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 1016

```
Int[((g_) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.
) + (e_.)*(x_) + (f_.)*(x_)^2)^(m_.), x_Symbol] :> Int[(d*(g/a) + f*h*(x/c)
)^m*(a + b*x + c*x^2)^(m + p), x] /; FreeQ[{a, b, c, d, e, f, g, h, p}, x]
&& EqQ[c*g^2 - b*g*h + a*h^2, 0] && EqQ[c^2*d*g^2 - a*c*e*g*h + a^2*f*h^2,
0] && IntegerQ[m]
```

Rubi steps

$$\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)\left(30+31x-12x^2\right)}\,dx = \int \frac{1}{(10-3x)\sqrt{6+17x+12x^2}}\,dx$$
$$= -\left(2\text{Subst}\left(\int \frac{1}{7056-x^2}\,dx, x, \frac{-206-291x}{\sqrt{6+17x+12x^2}}\right)\right)$$
$$= \frac{1}{42}\tanh^{-1}\left(\frac{206+291x}{84\sqrt{6+17x+12x^2}}\right)$$

**Mathematica** [A]
time = 0.15, size = 30, normalized size = 1.07

$$\frac{1}{21}\tanh^{-1}\left(\frac{6\sqrt{6+17x+12x^2}}{7(2+3x)}\right)$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)*(30 + 31*x - 12*x^2)),x]
```

```
[Out] ArcTanh[(6*Sqrt[6 + 17*x + 12*x^2])/(7*(2 + 3*x))]/21
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 162 vs. $2(22) = 44$.
time = 0.13, size = 163, normalized size = 5.82

| method | result |
|---|---|
| trager | $\frac{\ln\left(-\frac{206+291x+84\sqrt{12x^2+17x+6}}{3x-10}\right)}{42}$ |
| default | $-\frac{\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}}{588} - \frac{97\ln\left(\frac{\left(\frac{17}{2}+12x\right)\sqrt{12}}{12}+\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}\right)\sqrt{12}}{14112} + \frac{\text{arctanh}}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((12*x^2+17*x+6)^(1/2)/(2+3*x)/(-12*x^2+31*x+30),x,method=_RETURNVERBOSE
)
```

```
[Out] -1/588*(12*(x-10/3)^2+97*x-382/3)^(1/2)-97/14112*ln(1/12*(17/2+12*x)*12^(1/
2)+(12*(x-10/3)^2+97*x-382/3)^(1/2))*12^(1/2)+1/42*arctanh(1/28*(206/3+97*x
)/(12*(x-10/3)^2+97*x-382/3)^(1/2))-4/49*(12*(x+3/4)^2-x-3/4)^(1/2)+1/294*l
n(1/12*(17/2+12*x)*12^(1/2)+(12*(x+3/4)^2-x-3/4)^(1/2))*12^(1/2)+1/12*(12*(
x+2/3)^2+x+2/3)^(1/2)+1/288*ln(1/12*(17/2+12*x)*12^(1/2)+(12*(x+2/3)^2+x+2/
3)^(1/2))*12^(1/2)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)/(-12*x^2+31*x+30),x, algorithm="max
ima")
```

```
[Out] -integrate(sqrt(12*x^2 + 17*x + 6)/((12*x^2 - 31*x - 30)*(3*x + 2)), x)
```

**Fricas [B]** Leaf count of result is larger than twice the leaf count of optimal. 53 vs. $2(22) = 44$.
time = 0.37, size = 53, normalized size = 1.89

$$\frac{1}{84}\log\left(\frac{291\,x+84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)-\frac{1}{84}\log\left(\frac{291\,x-84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)/(-12*x^2+31*x+30),x, algorithm="fri
cas")
```

```
[Out] 1/84*log((291*x + 84*sqrt(12*x^2 + 17*x + 6) + 206)/x) - 1/84*log((291*x -
84*sqrt(12*x^2 + 17*x + 6) + 206)/x)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int\frac{\sqrt{12x^2+17x+6}}{36x^3-69x^2-152x-60}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x**2+17*x+6)**(1/2)/(2+3*x)/(-12*x**2+31*x+30),x)
```

```
[Out] -Integral(sqrt(12*x**2 + 17*x + 6)/(36*x**3 - 69*x**2 - 152*x - 60), x)
```

**Giac [B]** Leaf count of result is larger than twice the leaf count of optimal. 63 vs. $2(22) = 44$.
time = 5.63, size = 63, normalized size = 2.25

$$\frac{1}{42}\log\left(\left|-6\,\sqrt{3}\,x+20\,\sqrt{3}+3\,\sqrt{12\,x^2+17\,x+6}+42\right|\right)-\frac{1}{42}\log\left(\left|-6\,\sqrt{3}\,x+20\,\sqrt{3}+3\,\sqrt{12\,x^2+17\,x+6}-42\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)/(-12*x^2+31*x+30),x, algorithm="gia
c")
```

```
[Out] 1/42*log(abs(-6*sqrt(3)*x + 20*sqrt(3) + 3*sqrt(12*x^2 + 17*x + 6) + 42)) -
 1/42*log(abs(-6*sqrt(3)*x + 20*sqrt(3) + 3*sqrt(12*x^2 + 17*x + 6) - 42))
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.04

$$\int \frac{\sqrt{12\,x^2+17\,x+6}}{(3\,x+2)\,\left(-12\,x^2+31\,x+30\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)*(31*x - 12*x^2 + 30)),x)
[Out] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)*(31*x - 12*x^2 + 30)), x)
```

## 3.136 $\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)^2(30+31x-12x^2)^2}\,dx$

**Optimal**. Leaf size=84

$$-\frac{275+388x}{98(10-3x)\sqrt{6+17x+12x^2}}+\frac{3137\sqrt{6+17x+12x^2}}{38416(10-3x)}+\frac{97\tanh^{-1}\left(\frac{206+291x}{84\sqrt{6+17x+12x^2}}\right)}{3226944}$$

```
[Out] 97/3226944*arctanh(1/84*(206+291*x)/(12*x^2+17*x+6)^(1/2))+1/98*(-275-388*x
)/(10-3*x)/(12*x^2+17*x+6)^(1/2)+3137/38416*(12*x^2+17*x+6)^(1/2)/(10-3*x)
```

**Rubi** [A]
time = 0.05, antiderivative size = 84, normalized size of antiderivative = 1.00, number of steps used = 5, number of rules used = 5, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}} = 0.147$, Rules used = {1016, 754, 820, 738, 212}

$$-\frac{388x+275}{98(10-3x)\sqrt{12x^2+17x+6}}+\frac{3137\sqrt{12x^2+17x+6}}{38416(10-3x)}+\frac{97\tanh^{-1}\left(\frac{291x+206}{84\sqrt{12x^2+17x+6}}\right)}{3226944}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)^2*(30 + 31*x - 12*x^2)^2),x]
```

```
[Out] -1/98*(275 + 388*x)/((10 - 3*x)*Sqrt[6 + 17*x + 12*x^2]) + (3137*Sqrt[6 + 1
7*x + 12*x^2])/(38416*(10 - 3*x)) + (97*ArcTanh[(206 + 291*x)/(84*Sqrt[6 +
17*x + 12*x^2])])/3226944
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 754

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*(b*c*d - b^2*e + 2*a*c*e + c*(2*c*d - b*e)
*x)*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^
```

```
2))), x] + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d +
 e*x)^m*Simp[b*c*d*e*(2*p - m + 2) + b^2*e^2*(m + p + 2) - 2*c^2*d^2*(2*p +
 3) - 2*a*c*e^2*(m + 2*p + 3) - c*e*(2*c*d - b*e)*(m + 2*p + 4)*x, x]*(a +
b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4
*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p,
-1] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 820

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(e*f - d*g))*(d + e*x)^(m + 1)*((a +
 b*x + c*x^2)^(p + 1)/(2*(p + 1)*(c*d^2 - b*d*e + a*e^2))), x] - Dist[(b*(e
*f + d*g) - 2*(c*d*f + a*e*g))/(2*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(
m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && EqQ[Simplify[m
+ 2*p + 3], 0]
```

Rule 1016

```
Int[((g_) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.
) + (e_.)*(x_) + (f_.)*(x_)^2)^(m_.), x_Symbol] :> Int[(d*(g/a) + f*h*(x/c)
)^m*(a + b*x + c*x^2)^(m + p), x] /; FreeQ[{a, b, c, d, e, f, g, h, p}, x]
&& EqQ[c*g^2 - b*g*h + a*h^2, 0] && EqQ[c^2*d*g^2 - a*c*e*g*h + a^2*f*h^2,
0] && IntegerQ[m]
```

Rubi steps

$$\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)^2\left(30+31x-12x^2\right)^2}\,dx = \int \frac{1}{(10-3x)^2\left(6+17x+12x^2\right)^{3/2}}\,dx$$

$$= -\frac{275+388x}{98(10-3x)\sqrt{6+17x+12x^2}} - \frac{1}{882}\int \frac{-\frac{14859}{2}-10476x}{(10-3x)^2\sqrt{6+17x+12x}}$$

$$= -\frac{275+388x}{98(10-3x)\sqrt{6+17x+12x^2}} + \frac{3137\sqrt{6+17x+12x^2}}{38416(10-3x)} + \frac{97\int \frac{}{(10-}}{}$$

$$= -\frac{275+388x}{98(10-3x)\sqrt{6+17x+12x^2}} + \frac{3137\sqrt{6+17x+12x^2}}{38416(10-3x)} - \frac{97\text{Subst}}{}$$

$$= -\frac{275+388x}{98(10-3x)\sqrt{6+17x+12x^2}} + \frac{3137\sqrt{6+17x+12x^2}}{38416(10-3x)} + \frac{97\tanh}{}$$

**Mathematica** [**A**]
time = 0.29, size = 80, normalized size = 0.95

$$\frac{(88978+98767x-37644x^2)\sqrt{6+17x+12x^2}}{38416(-10+3x)(2+3x)(3+4x)}+\frac{97\tanh^{-1}\left(\frac{6\sqrt{6+17x+12x^2}}{7(2+3x)}\right)}{1613472}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)^2*(30 + 31*x - 12*x^2)^2),x]
```

```
[Out] ((88978 + 98767*x - 37644*x^2)*Sqrt[6 + 17*x + 12*x^2])/(38416*(-10 + 3*x)*
(2 + 3*x)*(3 + 4*x)) + (97*ArcTanh[(6*Sqrt[6 + 17*x + 12*x^2])/(7*(2 + 3*x)
)])/1613472
```

**Maple** [**B**] Leaf count of result is larger than twice the leaf count of optimal. 244 vs. $2(70) = 140$.
time = 0.12, size = 245, normalized size = 2.92

| method | result |
|---|---|
| risch | $-\frac{37644x^2-98767x-88978}{38416(3x-10)\sqrt{12x^2+17x+6}}+\frac{97\,\mathrm{arctanh}\left(\frac{\frac{206}{3}+97x}{28\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}}\right)}{3226944}$ |
| trager | $-\frac{(37644x^2-98767x-88978)\sqrt{12x^2+17x+6}}{38416(36x^3-69x^2-152x-60)}-\frac{97\ln\left(-\frac{84\sqrt{12x^2+17x+6}-206-291x}{3x-10}\right)}{3226944}$ |
| default | $-\frac{97\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}}{45177216}-\frac{7057\ln\left(\frac{\left(\frac{17}{2}+12x\right)\sqrt{12}}{12}+\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}\right)\sqrt{12}}{813189888}+-$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((12*x^2+17*x+6)^(1/2)/(2+3*x)^2/(-12*x^2+31*x+30)^2,x,method=_RETURNVER
BOSE)
```

```
[Out] -97/45177216*(12*(x-10/3)^2+97*x-382/3)^(1/2)-7057/813189888*ln(1/12*(17/2+
12*x)*12^(1/2)+(12*(x-10/3)^2+97*x-382/3)^(1/2))*12^(1/2)+97/3226944*arctan
h(1/28*(206/3+97*x)/(12*(x-10/3)^2+97*x-382/3)^(1/2))-1/67765824/(x-10/3)*(
12*(x-10/3)^2+97*x-382/3)^(3/2)+1/135531648*(17+24*x)*(12*(x-10/3)^2+97*x-3
82/3)^(1/2)+32/2401/(x+3/4)^2*(12*(x+3/4)^2-x-3/4)^(3/2)+384/117649*(12*(x+
3/4)^2-x-3/4)^(1/2)-16/117649*ln(1/12*(17/2+12*x)*12^(1/2)+(12*(x+3/4)^2-x-
3/4)^(1/2))*12^(1/2)-1/72/(x+2/3)^2*(12*(x+2/3)^2+x+2/3)^(3/2)+1/288*(12*(x
+2/3)^2+x+2/3)^(1/2)+1/6912*ln(1/12*(17/2+12*x)*12^(1/2)+(12*(x+2/3)^2+x+2/
3)^(1/2))*12^(1/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^2/(-12*x^2+31*x+30)^2,x, algorithm=
"maxima")
```

```
[Out] integrate(sqrt(12*x^2 + 17*x + 6)/((12*x^2 - 31*x - 30)^2*(3*x + 2)^2), x)
```

**Fricas** [A]
time = 0.38, size = 126, normalized size = 1.50

$$\frac{97\,(36\,x^3-69\,x^2-152\,x-60)\log\left(\frac{291\,x+84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)-97\,(36\,x^3-69\,x^2-152\,x-60)\log\left(\frac{291\,x-84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)-168\,(37644\,x^2-98767\,x-88978)\sqrt{12\,x^2+17\,x+6}}{6453888\,(36\,x^3-69\,x^2-152\,x-60)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^2/(-12*x^2+31*x+30)^2,x, algorithm=
"fricas")
```

```
[Out] 1/6453888*(97*(36*x^3 - 69*x^2 - 152*x - 60)*log((291*x + 84*sqrt(12*x^2 +
17*x + 6) + 206)/x) - 97*(36*x^3 - 69*x^2 - 152*x - 60)*log((291*x - 84*sqr
t(12*x^2 + 17*x + 6) + 206)/x) - 168*(37644*x^2 - 98767*x - 88978)*sqrt(12*
x^2 + 17*x + 6))/(36*x^3 - 69*x^2 - 152*x - 60)
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{\sqrt{(3x+2)\,(4x+3)}}{(3x-10)^2\,(3x+2)^2\,(4x+3)^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x**2+17*x+6)**(1/2)/(2+3*x)**2/(-12*x**2+31*x+30)**2,x)
```

```
[Out] Integral(sqrt((3*x + 2)*(4*x + 3))/((3*x - 10)**2*(3*x + 2)**2*(4*x + 3)**2
), x)
```

**Giac** [B] Leaf count of result is larger than twice the leaf count of optimal. 159 vs. 2(70) = 140.
time = 4.42, size = 159, normalized size = 1.89

$$\frac{1}{9680832}\sqrt{3}\left(\sqrt{3}\left(175672\sqrt{3}+97\log\left(\frac{7\sqrt{3}-12}{7\sqrt{3}+12}\right)\right)\mathrm{sgn}\left(\frac{1}{3x+2}\right)-\left(97\sqrt{3}\log\left(\frac{\left|-28\sqrt{3}+24\sqrt{\frac{1}{3x+2}+4}\right|}{4\left(7\sqrt{3}+6\sqrt{\frac{1}{3x+2}+4}\right)}\right)+134456\sqrt{\frac{1}{3x+2}+4}+\frac{28\left(\frac{221183}{3x+2}-18436\right)}{12\left(\frac{1}{3x+2}+4\right)^{\frac{3}{2}}-49\sqrt{\frac{1}{3x+2}+4}}\right)\mathrm{sgn}\left(\frac{1}{3x+2}\right)\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^2/(-12*x^2+31*x+30)^2,x, algorithm=
"giac")
```

```
[Out] 1/9680832*sqrt(3)*(sqrt(3)*(175672*sqrt(3) + 97*log((7*sqrt(3) - 12)/(7*sqr
t(3) + 12)))*sgn(1/(3*x + 2)) - (97*sqrt(3)*log(1/4*abs(-28*sqrt(3) + 24*sq
rt(1/(3*x + 2) + 4))/(7*sqrt(3) + 6*sqrt(1/(3*x + 2) + 4))) + 134456*sqrt(1
/(3*x + 2) + 4) + 28*(221183/(3*x + 2) - 18436)/(12*(1/(3*x + 2) + 4)^(3/2)
 - 49*sqrt(1/(3*x + 2) + 4)))*sgn(1/(3*x + 2)))
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\sqrt{12\,x^2+17\,x+6}}{{\left(3\,x+2\right)}^2\,{\left(-12\,x^2+31\,x+30\right)}^2}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)^2*(31*x - 12*x^2 + 30)^2),x)
```

```
[Out] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)^2*(31*x - 12*x^2 + 30)^2), x)
```

## 3.137 $\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)^3\left(30+31x-12x^2\right)^3}\,dx$

**Optimal**. Leaf size=139

$$-\frac{275+388x}{294(10-3x)^2\left(6+17x+12x^2\right)^{3/2}}+\frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12x^2}}-\frac{50555899\sqrt{6+17x+12x^2}}{19361664(10-3x)^2}-\frac{163\ldots}{}$$

```
[Out] 1/294*(-275-388*x)/(10-3*x)^2/(12*x^2+17*x+6)^(3/2)+40325/637540872192*arct
anh(1/84*(206+291*x)/(12*x^2+17*x+6)^(1/2))+1/8232*(738029+1042556*x)/(10-3
*x)^2/(12*x^2+17*x+6)^(1/2)-50555899/19361664*(12*x^2+17*x+6)^(1/2)/(10-3*x
)^2-1634466587/7589772288*(12*x^2+17*x+6)^(1/2)/(10-3*x)
```

**Rubi** [A]
time = 0.08, antiderivative size = 139, normalized size of antiderivative = 1.00, number of steps used = 7, number of rules used = 7, integrand size = 34, $\frac{\text{number of rules}}{\text{integrand size}} = 0.206$, Rules used = {1016, 754, 836, 848, 820, 738, 212}

$$-\frac{388x+275}{294(10-3x)^2\left(12x^2+17x+6\right)^{3/2}}-\frac{1634466587\sqrt{12x^2+17x+6}}{7589772288(10-3x)}-\frac{50555899\sqrt{12x^2+17x+6}}{19361664(10-3x)^2}+\frac{1042556x+738029}{8232(10-3x)^2\sqrt{12x^2+17x+6}}+\frac{40325\tanh^{-1}\left(\frac{291x+206}{84\sqrt{12x^2+17x+6}}\right)}{637540872192}$$

Antiderivative was successfully verified.

```
[In] Int[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)^3*(30 + 31*x - 12*x^2)^3),x]
```

```
[Out] -1/294*(275 + 388*x)/((10 - 3*x)^2*(6 + 17*x + 12*x^2)^(3/2)) + (738029 + 1
042556*x)/(8232*(10 - 3*x)^2*Sqrt[6 + 17*x + 12*x^2]) - (50555899*Sqrt[6 +
17*x + 12*x^2])/(19361664*(10 - 3*x)^2) - (1634466587*Sqrt[6 + 17*x + 12*x^
2])/(7589772288*(10 - 3*x)) + (40325*ArcTanh[(206 + 291*x)/(84*Sqrt[6 + 17*
x + 12*x^2])])/637540872192
```

Rule 212

```
Int[((a_) + (b_.)*(x_)^2)^(-1), x_Symbol] :> Simp[(1/(Rt[a, 2]*Rt[-b, 2]))*
ArcTanh[Rt[-b, 2]*(x/Rt[a, 2])], x] /; FreeQ[{a, b}, x] && NegQ[a/b] && (Gt
Q[a, 0] || LtQ[b, 0])
```

Rule 738

```
Int[1/(((d_.) + (e_.)*(x_))*Sqrt[(a_.) + (b_.)*(x_) + (c_.)*(x_)^2]), x_Sym
bol] :> Dist[-2, Subst[Int[1/(4*c*d^2 - 4*b*d*e + 4*a*e^2 - x^2), x], x, (2
*a*e - b*d - (2*c*d - b*e)*x)/Sqrt[a + b*x + c*x^2]], x] /; FreeQ[{a, b, c,
 d, e}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[2*c*d - b*e, 0]
```

Rule 754

```
Int[((d_.) + (e_.)*(x_))^(m_)*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_), x_S
ymbol] :> Simp[(d + e*x)^(m + 1)*(b*c*d - b^2*e + 2*a*c*e + c*(2*c*d - b*e)
*x)*((a + b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^
2))), x] + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d +
 e*x)^m*Simp[b*c*d*e*(2*p - m + 2) + b^2*e^2*(m + p + 2) - 2*c^2*d^2*(2*p +
 3) - 2*a*c*e^2*(m + 2*p + 3) - c*e*(2*c*d - b*e)*(m + 2*p + 4)*x, x]*(a +
b*x + c*x^2)^(p + 1), x], x] /; FreeQ[{a, b, c, d, e, m}, x] && NeQ[b^2 - 4
*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && NeQ[2*c*d - b*e, 0] && LtQ[p,
-1] && IntQuadraticQ[a, b, c, d, e, m, p, x]
```

Rule 820

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(-(e*f - d*g))*(d + e*x)^(m + 1)*((a +
 b*x + c*x^2)^(p + 1)/(2*(p + 1)*(c*d^2 - b*d*e + a*e^2))), x] - Dist[(b*(e
*f + d*g) - 2*(c*d*f + a*e*g))/(2*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(
m + 1)*(a + b*x + c*x^2)^p, x], x] /; FreeQ[{a, b, c, d, e, f, g, m, p}, x]
 && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && EqQ[Simplify[m
+ 2*p + 3], 0]
```

Rule 836

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_), x_Symbol] :> Simp[(d + e*x)^(m + 1)*(f*(b*c*d - b^2*e + 2
*a*c*e) - a*g*(2*c*d - b*e) + c*(f*(2*c*d - b*e) - g*(b*d - 2*a*e))*x)*((a
+ b*x + c*x^2)^(p + 1)/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2))), x]
 + Dist[1/((p + 1)*(b^2 - 4*a*c)*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^m*
(a + b*x + c*x^2)^(p + 1)*Simp[f*(b*c*d*e*(2*p - m + 2) + b^2*e^2*(p + m +
2) - 2*c^2*d^2*(2*p + 3) - 2*a*c*e^2*(m + 2*p + 3)) - g*(a*e*(b*e - 2*c*d*m
 + b*e*m) - b*d*(3*c*d - b*e + 2*c*d*p - b*e*p)) + c*e*(g*(b*d - 2*a*e) - f
*(2*c*d - b*e))*(m + 2*p + 4)*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, g,
m}, x] && NeQ[b^2 - 4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && LtQ[p, -1
] && (IntegerQ[m] || IntegerQ[p] || IntegersQ[2*m, 2*p])
```

Rule 848

```
Int[((d_.) + (e_.)*(x_))^(m_)*((f_.) + (g_.)*(x_))*((a_.) + (b_.)*(x_) + (c
_.)*(x_)^2)^(p_.), x_Symbol] :> Simp[(e*f - d*g)*(d + e*x)^(m + 1)*((a + b*
x + c*x^2)^(p + 1)/((m + 1)*(c*d^2 - b*d*e + a*e^2))), x] + Dist[1/((m + 1)
*(c*d^2 - b*d*e + a*e^2)), Int[(d + e*x)^(m + 1)*(a + b*x + c*x^2)^p*Simp[(
c*d*f - f*b*e + a*e*g)*(m + 1) + b*(d*g - e*f)*(p + 1) - c*(e*f - d*g)*(m +
 2*p + 3)*x, x], x], x] /; FreeQ[{a, b, c, d, e, f, g, p}, x] && NeQ[b^2 -
4*a*c, 0] && NeQ[c*d^2 - b*d*e + a*e^2, 0] && LtQ[m, -1] && (IntegerQ[m] ||
 IntegerQ[p] || IntegersQ[2*m, 2*p])
```

Rule 1016

```
Int[((g_) + (h_.)*(x_))^(m_.)*((a_) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_)*((d_.
) + (e_.)*(x_) + (f_.)*(x_)^2)^(m_.), x_Symbol] :> Int[(d*(g/a) + f*h*(x/c)
)^m*(a + b*x + c*x^2)^(m + p), x] /; FreeQ[{a, b, c, d, e, f, g, h, p}, x]
&& EqQ[c*g^2 - b*g*h + a*h^2, 0] && EqQ[c^2*d*g^2 - a*c*e*g*h + a^2*f*h^2,
0] && IntegerQ[m]
```

Rubi steps

$$\int \frac{\sqrt{6+17x+12x^2}}{(2+3x)^3\,(30+31x-12x^2)^3}\,dx = \int \frac{1}{(10-3x)^3\,(6+17x+12x^2)^{5/2}}\,dx$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} - \frac{\int \frac{\frac{109953}{2}-41904x}{(10-3x)^3(6+17x+12x^2)^{3/2}}\,dx}{2646}$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} + \frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12}}$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} + \frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12}}$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} + \frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12}}$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} + \frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12}}$$

$$= -\frac{275+388x}{294(10-3x)^2\,(6+17x+12x^2)^{3/2}} + \frac{738029+1042556x}{8232(10-3x)^2\sqrt{6+17x+12}}$$

**Mathematica** [A]
time = 0.37, size = 95, normalized size = 0.68

$$\frac{\sqrt{6+17x+12x^2}\,(2773753482408+10124325497244x+9848047480070x^2-1096520427663x^3-3206824169544x^4+706089565584x^5)}{7589772288(-10+3x)^2(2+3x)^2(3+4x)^2}+\frac{40325\tanh^{-1}\left(\frac{6\sqrt{6+17x+12x^2}}{7(2+3x)}\right)}{318770436096}$$

Antiderivative was successfully verified.

```
[In] Integrate[Sqrt[6 + 17*x + 12*x^2]/((2 + 3*x)^3*(30 + 31*x - 12*x^2)^3),x]
```

```
[Out] (Sqrt[6 + 17*x + 12*x^2]*(2773753482408 + 10124325497244*x + 9848047480070*
x^2 - 1096520427663*x^3 - 3206824169544*x^4 + 706089565584*x^5))/(758977228
```

```
8*(-10 + 3*x)^2*(2 + 3*x)^2*(3 + 4*x)^2) + (40325*ArcTanh[(6*Sqrt[6 + 17*x
+ 12*x^2])/(7*(2 + 3*x))])/318770436096
```

**Maple** [B] Leaf count of result is larger than twice the leaf count of optimal. 305 vs. $2(117) = 234$.
time = 0.14, size = 306, normalized size = 2.20

| method | result |
|---|---|
| risch | $\frac{706089565584x^5-3206824169544x^4-1096520427663x^3+9848047480070x^2+10124325497244x+2773753482408}{7589772288(12x^2+17x+6)^{\frac{3}{2}}(3x-10)^2}+\frac{40325\,\mathrm{arctanh}\Big(}{}$ |
| trager | $\frac{(706089565584x^5-3206824169544x^4-1096520427663x^3+9848047480070x^2+10124325497244x+2773753482408)\sqrt{12x^2+17x}}{7589772288(36x^3-69x^2-152x-60)^2}$ |
| default | $-\frac{1410048\sqrt{12\left(x+\frac{3}{4}\right)^2-x-\frac{3}{4}}}{282475249}-\frac{40325\sqrt{12\left(x-\frac{10}{3}\right)^2+97x-\frac{382}{3}}}{8925572210688}+\frac{40325\,\mathrm{arctanh}\left(\frac{20}{28\sqrt{12\left(x-\frac{1}{3}\right.}}\right.}{6375408721}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((12*x^2+17*x+6)^(1/2)/(2+3*x)^3/(-12*x^2+31*x+30)^3,x,method=_RETURNVER
BOSE)
```

```
[Out] -1410048/282475249*(12*(x+3/4)^2-x-3/4)^(1/2)-40325/8925572210688*(12*(x-10
/3)^2+97*x-382/3)^(1/2)+40325/637540872192*arctanh(1/28*(206/3+97*x)/(12*(x
-10/3)^2+97*x-382/3)^(1/2))-1/2592/(x+2/3)^3*(12*(x+2/3)^2+x+2/3)^(3/2)+47/
1152/(x+2/3)^2*(12*(x+2/3)^2+x+2/3)^(3/2)-128/352947/(x+3/4)^3*(12*(x+3/4)^
2-x-3/4)^(3/2)-230400/5764801/(x+3/4)^2*(12*(x+3/4)^2-x-3/4)^(3/2)-23/11059
2*ln(1/12*(17/2+12*x)*12^(1/2)+(12*(x+2/3)^2+x+2/3)^(1/2))*12^(1/2)+1/79692
609024/(x-10/3)^2*(12*(x-10/3)^2+97*x-382/3)^(3/2)+1261/62479005474816*(17+
24*x)*(12*(x-10/3)^2+97*x-382/3)^(1/2)-1261/31239502737408/(x-10/3)*(12*(x-
10/3)^2+97*x-382/3)^(3/2)+58752/282475249*ln(1/12*(17/2+12*x)*12^(1/2)+(12*
(x+3/4)^2-x-3/4)^(1/2))*12^(1/2)-570457/31239502737408*ln(1/12*(17/2+12*x)*
12^(1/2)+(12*(x-10/3)^2+97*x-382/3)^(1/2))*12^(1/2)-23/4608*(12*(x+2/3)^2+x
+2/3)^(1/2)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^3/(-12*x^2+31*x+30)^3,x, algorithm=
"maxima")
```

```
[Out] -integrate(sqrt(12*x^2 + 17*x + 6)/((12*x^2 - 31*x - 30)^3*(3*x + 2)^3), x)
```

**Fricas [A]**
time = 0.39, size = 186, normalized size = 1.34

$$\frac{40325\,(1296\,x^6-4968\,x^5-6183\,x^4+16656\,x^3+31384\,x^2+18240\,x+3600)\log\left(\frac{291\,x+84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)-40325\,(1296\,x^6-4968\,x^5-6183\,x^4+16656\,x^3+31384\,x^2+18240\,x+3600)\log\left(\frac{291\,x-84\,\sqrt{12\,x^2+17\,x+6}+206}{x}\right)+168\,(706089565584\,x^5-3206824169544\,x^4-1096520427663\,x^3+9848047480070\,x^2+10124325497244\,x+2773753482408)\sqrt{12\,x^2+17\,x+6}}{1275081744384\,(1296\,x^6-4968\,x^5-6183\,x^4+16656\,x^3+31384\,x^2+18240\,x+3600)}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^3/(-12*x^2+31*x+30)^3,x, algorithm=
"fricas")
```

```
[Out] 1/1275081744384*(40325*(1296*x^6 - 4968*x^5 - 6183*x^4 + 16656*x^3 + 31384*
x^2 + 18240*x + 3600)*log((291*x + 84*sqrt(12*x^2 + 17*x + 6) + 206)/x) - 4
0325*(1296*x^6 - 4968*x^5 - 6183*x^4 + 16656*x^3 + 31384*x^2 + 18240*x + 36
00)*log((291*x - 84*sqrt(12*x^2 + 17*x + 6) + 206)/x) + 168*(706089565584*x
^5 - 3206824169544*x^4 - 1096520427663*x^3 + 9848047480070*x^2 + 1012432549
7244*x + 2773753482408)*sqrt(12*x^2 + 17*x + 6))/(1296*x^6 - 4968*x^5 - 618
3*x^4 + 16656*x^3 + 31384*x^2 + 18240*x + 3600)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$-\int \frac{\sqrt{12x^2+17x+6}}{46656x^9-268272x^8-76788x^7+1703619x^6+1218456x^5-3669588x^4-6898688x^3-4903920x^2-1641600x-216000}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x**2+17*x+6)**(1/2)/(2+3*x)**3/(-12*x**2+31*x+30)**3,x)
```

```
[Out] -Integral(sqrt(12*x**2 + 17*x + 6)/(46656*x**9 - 268272*x**8 - 76788*x**7 +
 1703619*x**6 + 1218456*x**5 - 3669588*x**4 - 6898688*x**3 - 4903920*x**2 -
 1641600*x - 216000), x)
```

**Giac [A]**
time = 4.27, size = 232, normalized size = 1.67

$$\frac{\sqrt{3}\left(282273\,\sqrt{3}\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)^3-11460924\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)^2-37551180\,\sqrt{3}\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)-83365264\right)}{159385218048\left(3\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)^2-40\,\sqrt{3}\left(2\,\sqrt{3}\,x-\sqrt{12\,x^2+17\,x+6}\right)-188\right)^2}+\frac{(8\,(2860316794\,x+6078171227)x+34383350229)x+8090114146}{2213683584\,(12\,x^2+17\,x+6)^{\frac{3}{2}}}+\frac{40325}{637540872192}\log\left(\left|-6\,\sqrt{3}\,x+20\,\sqrt{3}+3\,\sqrt{12\,x^2+17\,x+6}+42\right|\right)-\frac{40325}{637540872192}\log\left(\left|-6\,\sqrt{3}\,x+20\,\sqrt{3}+3\,\sqrt{12\,x^2+17\,x+6}-42\right|\right)$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((12*x^2+17*x+6)^(1/2)/(2+3*x)^3/(-12*x^2+31*x+30)^3,x, algorithm=
"giac")
```

```
[Out] 1/159385218048*sqrt(3)*(282273*sqrt(3)*(2*sqrt(3)*x - sqrt(12*x^2 + 17*x +
6))^3 - 11460924*(2*sqrt(3)*x - sqrt(12*x^2 + 17*x + 6))^2 - 37551180*sqrt(
3)*(2*sqrt(3)*x - sqrt(12*x^2 + 17*x + 6)) - 83365264)/(3*(2*sqrt(3)*x - sq
rt(12*x^2 + 17*x + 6))^2 - 40*sqrt(3)*(2*sqrt(3)*x - sqrt(12*x^2 + 17*x + 6
)) - 188)^2 + 1/2213683584*((8*(2860316794*x + 6078171227)*x + 34383350229)
```

```
*x + 8090114146)/(12*x^2 + 17*x + 6)^(3/2) + 40325/637540872192*log(abs(-6*
sqrt(3)*x + 20*sqrt(3) + 3*sqrt(12*x^2 + 17*x + 6) + 42)) - 40325/637540872
192*log(abs(-6*sqrt(3)*x + 20*sqrt(3) + 3*sqrt(12*x^2 + 17*x + 6) - 42))
```

**Mupad** [F]
time = 0.00, size = -1, normalized size = -0.01

$$\int \frac{\sqrt{12\,x^2+17\,x+6}}{{\left(3\,x+2\right)}^3\,{\left(-12\,x^2+31\,x+30\right)}^3} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)^3*(31*x - 12*x^2 + 30)^3),x)
[Out] int((17*x + 12*x^2 + 6)^(1/2)/((3*x + 2)^3*(31*x - 12*x^2 + 30)^3), x)
```

# 3.138 $\int (-3+2x) \left(-3x+x^2\right)^{2/3} \, dx$

**Optimal**. Leaf size=15

$$\frac{3}{5} \left(-3x+x^2\right)^{5/3}$$

```
[Out] 3/5*(x^2-3*x)^(5/3)
```

**Rubi** [A]
time = 0.00, antiderivative size = 15, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, integrand size = 17, $\frac{\text{number of rules}}{\text{integrand size}} = 0.059$, Rules used = $\{643\}$

$$\frac{3}{5} \left(x^2-3x\right)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Int[(-3 + 2*x)*(-3*x + x^2)^(2/3),x]
```

```
[Out] (3*(-3*x + x^2)^(5/3))/5
```

Rule 643

```
Int[((d_) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol
] :> Simp[d*((a + b*x + c*x^2)^(p + 1)/(b*(p + 1))), x] /; FreeQ[{a, b, c,
d, e, p}, x] && EqQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rubi steps

$$\int (-3+2x) \left(-3x+x^2\right)^{2/3} \, dx = \frac{3}{5} \left(-3x+x^2\right)^{5/3}$$

**Mathematica** [A]
time = 9.31, size = 13, normalized size = 0.87

$$\frac{3}{5}((-3+x)x)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(-3 + 2*x)*(-3*x + x^2)^(2/3),x]
```

```
[Out] (3*((-3 + x)*x)^(5/3))/5
```

**Maple** [A]
time = 0.11, size = 12, normalized size = 0.80

| method | result | size |
|---|---|---|
| derivativedivides | $\frac{3(x^2-3x)^{\frac{5}{3}}}{5}$ | 12 |
| default | $\frac{3(x^2-3x)^{\frac{5}{3}}}{5}$ | 12 |
| gosper | $\frac{3(x-3)x(x^2-3x)^{\frac{2}{3}}}{5}$ | 16 |
| trager | $\frac{3(x-3)x(x^2-3x)^{\frac{2}{3}}}{5}$ | 16 |
| risch | $\frac{3(x-3)^2x^2}{5((x-3)x)^{\frac{1}{3}}}$ | 18 |
| meijerg | $-\frac{9\,3^{\frac{2}{3}}\operatorname{signum}(x-3)^{\frac{2}{3}}x^{\frac{5}{3}}\operatorname{hypergeom}\left(\left[-\frac{2}{3},\frac{5}{3}\right],\left[\frac{8}{3}\right],\frac{x}{3}\right)}{5(-\operatorname{signum}(x-3))^{\frac{2}{3}}}+\frac{3\,3^{\frac{2}{3}}\operatorname{signum}(x-3)^{\frac{2}{3}}x^{\frac{8}{3}}\operatorname{hypergeom}\left(\left[-\frac{2}{3},\frac{8}{3}\right],\left[\frac{11}{3}\right],\frac{x}{3}\right)}{4(-\operatorname{signum}(x-3))^{\frac{2}{3}}}$ | 64 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x-3)*(x^2-3*x)^(2/3),x,method=_RETURNVERBOSE)
[Out] 3/5*(x^2-3*x)^(5/3)
```

**Maxima [A]**
time = 0.27, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^2-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-3+2*x)*(x^2-3*x)^(2/3),x, algorithm="maxima")
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Fricas [A]**
time = 0.33, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^2-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-3+2*x)*(x^2-3*x)^(2/3),x, algorithm="fricas")
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Sympy [B]** Leaf count of result is larger than twice the leaf count of optimal. 31 vs. $2(12) = 24$.
time = 0.07, size = 31, normalized size = 2.07

$$\frac{3x^2(x^2-3x)^{\frac{2}{3}}}{5}-\frac{9x(x^2-3x)^{\frac{2}{3}}}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-3+2*x)*(x**2-3*x)**(2/3),x)
```

```
[Out] 3*x**2*(x**2 - 3*x)**(2/3)/5 - 9*x*(x**2 - 3*x)**(2/3)/5
```

**Giac** [A]
time = 3.68, size = 11, normalized size = 0.73

$$\frac{3}{5} \left(x^2 - 3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((-3+2*x)*(x^2-3*x)^(2/3),x, algorithm="giac")
```

```
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Mupad** [B]
time = 3.72, size = 15, normalized size = 1.00

$$\frac{3\,x\,\left(x^2-3\,x\right)^{2/3}\,\left(x-3\right)}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x - 3)*(x^2 - 3*x)^(2/3),x)
```

```
[Out] (3*x*(x^2 - 3*x)^(2/3)*(x - 3))/5
```

# 3.139 $\int ((-3+x)x)^{2/3}(-3+2x)\,dx$

**Optimal**. Leaf size=16

$$\frac{3}{5}(-((3-x)x))^{5/3}$$

[Out] 3/5*(-(3-x)*x)^(5/3)

**Rubi** [A]
time = 0.00, antiderivative size = 16, normalized size of antiderivative = 1.00, number of steps used = 1, number of rules used = 1, integrand size = 15, $\frac{\text{number of rules}}{\text{integrand size}} = 0.067$, Rules used = $\{1602\}$

$$\frac{3}{5}(-((3-x)x))^{5/3}$$

Antiderivative was successfully verified.

[In] Int[((-3 + x)*x)^(2/3)*(-3 + 2*x),x]

[Out] (3*(-((3 - x)*x))^(5/3))/5

Rule 1602

```
Int[(Pp_)*(Qq_)^(m_.), x_Symbol] :> With[{p = Expon[Pp, x], q = Expon[Qq, x
]}, Simp[Coeff[Pp, x, p]*x^(p - q + 1)*(Qq^(m + 1)/((p + m*q + 1)*Coeff[Qq,
 x, q])), x] /; NeQ[p + m*q + 1, 0] && EqQ[(p + m*q + 1)*Coeff[Qq, x, q]*Pp
, Coeff[Pp, x, p]*x^(p - q)*((p - q + 1)*Qq + (m + 1)*x*D[Qq, x])]] /; Free
Q[m, x] && PolyQ[Pp, x] && PolyQ[Qq, x] && NeQ[m, -1]
```

Rubi steps

$$\int ((-3+x)x)^{2/3}(-3+2x)\,dx = \frac{3}{5}(-(3-x)x)^{5/3}$$

**Mathematica** [A]
time = 0.00, size = 13, normalized size = 0.81

$$\frac{3}{5}((-3+x)x)^{5/3}$$

Antiderivative was successfully verified.

[In] Integrate[((-3 + x)*x)^(2/3)*(-3 + 2*x),x]

[Out] (3*((-3 + x)*x)^(5/3))/5

**Maple** [A]
time = 0.08, size = 10, normalized size = 0.62

| method | result | size |
|---|---|---|
| derivativedivides | $\frac{3((x-3)x)^{\frac{5}{3}}}{5}$ | 10 |
| default | $\frac{3((x-3)x)^{\frac{5}{3}}}{5}$ | 10 |
| gosper | $\frac{3(x-3)x((x-3)x)^{\frac{2}{3}}}{5}$ | 14 |
| trager | $\frac{3(x-3)x(x^2-3x)^{\frac{2}{3}}}{5}$ | 16 |
| risch | $\frac{3(x-3)^2x^2}{5((x-3)x)^{\frac{1}{3}}}$ | 18 |
| meijerg | $-\frac{9\,3^{\frac{2}{3}}\operatorname{signum}(x-3)^{\frac{2}{3}}x^{\frac{5}{3}}\operatorname{hypergeom}\left(\left[-\frac{2}{3},\frac{5}{3}\right],\left[\frac{8}{3}\right],\frac{x}{3}\right)}{5(-\operatorname{signum}(x-3))^{\frac{2}{3}}}+\frac{3\,3^{\frac{2}{3}}\operatorname{signum}(x-3)^{\frac{2}{3}}x^{\frac{8}{3}}\operatorname{hypergeom}\left(\left[-\frac{2}{3},\frac{8}{3}\right],\left[\frac{11}{3}\right],\frac{x}{3}\right)}{4(-\operatorname{signum}(x-3))^{\frac{2}{3}}}$ | 64 |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(((x-3)*x)^(2/3)*(2*x-3),x,method=_RETURNVERBOSE)
[Out] 3/5*((x-3)*x)^(5/3)
```

**Maxima** [A]
time = 0.27, size = 9, normalized size = 0.56

$$\frac{3}{5}\,((x-3)x)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((-3+x)*x)^(2/3)*(-3+2*x),x, algorithm="maxima")
[Out] 3/5*((x - 3)*x)^(5/3)
```

**Fricas** [A]
time = 0.36, size = 11, normalized size = 0.69

$$\frac{3}{5}\,\left(x^2-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((-3+x)*x)^(2/3)*(-3+2*x),x, algorithm="fricas")
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Sympy** [A]
time = 3.34, size = 10, normalized size = 0.62

$$\frac{3(x(x-3))^{\frac{5}{3}}}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((-3+x)*x)**(2/3)*(-3+2*x),x)
```

```
[Out] 3*(x*(x - 3))**(5/3)/5
```

**Giac** [A]
time = 2.62, size = 11, normalized size = 0.69

$$\frac{3}{5}\left(x^{2}-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(((-3+x)*x)^(2/3)*(-3+2*x),x, algorithm="giac")
```

```
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Mupad** [B]
time = 3.66, size = 13, normalized size = 0.81

$$\frac{3\,x\,{\left(x\,\left(x-3\right)\right)}^{2/3}\,\left(x-3\right)}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((2*x - 3)*(x*(x - 3))^(2/3),x)
```

```
[Out] (3*x*(x*(x - 3))^(2/3)*(x - 3))/5
```

# 3.140 $\int \frac{x(9-9x+2x^2)}{\sqrt[3]{-3x+x^2}}\,dx$

**Optimal**. Leaf size=15

$$\frac{3}{5}\left(-3x+x^2\right)^{5/3}$$

```
[Out] 3/5*(x^2-3*x)^(5/3)
```

**Rubi** [A]
time = 0.02, antiderivative size = 15, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 23, $\frac{\text{number of rules}}{\text{integrand size}} = 0.087$, Rules used = {1645, 643}

$$\frac{3}{5}\left(x^2-3x\right)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Int[(x*(9 - 9*x + 2*x^2))/(-3*x + x^2)^(1/3),x]
```

```
[Out] (3*(-3*x + x^2)^(5/3))/5
```

Rule 643

```
Int[((d_) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol
] :> Simp[d*((a + b*x + c*x^2)^(p + 1)/(b*(p + 1))), x] /; FreeQ[{a, b, c,
d, e, p}, x] && EqQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 1645

```
Int[(Pq_)*((e_.)*(x_))^(m_.)*((b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :
> Dist[e, Int[(e*x)^(m - 1)*PolynomialQuotient[Pq, b + c*x, x]*(b*x + c*x^2
)^(p + 1), x], x] /; FreeQ[{b, c, e, m, p}, x] && PolyQ[Pq, x] && EqQ[Polyn
omialRemainder[Pq, b + c*x, x], 0]
```

Rubi steps

$$\begin{aligned} \int \frac{x(9-9x+2x^2)}{\sqrt[3]{-3x+x^2}}\,dx &= \int (-3+2x)\left(-3x+x^2\right)^{2/3}\,dx\\ &= \frac{3}{5}\left(-3x+x^2\right)^{5/3} \end{aligned}$$

**Mathematica** [A]
time = 0.01, size = 13, normalized size = 0.87

$$\frac{3}{5}((-3+x)x)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*(9 - 9*x + 2*x^2))/(-3*x + x^2)^(1/3),x]
[Out] (3*((-3 + x)*x)^(5/3))/5
```

**Maple [A]**
time = 0.10, size = 16, normalized size = 1.07

| method | result |
|---|---|
| trager | $\frac{3(x-3)x\left(x^2-3x\right)^{\frac{2}{3}}}{5}$ |
| risch | $\frac{3(x-3)^2 x^2}{5((x-3)x)^{\frac{1}{3}}}$ |
| gosper | $\frac{3(x-3)^2 x^2}{5(x^2-3x)^{\frac{1}{3}}}$ |
| meijerg | $\frac{2\,3^{\frac{2}{3}}(-\mathrm{signum}(x-3))^{\frac{1}{3}} x^{\frac{11}{3}}\,\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{11}{3}\right],\left[\frac{14}{3}\right],\frac{x}{3}\right)}{11\mathrm{signum}(x-3)^{\frac{1}{3}}} - \frac{9\,3^{\frac{2}{3}}(-\mathrm{signum}(x-3))^{\frac{1}{3}} x^{\frac{8}{3}}\,\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{8}{3}\right],\left[\frac{11}{3}\right],\frac{x}{3}\right)}{8\mathrm{signum}(x-3)^{\frac{1}{3}}} + \frac{9\,3^{\frac{2}{3}}(-\mathrm{signu}}{}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(2*x^2-9*x+9)/(x^2-3*x)^(1/3),x,method=_RETURNVERBOSE)
[Out] 3/5*(x-3)*x*(x^2-3*x)^(2/3)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/(x^2-3*x)^(1/3),x, algorithm="maxima")
[Out] integrate((2*x^2 - 9*x + 9)*x/(x^2 - 3*x)^(1/3), x)
```

**Fricas [A]**
time = 0.36, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^2-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/(x^2-3*x)^(1/3),x, algorithm="fricas")
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{x(x-3)\,(2x-3)}{\sqrt[3]{x\,(x-3)}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x**2-9*x+9)/(x**2-3*x)**(1/3),x)
```

```
[Out] Integral(x*(x - 3)*(2*x - 3)/(x*(x - 3))**(1/3), x)
```

**Giac** [A]
time = 2.66, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^2-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/(x^2-3*x)^(1/3),x, algorithm="giac")
```

```
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Mupad** [B]
time = 3.68, size = 15, normalized size = 1.00

$$\frac{3\,x\,\left(x^2-3\,x\right)^{2/3}\,\left(x-3\right)}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(2*x^2 - 9*x + 9))/(x^2 - 3*x)^(1/3),x)
```

```
[Out] (3*x*(x^2 - 3*x)^(2/3)*(x - 3))/5
```

# 3.141 $\int \frac{x(9-9x+2x^2)}{\sqrt[3]{(-3+x)x}}\,dx$

**Optimal**. Leaf size=15

$$\frac{3}{5}\left(-3x+x^2\right)^{5/3}$$

```
[Out] 3/5*(x^2-3*x)^(5/3)
```

**Rubi** [A]
time = 0.03, antiderivative size = 15, normalized size of antiderivative = 1.00, number of steps used = 3, number of rules used = 3, integrand size = 21, $\frac{\text{number of rules}}{\text{integrand size}} = 0.143$, Rules used = {1976, 1645, 643}

$$\frac{3}{5}\left(x^2-3x\right)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Int[(x*(9 - 9*x + 2*x^2))/((-3 + x)*x)^(1/3),x]
```

```
[Out] (3*(-3*x + x^2)^(5/3))/5
```

Rule 643

```
Int[((d_) + (e_.)*(x_))*((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol
] :> Simp[d*((a + b*x + c*x^2)^(p + 1)/(b*(p + 1))), x] /; FreeQ[{a, b, c,
d, e, p}, x] && EqQ[2*c*d - b*e, 0] && NeQ[p, -1]
```

Rule 1645

```
Int[(Pq_)*((e_.)*(x_))^(m_.)*((b_.)*(x_) + (c_.)*(x_)^2)^(p_.), x_Symbol] :
> Dist[e, Int[(e*x)^(m - 1)*PolynomialQuotient[Pq, b + c*x, x]*(b*x + c*x^2
)^(p + 1), x], x] /; FreeQ[{b, c, e, m, p}, x] && PolyQ[Pq, x] && EqQ[Polyn
omialRemainder[Pq, b + c*x, x], 0]
```

Rule 1976

```
Int[(u_.)*((e_.)*((a_.) + (b_.)*(x_)^(n_.))*((c_) + (d_.)*(x_)^(n_.)))^(p_)
, x_Symbol] :> Int[u*(a*c*e + (b*c + a*d)*e*x^n + b*d*e*x^(2*n))^p, x] /; F
reeQ[{a, b, c, d, e, n, p}, x]
```

Rubi steps

$$\int \frac{x(9-9x+2x^2)}{\sqrt[3]{(-3+x)x}}\,dx = \int \frac{x(9-9x+2x^2)}{\sqrt[3]{-3x+x^2}}\,dx$$
$$= \int (-3+2x)\left(-3x+x^2\right)^{2/3}\,dx$$
$$= \frac{3}{5}\left(-3x+x^2\right)^{5/3}$$

**Mathematica [A]**
time = 0.00, size = 13, normalized size = 0.87

$$\frac{3}{5}((-3+x)x)^{5/3}$$

Antiderivative was successfully verified.

```
[In] Integrate[(x*(9 - 9*x + 2*x^2))/((-3 + x)*x)^(1/3),x]
```

```
[Out] (3*((-3 + x)*x)^(5/3))/5
```

**Maple [A]**
time = 0.07, size = 16, normalized size = 1.07

| method | result |
|---|---|
| trager | $\frac{3(x-3)x\left(x^2-3x\right)^{\frac{2}{3}}}{5}$ |
| gosper | $\frac{3(x-3)^2x^2}{5((x-3)x)^{\frac{1}{3}}}$ |
| risch | $\frac{3(x-3)^2x^2}{5((x-3)x)^{\frac{1}{3}}}$ |
| meijerg | $\frac{2\,3^{\frac{2}{3}}(-\mathrm{signum}(x-3))^{\frac{1}{3}}x^{\frac{11}{3}}\,\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{11}{3}\right],\left[\frac{14}{3}\right],\frac{x}{3}\right)}{11\mathrm{signum}(x-3)^{\frac{1}{3}}} - \frac{9\,3^{\frac{2}{3}}(-\mathrm{signum}(x-3))^{\frac{1}{3}}x^{\frac{8}{3}}\,\mathrm{hypergeom}\left(\left[\frac{1}{3},\frac{8}{3}\right],\left[\frac{11}{3}\right],\frac{x}{3}\right)}{8\mathrm{signum}(x-3)^{\frac{1}{3}}} + 9\,3^{\frac{2}{3}}(-\mathrm{signum}$ |

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int(x*(2*x^2-9*x+9)/((x-3)*x)^(1/3),x,method=_RETURNVERBOSE)
```

```
[Out] 3/5*(x-3)*x*(x^2-3*x)^(2/3)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/((-3+x)*x)^(1/3),x, algorithm="maxima")
```

```
[Out] integrate((2*x^2 - 9*x + 9)*x/((x - 3)*x)^(1/3), x)
```

**Fricas [A]**
time = 0.34, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^{2}-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/((-3+x)*x)^(1/3),x, algorithm="fricas")
```

```
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\int\frac{x(x-3)\,(2x-3)}{\sqrt[3]{x\,(x-3)}}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x**2-9*x+9)/((-3+x)*x)**(1/3),x)
```

```
[Out] Integral(x*(x - 3)*(2*x - 3)/(x*(x - 3))**(1/3), x)
```

**Giac [A]**
time = 3.37, size = 11, normalized size = 0.73

$$\frac{3}{5}\left(x^{2}-3\,x\right)^{\frac{5}{3}}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate(x*(2*x^2-9*x+9)/((-3+x)*x)^(1/3),x, algorithm="giac")
```

```
[Out] 3/5*(x^2 - 3*x)^(5/3)
```

**Mupad [B]**
time = 3.61, size = 13, normalized size = 0.87

$$\frac{3\,x\,\left(x\,\left(x-3\right)\right)^{2/3}\,\left(x-3\right)}{5}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((x*(2*x^2 - 9*x + 9))/(x*(x - 3))^(1/3),x)
```

```
[Out] (3*x*(x*(x - 3))^(2/3)*(x - 3))/5
```

# 3.142 $\int \frac{g+hx}{\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}\,(g^2+3h^2x^2)}\,dx$

**Optimal**. Leaf size=242

$$\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\tan^{-1}\left(\frac{1}{\sqrt{3}}-\frac{2^{2/3}\left(1-\frac{3hx}{g}\right)^{2/3}}{\sqrt{3}\sqrt[3]{1+\frac{3hx}{g}}}\right)}{2^{2/3}\sqrt{3}\,h\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}}+\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\log\left(g^2+3h^2x^2\right)}{6\ 2^{2/3}h\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}}-\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\log\left(\left(1-\right.\right.}{2\ 2^{2/3}h\sqrt[3]{-}}$$

[Out] 1/12*(1-9*h^2*x^2/g^2)^(1/3)*ln(3*h^2*x^2+g^2)*2^(1/3)/h/(-c*g^2/h^2+9*c*x^2)^(1/3)-1/4*(1-9*h^2*x^2/g^2)^(1/3)*ln((1-3*h*x/g)^(2/3)+2^(1/3)*(1+3*h*x/g)^(1/3))*2^(1/3)/h/(-c*g^2/h^2+9*c*x^2)^(1/3)-1/6*(1-9*h^2*x^2/g^2)^(1/3)*arctan(-1/3*3^(1/2)+1/3*2^(2/3)*(1-3*h*x/g)^(2/3)/(1+3*h*x/g)^(1/3)*3^(1/2))*2^(1/3)/h/(-c*g^2/h^2+9*c*x^2)^(1/3)*3^(1/2)

**Rubi** [A]
time = 0.06, antiderivative size = 242, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 40, $\frac{\text{number of rules}}{\text{integrand size}} = 0.050$, Rules used = {1023, 1022}

$$\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\,\text{ArcTan}\left(\frac{1}{\sqrt{3}}-\frac{2^{2/3}\left(1-\frac{3hx}{g}\right)^{2/3}}{\sqrt{3}\sqrt[3]{\frac{3hx}{g}+1}}\right)}{2^{2/3}\sqrt{3}\,h\sqrt[3]{9cx^2-\frac{cg^2}{h^2}}}+\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\log\left(g^2+3h^2x^2\right)}{6\ 2^{2/3}h\sqrt[3]{9cx^2-\frac{cg^2}{h^2}}}-\frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\log\left(\left(1-\frac{3hx}{g}\right)^{2/3}+\sqrt[3]{2}\sqrt[3]{\frac{3hx}{g}+1}\right)}{2\ 2^{2/3}h\sqrt[3]{9cx^2-\frac{cg^2}{h^2}}}$$

Antiderivative was successfully verified.

[In] Int[(g + h*x)/((-((c*g^2)/h^2) + 9*c*x^2)^(1/3)*(g^2 + 3*h^2*x^2)),x]

[Out] ((1 - (9*h^2*x^2)/g^2)^(1/3)*ArcTan[1/Sqrt[3] - (2^(2/3)*(1 - (3*h*x)/g)^(2/3))/(Sqrt[3]*(1 + (3*h*x)/g)^(1/3))])/(2^(2/3)*Sqrt[3]*h*(-((c*g^2)/h^2) + 9*c*x^2)^(1/3)) + ((1 - (9*h^2*x^2)/g^2)^(1/3)*Log[g^2 + 3*h^2*x^2])/(6*2^(2/3)*h*(-((c*g^2)/h^2) + 9*c*x^2)^(1/3)) - ((1 - (9*h^2*x^2)/g^2)^(1/3)*Log[(1 - (3*h*x)/g)^(2/3) + 2^(1/3)*(1 + (3*h*x)/g)^(1/3)])/(2*2^(2/3)*h*(-((c*g^2)/h^2) + 9*c*x^2)^(1/3))

Rule 1022

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)^(1/3)*((d_) + (f_.)*(x_)^2))
, x_Symbol] :> Simp[Sqrt[3]*h*(ArcTan[1/Sqrt[3] - 2^(2/3)*((1 - 3*h*(x/g))^
(2/3)/(Sqrt[3]*(1 + 3*h*(x/g))^(1/3)))]/(2^(2/3)*a^(1/3)*f)), x] + (-Simp[3
```

```
*h*(Log[(1 - 3*h*(x/g))^(2/3) + 2^(1/3)*(1 + 3*h*(x/g))^(1/3)]/(2^(5/3)*a^(
1/3)*f)), x] + Simp[h*(Log[d + f*x^2]/(2^(5/3)*a^(1/3)*f)), x]) /; FreeQ[{a
, c, d, f, g, h}, x] && EqQ[c*d + 3*a*f, 0] && EqQ[c*g^2 + 9*a*h^2, 0] && G
tQ[a, 0]
```

Rule 1023

```
Int[((g_) + (h_.)*(x_))/(((a_) + (c_.)*(x_)^2)^(1/3)*((d_) + (f_.)*(x_)^2))
, x_Symbol] :> Dist[(1 + c*(x^2/a))^(1/3)/(a + c*x^2)^(1/3), Int[(g + h*x)/
((1 + c*(x^2/a))^(1/3)*(d + f*x^2)), x], x] /; FreeQ[{a, c, d, f, g, h}, x]
 && EqQ[c*d + 3*a*f, 0] && EqQ[c*g^2 + 9*a*h^2, 0] &&  !GtQ[a, 0]
```

Rubi steps

$$\int \frac{g+hx}{\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}\,(g^2+3h^2x^2)}\,dx = \frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\int \frac{g+hx}{(g^2+3h^2x^2)\sqrt[3]{1-\frac{9h^2x^2}{g^2}}}\,dx}{\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}}$$

$$= \frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\tan^{-1}\left(\frac{1}{\sqrt{3}}-\frac{2^{2/3}\left(1-\frac{3hx}{g}\right)^{2/3}}{\sqrt{3}\sqrt[3]{1+\frac{3hx}{g}}}\right)}{2^{2/3}\sqrt{3}\,h\sqrt[3]{-\frac{cg^2}{h^2}+9cx^2}} + \frac{\sqrt[3]{1-\frac{9h^2x^2}{g^2}}\ \log}{6\ 2^{2/3}h\sqrt[3]{-\frac{c}{h}}}$$

**Mathematica [A]**
time = 0.72, size = 381, normalized size = 1.57

$$\frac{\sqrt[3]{-\frac{2g^2}{h^2}+18x^2}\left(2\sqrt{3}\tan^{-1}\left(\frac{\sqrt{3}\sqrt[3]{g}h^{2/3}\sqrt[3]{-\frac{g^2}{h^2}+9x^2}}{2^{2/3}g-3\ 2^{2/3}hx+\sqrt[3]{g}h^{2/3}\sqrt[3]{-\frac{g^2}{h^2}+9x^2}}\right)-2\log\left(\sqrt[3]{g}\sqrt[3]{-\frac{g^2}{h^2}+9x^2}\right)+\log\left(g^{2/3}\left(-\frac{g^2}{h^2}+9x^2\right)^{2/3}\right)+2\log\left(2^{2/3}g-3\ 2^{2/3}hx-2\sqrt[3]{g}h^{2/3}\sqrt[3]{-\frac{g^2}{h^2}+9x^2}\right)-\log\left(\sqrt[3]{2}g^2-6\sqrt[3]{2}ghx+9\sqrt[3]{2}h^2x^2+2^{2/3}g^{4/3}h^{2/3}\sqrt[3]{-\frac{g^2}{h^2}+9x^2}-3\ 2^{2/3}\sqrt[3]{g}h^{5/3}x\sqrt[3]{-\frac{g^2}{h^2}+9x^2}+2g^{2/3}h^{4/3}\left(-\frac{g^2}{h^2}+9x^2\right)^{2/3}\right)\right)}{12g^{2/3}\sqrt[3]{h}\sqrt[3]{c\left(-\frac{g^2}{h^2}+9x^2\right)}}$$

Antiderivative was successfully verified.

```
[In] Integrate[(g + h*x)/((-((c*g^2)/h^2) + 9*c*x^2)^(1/3)*(g^2 + 3*h^2*x^2)),x]
```

```
[Out] (((-2*g^2)/h^2 + 18*x^2)^(1/3)*(2*Sqrt[3]*ArcTan[(Sqrt[3]*g^(1/3)*h^(2/3)*(
-(g^2/h^2) + 9*x^2)^(1/3))/(2^(2/3)*g - 3*2^(2/3)*h*x + g^(1/3)*h^(2/3)*(-(
g^2/h^2) + 9*x^2)^(1/3))] - 2*Log[g^(1/3)*(-(g^2/h^2) + 9*x^2)^(1/3)] + Log
[g^(2/3)*(-(g^2/h^2) + 9*x^2)^(2/3)] + 2*Log[2^(2/3)*g - 3*2^(2/3)*h*x - 2*
```

```
g^(1/3)*h^(2/3)*(-(g^2/h^2) + 9*x^2)^(1/3)] - Log[2^(1/3)*g^2 - 6*2^(1/3)*g
*h*x + 9*2^(1/3)*h^2*x^2 + 2^(2/3)*g^(4/3)*h^(2/3)*(-(g^2/h^2) + 9*x^2)^(1/
3) - 3*2^(2/3)*g^(1/3)*h^(5/3)*x*(-(g^2/h^2) + 9*x^2)^(1/3) + 2*g^(2/3)*h^(
4/3)*(-(g^2/h^2) + 9*x^2)^(2/3)]))/(12*g^(2/3)*h^(1/3)*(c*(-(g^2/h^2) + 9*x
^2))^(1/3))
```

**Maple** [F]
time = 0.03, size = 0, normalized size = 0.00

$$\int \frac{hx+g}{\left(-\frac{c\,g^{2}}{h^{2}}+9c\,x^{2}\right)^{\frac{1}{3}}\left(3h^{2}x^{2}+g^{2}\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)/(-c*g^2/h^2+9*c*x^2)^(1/3)/(3*h^2*x^2+g^2),x)
```

```
[Out] int((h*x+g)/(-c*g^2/h^2+9*c*x^2)^(1/3)/(3*h^2*x^2+g^2),x)
```

**Maxima** [F]
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(-c*g^2/h^2+9*c*x^2)^(1/3)/(3*h^2*x^2+g^2),x, algorithm="
maxima")
```

```
[Out] integrate((h*x + g)/((3*h^2*x^2 + g^2)*(9*c*x^2 - c*g^2/h^2)^(1/3)), x)
```

**Fricas** [F(-1)] Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(-c*g^2/h^2+9*c*x^2)^(1/3)/(3*h^2*x^2+g^2),x, algorithm="
fricas")
```

```
[Out] Timed out
```

**Sympy** [F]
time = 0.00, size = 0, normalized size = 0.00

$$\int \frac{g+hx}{\sqrt[3]{c\left(-\frac{g}{h}+3x\right)\left(\frac{g}{h}+3x\right)}\left(g^{2}+3h^{2}x^{2}\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(-c*g**2/h**2+9*c*x**2)**(1/3)/(3*h**2*x**2+g**2),x)
```

```
[Out] Integral((g + h*x)/((c*(-g/h + 3*x)*(g/h + 3*x))**(1/3)*(g**2 + 3*h**2*x**2
)), x)
```

**Giac [F]**
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(-c*g^2/h^2+9*c*x^2)^(1/3)/(3*h^2*x^2+g^2),x, algorithm="
giac")
```

```
[Out] integrate((h*x + g)/((3*h^2*x^2 + g^2)*(9*c*x^2 - c*g^2/h^2)^(1/3)), x)
```

**Mupad [F]**
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{g+h\,x}{\left(g^2+3\,h^2\,x^2\right)\,{\left(9\,c\,x^2-\frac{c\,g^2}{h^2}\right)}^{1/3}} \,d x$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((g + h*x)/((g^2 + 3*h^2*x^2)*(9*c*x^2 - (c*g^2)/h^2)^(1/3)),x)
```

```
[Out] int((g + h*x)/((g^2 + 3*h^2*x^2)*(9*c*x^2 - (c*g^2)/h^2)^(1/3)), x)
```

## 3.143 $\int \frac{g+hx}{\sqrt[3]{\frac{-c^2g^2+bcgh+2b^2h^2}{9ch^2}+bx+cx^2}\left(\frac{f\left(b^2-\frac{-c^2g^2+bcgh+2b^2h^2}{3h^2}\right)}{c^2}+\cdots\right.}$

**Optimal**. Leaf size=488

$$\frac{3\sqrt[6]{3}\,h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh)(cg+bh)}{ch^2}-9bx-9cx^2\right)}{(2cg-bh)^2}}\tan^{-1}\left(\frac{1}{\sqrt{3}}-\frac{2^{2/3}\left(1-\frac{3h(b+2cx)}{2cg-bh}\right)^{2/3}}{\sqrt{3}\sqrt[3]{1+\frac{3h(b+2cx)}{2cg-bh}}}\right)}{f\sqrt[3]{-\frac{(cg-2bh)(cg+bh)}{ch^2}+9bx+9cx^2}}+\frac{3^{2/3}h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh}{c}\cdots\right.}{\cdots}}}{2\cdots}$$

```
[Out] -3*3^(1/6)*h*(c*h^2*((-2*b*h+c*g)*(b*h+c*g)/c/h^2-9*b*x-9*c*x^2)/(-b*h+2*c*
g)^2)^(1/3)*arctan(-1/3*3^(1/2)+1/3*2^(2/3)*(1-3*h*(2*c*x+b)/(-b*h+2*c*g))^
(2/3)/(1+3*h*(2*c*x+b)/(-b*h+2*c*g))^(1/3)*3^(1/2))/f/(-(-2*b*h+c*g)*(b*h+c
*g)/c/h^2+9*b*x+9*c*x^2)^(1/3)+1/2*3^(2/3)*h*(c*h^2*((-2*b*h+c*g)*(b*h+c*g)
/c/h^2-9*b*x-9*c*x^2)/(-b*h+2*c*g)^2)^(1/3)*ln(1/3*f*(b^2*h^2-b*c*g*h+c^2*g
^2)/c^2/h^2+b*f*x/c+f*x^2)/f/(-(-2*b*h+c*g)*(b*h+c*g)/c/h^2+9*b*x+9*c*x^2)^
(1/3)-3/2*3^(2/3)*h*(c*h^2*((-2*b*h+c*g)*(b*h+c*g)/c/h^2-9*b*x-9*c*x^2)/(-b
*h+2*c*g)^2)^(1/3)*ln((1-3*h*(2*c*x+b)/(-b*h+2*c*g))^(2/3)+2^(1/3)*(1+3*h*(
2*c*x+b)/(-b*h+2*c*g))^(1/3))/f/(-(-2*b*h+c*g)*(b*h+c*g)/c/h^2+9*b*x+9*c*x^
2)^(1/3)
```

**Rubi** [A]

time = 0.24, antiderivative size = 488, normalized size of antiderivative = 1.00, number of steps used = 2, number of rules used = 2, integrand size = 104, $\frac{\text{number of rules}}{\text{integrand size}} = 0.019$, Rules used = {1055, 1054}

$$\frac{3\sqrt[6]{3}\,h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh)(bh+cg)}{ch^2}-9bx-9cx^2\right)}{(2cg-bh)^2}}\text{ArcTan}\left(\frac{1}{\sqrt{3}}-\frac{2^{2/3}\left(1-\frac{3h(b+2cx)}{2cg-bh}\right)^{2/3}}{\sqrt{3}\sqrt[3]{\frac{3h(b+2cx)}{2cg-bh}+1}}\right)}{f\sqrt[3]{-\frac{(cg-2bh)(bh+cg)}{ch^2}+9bx+9cx^2}}+\frac{3^{2/3}h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh)(bh+cg)}{ch^2}-9bx-9cx^2\right)}{(2cg-bh)^2}}\log\left(\frac{f(b^2h^2-bcgh+c^2g^2)}{3c^2h^2}+\frac{bfx}{c}+fx^2\right)}{2f\sqrt[3]{-\frac{(cg-2bh)(bh+cg)}{ch^2}+9bx+9cx^2}}-\frac{3\,3^{2/3}h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh)(bh+cg)}{ch^2}-9bx-9cx^2\right)}{(2cg-bh)^2}}\log\left(\left(1-\frac{3h(b+2cx)}{2cg-bh}\right)^{2/3}+\sqrt[3]{2}\sqrt[3]{\frac{3h(b+2cx)}{2cg-bh}+1}\right)}{2f\sqrt[3]{-\frac{(cg-2bh)(bh+cg)}{ch^2}+9bx+9cx^2}}$$

Antiderivative was successfully verified.

```
[In] Int[(g + h*x)/((((-(c^2*g^2) + b*c*g*h + 2*b^2*h^2)/(9*c*h^2) + b*x + c*x^2)
^(1/3)*((f*(b^2 - (-(c^2*g^2) + b*c*g*h + 2*b^2*h^2)/(3*h^2)))/c^2 + (b*f*x
)/c + f*x^2)),x]
```

```
[Out] (3*3^(1/6)*h*((c*h^2*(((c*g - 2*b*h)*(c*g + b*h))/(c*h^2) - 9*b*x - 9*c*x^2
))/(2*c*g - b*h)^2)^(1/3)*ArcTan[1/Sqrt[3] - (2^(2/3)*(1 - (3*h*(b + 2*c*x)
)/(2*c*g - b*h))^(2/3))/(Sqrt[3]*(1 + (3*h*(b + 2*c*x))/(2*c*g - b*h))^(1/3
))])/(f*(-(((c*g - 2*b*h)*(c*g + b*h))/(c*h^2)) + 9*b*x + 9*c*x^2)^(1/3)) +
 (3^(2/3)*h*((c*h^2*(((c*g - 2*b*h)*(c*g + b*h))/(c*h^2) - 9*b*x - 9*c*x^2)
```

```
)/(2*c*g - b*h)^2)^(1/3)*Log[(f*(c^2*g^2 - b*c*g*h + b^2*h^2))/(3*c^2*h^2)
+ (b*f*x)/c + f*x^2])/(2*f*(-(((c*g - 2*b*h)*(c*g + b*h))/(c*h^2)) + 9*b*x
+ 9*c*x^2)^(1/3)) - (3*3^(2/3)*h*((c*h^2*(((c*g - 2*b*h)*(c*g + b*h))/(c*h^
2) - 9*b*x - 9*c*x^2))/(2*c*g - b*h)^2)^(1/3)*Log[(1 - (3*h*(b + 2*c*x))/(2
*c*g - b*h))^(2/3) + 2^(1/3)*(1 + (3*h*(b + 2*c*x))/(2*c*g - b*h))^(1/3)])/
(2*f*(-(((c*g - 2*b*h)*(c*g + b*h))/(c*h^2)) + 9*b*x + 9*c*x^2)^(1/3))
```

Rule 1054

```
Int[((g_.) + (h_.)*(x_))/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(1/3)*((d_.)
+ (e_.)*(x_) + (f_.)*(x_)^2)), x_Symbol] :> With[{q = (-9*c*(h^2/(2*c*g - b
*h)^2))^(1/3)}, Simp[Sqrt[3]*h*q*(ArcTan[1/Sqrt[3] - 2^(2/3)*((1 - (3*h*(b
+ 2*c*x))/(2*c*g - b*h))^(2/3)/(Sqrt[3]*(1 + (3*h*(b + 2*c*x))/(2*c*g - b*h
))^(1/3)))]/f), x] + (-Simp[3*h*q*(Log[(1 - 3*h*((b + 2*c*x)/(2*c*g - b*h))
)^(2/3) + 2^(1/3)*(1 + 3*h*((b + 2*c*x)/(2*c*g - b*h)))^(1/3)]/(2*f)), x] +
 Simp[h*q*(Log[d + e*x + f*x^2]/(2*f)), x])] /; FreeQ[{a, b, c, d, e, f, g,
 h}, x] && EqQ[c*e - b*f, 0] && EqQ[c^2*d - f*(b^2 - 3*a*c), 0] && EqQ[c^2*
g^2 - b*c*g*h - 2*b^2*h^2 + 9*a*c*h^2, 0] && GtQ[-9*c*(h^2/(2*c*g - b*h)^2)
, 0]
```

Rule 1055

```
Int[((g_.) + (h_.)*(x_))/(((a_.) + (b_.)*(x_) + (c_.)*(x_)^2)^(1/3)*((d_.)
+ (e_.)*(x_) + (f_.)*(x_)^2)), x_Symbol] :> With[{q = -c/(b^2 - 4*a*c)}, Di
st[(q*(a + b*x + c*x^2))^(1/3)/(a + b*x + c*x^2)^(1/3), Int[(g + h*x)/((q*a
 + b*q*x + c*q*x^2)^(1/3)*(d + e*x + f*x^2)), x], x]] /; FreeQ[{a, b, c, d,
 e, f, g, h}, x] && EqQ[c*e - b*f, 0] && EqQ[c^2*d - f*(b^2 - 3*a*c), 0] &&
 EqQ[c^2*g^2 - b*c*g*h - 2*b^2*h^2 + 9*a*c*h^2, 0] &&  !GtQ[4*a - b^2/c, 0]
```

Rubi steps

$$\int \frac{g+hx}{\sqrt[3]{\frac{-c^2g^2+bcgh+2b^2h^2}{9ch^2}+bx+cx^2}\left(\frac{f\left(b^2-\frac{-c^2g^2+bcgh+2b^2h^2}{3h^2}\right)}{c^2}+\frac{bfx}{c}+fx^2\right)}\,dx = \frac{\sqrt[3]{-\frac{c\left(\frac{-c^2g^2+bcgh+2b^2h^2}{9ch^2}+\cdots\right.}{b^2-\frac{4(-c^2g^2+bcgh\cdots}{9h^2}}}\cdots}{\cdots}$$

$$= \frac{3\sqrt[6]{3}\,h\sqrt[3]{\frac{ch^2\left(\frac{(cg-2bh)(cg+\cdots}{ch^2}\right.}{(2cg\cdots}}\cdots}{f\cdots}$$

**Mathematica** [A]
time = 2.30, size = 593, normalized size = 1.22

Antiderivative was successfully verified.

```
[In] Integrate[(g + h*x)/((((-(c^2*g^2) + b*c*g*h + 2*b^2*h^2)/(9*c*h^2) + b*x +
c*x^2)^(1/3)*((f*(b^2 - (-(c^2*g^2) + b*c*g*h + 2*b^2*h^2)/(3*h^2)))/c^2 +
(b*f*x)/c + f*x^2)),x]
```

```
[Out] (3^(2/3)*c^(1/3)*h^(5/3)*(2*Sqrt[3]*ArcTan[(Sqrt[3]*c^(1/3)*h^(2/3)*(2*c*g
- b*h)^(1/3)*((2*b^2)/c - (c*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2)^(1/3))/(
-4*b*h + 2*c*(g - 3*h*x) + c^(1/3)*h^(2/3)*(2*c*g - b*h)^(1/3)*((2*b^2)/c -
 (c*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2)^(1/3))] - 2*Log[Sqrt[h]*((2*b^2)/
c - (c*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2)^(1/3)] + Log[h*((2*b^2)/c - (c
*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2)^(2/3)] + 2*Log[Sqrt[c]*(c*g - 2*b*h
- 3*c*h*x - c^(1/3)*h^(2/3)*(2*c*g - b*h)^(1/3)*((2*b^2)/c - (c*g^2)/h^2 +
(b*g)/h + 9*b*x + 9*c*x^2)^(1/3))] - Log[c*(4*b^2*h^2 - 4*b*c*h*(g - 3*h*x)
 + c^2*(g - 3*h*x)^2 - 2*b*c^(1/3)*h^(5/3)*(2*c*g - b*h)^(1/3)*((2*b^2)/c -
 (c*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2)^(1/3) + c^(4/3)*h^(2/3)*(2*c*g -
b*h)^(1/3)*(g - 3*h*x)*((2*b^2)/c - (c*g^2)/h^2 + (b*g)/h + 9*b*x + 9*c*x^2
)^(1/3) + c^(2/3)*h^(4/3)*(2*c*g - b*h)^(2/3)*((2*b^2)/c - (c*g^2)/h^2 + (b
*g)/h + 9*b*x + 9*c*x^2)^(2/3))]))/(2*f*(2*c*g - b*h)^(2/3))
```

**Maple** [F]

time = 0.17, size = 0, normalized size = 0.00

$$\int \frac{hx+g}{\left(\frac{2b^2h^2+bcgh-c^2g^2}{9c\,h^2}+bx+c\,x^2\right)^{\frac{1}{3}}\left(\frac{f\left(b^2+\frac{-2b^2h^2-bcgh+c^2g^2}{3h^2}\right)}{c^2}+\frac{bfx}{c}+f\,x^2\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((h*x+g)/(1/9*(2*b^2*h^2+b*c*g*h-c^2*g^2)/c/h^2+b*x+c*x^2)^(1/3)/(f*(b^2
+1/3*(-2*b^2*h^2-b*c*g*h+c^2*g^2)/h^2)/c^2+b*f*x/c+f*x^2),x)
```

```
[Out] int((h*x+g)/(1/9*(2*b^2*h^2+b*c*g*h-c^2*g^2)/c/h^2+b*x+c*x^2)^(1/3)/(f*(b^2
+1/3*(-2*b^2*h^2-b*c*g*h+c^2*g^2)/h^2)/c^2+b*f*x/c+f*x^2),x)
```

**Maxima [F]**
time = 0.00, size = 0, normalized size = 0.00

Failed to integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(1/9*(2*b^2*h^2+b*c*g*h-c^2*g^2)/c/h^2+b*x+c*x^2)^(1/3)/(
f*(b^2+1/3*(-2*b^2*h^2-b*c*g*h+c^2*g^2)/h^2)/c^2+b*f*x/c+f*x^2),x, algorith
m="maxima")
```

```
[Out] 3*integrate((h*x + g)/((c*x^2 + b*x - 1/9*(c^2*g^2 - b*c*g*h - 2*b^2*h^2)/(
c*h^2))^(1/3)*(3*f*x^2 + 3*b*f*x/c + (3*b^2 + (c^2*g^2 - b*c*g*h - 2*b^2*h^
2)/h^2)*f/c^2)), x)
```

**Fricas [F(-1)]** Timed out
time = 0.00, size = 0, normalized size = 0.00

Timed out

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(1/9*(2*b^2*h^2+b*c*g*h-c^2*g^2)/c/h^2+b*x+c*x^2)^(1/3)/(
f*(b^2+1/3*(-2*b^2*h^2-b*c*g*h+c^2*g^2)/h^2)/c^2+b*f*x/c+f*x^2),x, algorith
m="fricas")
```

```
[Out] Timed out
```

**Sympy [F]**
time = 0.00, size = 0, normalized size = 0.00

$$\frac{3\cdot 3^{\frac{2}{3}}c^2h^2\left(\int \frac{g}{b^2h^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}-bcgh\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+3bch^2x\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+c^2g^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+3c^2h^2x^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}}\,dx+\int \frac{hx}{b^2h^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}-bcgh\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+3bch^2x\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+c^2g^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}+3c^2h^2x^2\sqrt[3]{\frac{2b^2}{c}+\frac{bg}{h}+9bx-\frac{cg^2}{h^2}+9cx^2}}\,dx\right)}{f}$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(1/9*(2*b**2*h**2+b*c*g*h-c**2*g**2)/c/h**2+b*x+c*x**2)**
(1/3)/(f*(b**2+1/3*(-2*b**2*h**2-b*c*g*h+c**2*g**2)/h**2)/c**2+b*f*x/c+f*x*
*2),x)
```

```
[Out] 3*3**(2/3)*c**2*h**2*(Integral(g/(b**2*h**2*(2*b**2/c + b*g/h + 9*b*x - c*g
**2/h**2 + 9*c*x**2)**(1/3) - b*c*g*h*(2*b**2/c + b*g/h + 9*b*x - c*g**2/h*
*2 + 9*c*x**2)**(1/3) + 3*b*c*h**2*x*(2*b**2/c + b*g/h + 9*b*x - c*g**2/h**
2 + 9*c*x**2)**(1/3) + c**2*g**2*(2*b**2/c + b*g/h + 9*b*x - c*g**2/h**2 +
9*c*x**2)**(1/3) + 3*c**2*h**2*x**2*(2*b**2/c + b*g/h + 9*b*x - c*g**2/h**2
 + 9*c*x**2)**(1/3)), x) + Integral(h*x/(b**2*h**2*(2*b**2/c + b*g/h + 9*b*
x - c*g**2/h**2 + 9*c*x**2)**(1/3) - b*c*g*h*(2*b**2/c + b*g/h + 9*b*x - c*
g**2/h**2 + 9*c*x**2)**(1/3) + 3*b*c*h**2*x*(2*b**2/c + b*g/h + 9*b*x - c*g
**2/h**2 + 9*c*x**2)**(1/3) + c**2*g**2*(2*b**2/c + b*g/h + 9*b*x - c*g**2/
h**2 + 9*c*x**2)**(1/3) + 3*c**2*h**2*x**2*(2*b**2/c + b*g/h + 9*b*x - c*g*
*2/h**2 + 9*c*x**2)**(1/3)), x))/f
```

**Giac** [**F**]
time = 0.00, size = 0, normalized size = 0.00

could not integrate

Verification of antiderivative is not currently implemented for this CAS.

```
[In] integrate((h*x+g)/(1/9*(2*b^2*h^2+b*c*g*h-c^2*g^2)/c/h^2+b*x+c*x^2)^(1/3)/(
f*(b^2+1/3*(-2*b^2*h^2-b*c*g*h+c^2*g^2)/h^2)/c^2+b*f*x/c+f*x^2),x, algorith
m="giac")
```

```
[Out] integrate(3*(h*x + g)/((c*x^2 + b*x - 1/9*(c^2*g^2 - b*c*g*h - 2*b^2*h^2)/(
c*h^2))^(1/3)*(3*f*x^2 + 3*b*f*x/c + (3*b^2 + (c^2*g^2 - b*c*g*h - 2*b^2*h^
2)/h^2)*f/c^2)), x)
```

**Mupad** [**F**]
time = 0.00, size = -1, normalized size = -0.00

$$\int \frac{g+h\,x}{{\left(b\,x+c\,x^2+\frac{\frac{2\,b^2\,h^2}{9}+\frac{b\,c\,g\,h}{9}-\frac{c^2\,g^2}{9}}{c\,h^2}\right)}^{1/3}\,\left(f\,x^2-\frac{f\,\left(\frac{\frac{2\,b^2\,h^2}{3}+\frac{b\,c\,g\,h}{3}-\frac{c^2\,g^2}{3}}{h^2}-b^2\right)}{c^2}+\frac{b\,f\,x}{c}\right)}\,dx$$

Verification of antiderivative is not currently implemented for this CAS.

```
[In] int((g + h*x)/((b*x + c*x^2 + ((2*b^2*h^2)/9 - (c^2*g^2)/9 + (b*c*g*h)/9)/(
c*h^2))^(1/3)*(f*x^2 - (f*(((2*b^2*h^2)/3 - (c^2*g^2)/3 + (b*c*g*h)/3)/h^2
- b^2))/c^2 + (b*f*x)/c)),x)
```

```
[Out] int((g + h*x)/((b*x + c*x^2 + ((2*b^2*h^2)/9 - (c^2*g^2)/9 + (b*c*g*h)/9)/(
c*h^2))^(1/3)*(f*x^2 - (f*(((2*b^2*h^2)/3 - (c^2*g^2)/3 + (b*c*g*h)/3)/h^2
- b^2))/c^2 + (b*f*x)/c)), x)
```

# Chapter 4

# Appendix

## Local contents

# 4.1 Download section

The following zip files contain the raw integrals used in this test.

**Mathematica format** Mathematica_syntax.zip

**Maple and Mupad format** Maple_syntax.zip

**Sympy format** SYMPY_syntax.zip

**Sage math format** SAGE_syntax.zip

# 4.2 Listing of Grading functions

The following are the current version of the grading functions used for grading the quality of the antiderivative with reference to the optimal antiderivative included in the test suite.

There is a version for Maple and for Mathematica/Rubi. There is a version for grading Sympy and version for use with Sagemath.

The following are links to the current source code.

The following are the listings of source code of the grading functions.

## 4.2.1 Mathematica and Rubi grading function

```
(* Original version thanks to Albert Rich emailed on 03/21/2017 *)
(* ::Package:: *)

(* Nasser: April 7,2022. add second output which gives reason for the grade *)
(*                       Small rewrite of logic in main function to make it*)
(*                       match Maple's logic. No change in functionality otherwise*)

(* ::Subsection:: *)
(*GradeAntiderivative[result,optimal]*)


(* ::Text:: *)
(*If result and optimal are mathematical expressions, *)
(*         GradeAntiderivative[result,optimal] returns*)
(*  "F" if the result fails to integrate an expression that*)
(*      is integrable*)
(*  "C" if result involves higher level functions than necessary*)
(*  "B" if result is more than twice the size of the optimal*)
(*      antiderivative*)
(*  "A" if result can be considered optimal*)
```

```
GradeAntiderivative[result_,optimal_] := Module[{expnResult,expnOptimal,leafCountResult,leafC
    expnResult  = ExpnType[result];
    expnOptimal = ExpnType[optimal];
    leafCountResult = LeafCount[result];
    leafCountOptimal = LeafCount[optimal];

    (*Print["expnResult=",expnResult," expnOptimal=",expnOptimal];*)
    If[expnResult<=expnOptimal,
        If[Not[FreeQ[result,Complex]], (*result contains complex*)
            If[Not[FreeQ[optimal,Complex]], (*optimal contains complex*)
                If[leafCountResult<=2*leafCountOptimal,
                    finalresult={"A","none"}
                    ,(*ELSE*)
                    finalresult={"B","Both result and optimal contain complex but leaf count
                  ]
            ,(*ELSE*)
                finalresult={"C","Result contains complex when optimal does not."}
            ]
        ,(*ELSE*)(*result does not contains complex*)
            If[leafCountResult<=2*leafCountOptimal,
               finalresult={"A","none"}
            ,(*ELSE*)
               finalresult={"B","Leaf count is larger than twice the leaf count of optimal. $
              ]
        ]
    ,(*ELSE*) (*expnResult>expnOptimal*)
        If[FreeQ[result,Integrate] && FreeQ[result,Int],
            finalresult={"C","Result contains higher order function than in optimal. Order "<
            ,
            finalresult={"F","Contains unresolved integral."}
        ]
    ];

    finalresult
]

(* ::Text:: *)
(*The following summarizes the type number assigned an *)
(*expression based on the functions it involves*)
(*1 = rational function*)
(*2 = algebraic function*)
(*3 = elementary function*)
(*4 = special function*)
(*5 = hyperpergeometric function*)
(*6 = appell function*)
(*7 = rootsum function*)
(*8 = integrate function*)
```

```
(*9 = unknown function*)


ExpnType[expn_] :=
  If[AtomQ[expn],
    1,
  If[ListQ[expn],
    Max[Map[ExpnType,expn]],
  If[Head[expn]===Power,
    If[IntegerQ[expn[[2]]],
      ExpnType[expn[[1]]],
    If[Head[expn[[2]]]===Rational,
      If[IntegerQ[expn[[1]]] || Head[expn[[1]]]===Rational,
        1,
      Max[ExpnType[expn[[1]]],2]],
    Max[ExpnType[expn[[1]]],ExpnType[expn[[2]]],3]]],
  If[Head[expn]===Plus || Head[expn]===Times,
    Max[ExpnType[First[expn]],ExpnType[Rest[expn]]],
  If[ElementaryFunctionQ[Head[expn]],
    Max[3,ExpnType[expn[[1]]]],
  If[SpecialFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],4]],
  If[HypergeometricFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],5]],
  If[AppellFunctionQ[Head[expn]],
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],6]],
  If[Head[expn]===RootSum,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
  If[Head[expn]===Integrate || Head[expn]===Int,
    Apply[Max,Append[Map[ExpnType,Apply[List,expn]],8]],
  9]]]]]]]]]]


ElementaryFunctionQ[func_] :=
  MemberQ[{
  Exp,Log,
  Sin,Cos,Tan,Cot,Sec,Csc,
  ArcSin,ArcCos,ArcTan,ArcCot,ArcSec,ArcCsc,
  Sinh,Cosh,Tanh,Coth,Sech,Csch,
  ArcSinh,ArcCosh,ArcTanh,ArcCoth,ArcSech,ArcCsch
},func]


SpecialFunctionQ[func_] :=
  MemberQ[{
  Erf, Erfc, Erfi,
  FresnelS, FresnelC,
```

```
  ExpIntegralE, ExpIntegralEi, LogIntegral,
  SinIntegral, CosIntegral, SinhIntegral, CoshIntegral,
  Gamma, LogGamma, PolyGamma,
  Zeta, PolyLog, ProductLog,
  EllipticF, EllipticE, EllipticPi
},func]


HypergeometricFunctionQ[func_] :=
  MemberQ[{Hypergeometric1F1,Hypergeometric2F1,HypergeometricPFQ},func]


AppellFunctionQ[func_] :=
  MemberQ[{AppellF1},func]
```

### 4.2.2 Maple grading function

```
# File: GradeAntiderivative.mpl
# Original version thanks to Albert Rich emailed on 03/21/2017

#Nasser 03/22/2017  Use Maple leaf count instead since buildin
#Nasser 03/23/2017  missing 'ln' for ElementaryFunctionQ added
#Nasser 03/24/2017  corrected the check for complex result
#Nasser 10/27/2017  check for leafsize and do not call ExpnType()
#                   if leaf size is "too large". Set at 500,000
#Nasser 12/22/2019  Added debug flag, added 'dilog' to special functions
#                   see problem 156, file Apostol_Problems
#Nasser 4/07/2022   add second output which gives reason for the grade

GradeAntiderivative := proc(result,optimal)
local leaf_count_result,
        leaf_count_optimal,
        ExpnType_result,
        ExpnType_optimal,
        debug:=false;

        leaf_count_result:=leafcount(result);
        #do NOT call ExpnType() if leaf size is too large. Recursion problem
        if leaf_count_result > 500000 then
            return "B","result has leaf size over 500,000. Avoiding possible recursion issues
        fi;

        leaf_count_optimal := leafcount(optimal);
        ExpnType_result  := ExpnType(result);
        ExpnType_optimal := ExpnType(optimal);
```

```
        if debug then
            print("ExpnType_result",ExpnType_result," ExpnType_optimal=",ExpnType_optimal);
        fi;

# If result and optimal are mathematical expressions,
#  GradeAntiderivative[result,optimal] returns
#   "F" if the result fails to integrate an expression that
#       is integrable
#   "C" if result involves higher level functions than necessary
#   "B" if result is more than twice the size of the optimal
#       antiderivative
#   "A" if result can be considered optimal

    #This check below actually is not needed, since I only
    #call this grading only for passed integrals. i.e. I check
    #for "F" before calling this. But no harm of keeping it here.
    #just in case.


    if not type(result,freeof('int')) then
        return "F","Result contains unresolved integral";
    fi;


    if ExpnType_result<=ExpnType_optimal then
        if debug then
             print("ExpnType_result<=ExpnType_optimal");
        fi;
        if is_contains_complex(result) then
            if is_contains_complex(optimal) then
                if debug then
                        print("both result and optimal complex");
                fi;
                if leaf_count_result<=2*leaf_count_optimal then
                    return "A","";
                else
                   return "B",cat("Both result and optimal contain complex but leaf count of re
                                  convert(leaf_count_result,string)," vs. $2 (",
                                  convert(leaf_count_optimal,string)," ) = ",convert(2*leaf_co
                end if
            else #result contains complex but optimal is not
                if debug then
                        print("result contains complex but optimal is not");
                fi;
                return "C","Result contains complex when optimal does not.";
            fi;
        else # result do not contain complex
```

```
             # this assumes optimal do not as well. No check is needed here.
            if debug then
                   print("result do not contain complex, this assumes optimal do not as well"
            fi;
            if leaf_count_result<=2*leaf_count_optimal then
                if debug then
                    print("leaf_count_result<=2*leaf_count_optimal");
                fi;
                return "A","";
            else
                if debug then
                    print("leaf_count_result>2*leaf_count_optimal");
                fi;
                return "B",cat("Leaf count of result is larger than twice the leaf count of o
                               convert(leaf_count_result,string),"$ vs. $2(",
                               convert(leaf_count_optimal,string),")=",convert(2*leaf_cou
            fi;
        fi;
    else #ExpnType(result) > ExpnType(optimal)
        if debug then
            print("ExpnType(result) > ExpnType(optimal)");
        fi;
        return "C",cat("Result contains higher order function than in optimal. Order ",
                       convert(ExpnType_result,string)," vs. order ",
                       convert(ExpnType_optimal,string),".");
    fi;

end proc:

#
# is_contains_complex(result)
# takes expressions and returns true if it contains "I" else false
#
#Nasser 032417
is_contains_complex:= proc(expression)
  return (has(expression,I));
end proc:

# The following summarizes the type number assigned an expression
# based on the functions it involves
# 1 = rational function
# 2 = algebraic function
# 3 = elementary function
# 4 = special function
# 5 = hyperpergeometric function
# 6 = appell function
# 7 = rootsum function
```

```
# 8 = integrate function
# 9 = unknown function

ExpnType := proc(expn)
  if type(expn,'atomic') then
    1
  elif type(expn,'list') then
    apply(max,map(ExpnType,expn))
  elif type(expn,'sqrt') then
    if type(op(1,expn),'rational') then
       1
    else
       max(2,ExpnType(op(1,expn)))
    end if
  elif type(expn,'`^`') then
    if type(op(2,expn),'integer') then
      ExpnType(op(1,expn))
    elif type(op(2,expn),'rational') then
      if type(op(1,expn),'rational') then
         1
      else
         max(2,ExpnType(op(1,expn)))
      end if
    else
         max(3,ExpnType(op(1,expn)),ExpnType(op(2,expn)))
    end if
  elif type(expn,'`+`') or type(expn,'`*`') then
    max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
  elif ElementaryFunctionQ(op(0,expn)) then
    max(3,ExpnType(op(1,expn)))
  elif SpecialFunctionQ(op(0,expn)) then
    max(4,apply(max,map(ExpnType,[op(expn)])))
  elif HypergeometricFunctionQ(op(0,expn)) then
    max(5,apply(max,map(ExpnType,[op(expn)])))
  elif AppellFunctionQ(op(0,expn)) then
    max(6,apply(max,map(ExpnType,[op(expn)])))
  elif op(0,expn)='int' then
    max(8,apply(max,map(ExpnType,[op(expn)]))) else
  9
  end if
end proc:


ElementaryFunctionQ := proc(func)
  member(func,[
        exp,log,ln,
        sin,cos,tan,cot,sec,csc,
```

```
        arcsin,arccos,arctan,arccot,arcsec,arccsc,
        sinh,cosh,tanh,coth,sech,csch,
        arcsinh,arccosh,arctanh,arccoth,arcsech,arccsch])
end proc:

SpecialFunctionQ := proc(func)
  member(func,[
        erf,erfc,erfi,
        FresnelS,FresnelC,
        Ei,Ei,Li,Si,Ci,Shi,Chi,
        GAMMA,lnGAMMA,Psi,Zeta,polylog,dilog,LambertW,
        EllipticF,EllipticE,EllipticPi])
end proc:

HypergeometricFunctionQ := proc(func)
  member(func,[Hypergeometric1F1,hypergeom,HypergeometricPFQ])
end proc:

AppellFunctionQ := proc(func)
  member(func,[AppellF1])
end proc:

# u is a sum or product.  rest(u) returns all but the
# first term or factor of u.
rest := proc(u) local v;
  if nops(u)=2 then
     op(2,u)
  else
     apply(op(0,u),op(2..nops(u),u))
  end if
end proc:

#leafcount(u) returns the number of nodes in u.
#Nasser 3/23/17  Replaced by build-in leafCount from package in Maple
leafcount := proc(u)
   MmaTranslator[Mma][LeafCount](u);
end proc:
```

### 4.2.3 Sympy grading function

```python
#Dec 24, 2019. Nasser M. Abbasi:
#             Port of original Maple grading function by
#             Albert Rich to use with Sympy/Python
#Dec 27, 2019  Nasser. Added `RootSum`. See problem 177, Timofeev file
#             added 'exp_polar'
from sympy import *

def leaf_count(expr):
   #sympy do not have leaf count function. This is approximation
   return round(1.7*count_ops(expr))

def is_sqrt(expr):
   if isinstance(expr,Pow):
      if expr.args[1] == Rational(1,2):
         return True
      else:
         return False
   else:
      return False

def is_elementary_function(func):
   return func in [exp,log,ln,sin,cos,tan,cot,sec,csc,
         asin,acos,atan,acot,asec,acsc,sinh,cosh,tanh,coth,sech,csch,
         asinh,acosh,atanh,acoth,asech,acsch
      ]

def is_special_function(func):
   return func in [ erf,erfc,erfi,
          fresnels,fresnelc,Ei,Ei,Li,Si,Ci,Shi,Chi,
          gamma,loggamma,digamma,zeta,polylog,LambertW,
          elliptic_f,elliptic_e,elliptic_pi,exp_polar
       ]

def is_hypergeometric_function(func):
   return func in [hyper]

def is_appell_function(func):
   return func in [appellf1]

def is_atom(expn):
   try:
      if expn.isAtom or isinstance(expn,int) or isinstance(expn,float):
        return True
      else:
        return False
```

```
    except AttributeError as error:
        return False

def expnType(expn):
    debug=False
    if debug:
        print("expn=",expn,"type(expn)=",type(expn))

    if is_atom(expn):
       return 1
    elif isinstance(expn,list):
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
            return 1
        else:
            return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
    elif isinstance(expn,Pow):    #type(expn,'`^`')
        if isinstance(expn.args[1],Integer):   #type(op(2,expn),'integer')
            return expnType(expn.args[0])    #ExpnType(op(1,expn))
        elif isinstance(expn.args[1],Rational):   #type(op(2,expn),'rational')
            if isinstance(expn.args[0],Rational): #type(op(1,expn),'rational')
                return 1
            else:
                return max(2,expnType(expn.args[0]))  #max(2,ExpnType(op(1,expn)))
        else:
            return max(3,expnType(expn.args[0]),expnType(expn.args[1])) #max(3,ExpnType(op(1,expn)),ExpnT
    elif isinstance(expn,Add) or isinstance(expn,Mul): #type(expn,'`+`') or type(expn,'`*`')
        m1 = expnType(expn.args[0])
        m2 = expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.func):  #ElementaryFunctionQ(op(0,expn))
        return max(3,expnType(expn.args[0]))  #max(3,ExpnType(op(1,expn)))
    elif is_special_function(expn.func): #SpecialFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(4,m1)    #max(4,apply(max,map(ExpnType,[op(expn)])))
    elif is_hypergeometric_function(expn.func): #HypergeometricFunctionQ(op(0,expn))
        m1 = max(map(expnType, list(expn.args)))
        return max(5,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif is_appell_function(expn.func):
        m1 = max(map(expnType, list(expn.args)))
        return max(6,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    elif isinstance(expn,RootSum):
        m1 = max(map(expnType, list(expn.args))) #Apply[Max,Append[Map[ExpnType,Apply[List,expn]],7]],
        return max(7,m1)
    elif str(expn).find("Integral") != -1:
```

```
        m1 = max(map(expnType, list(expn.args)))
        return max(8,m1)    #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):

    #print ("Enter grade_antiderivative for sagemath")
    #print("Enter grade_antiderivative, result=",result," optimal=",optimal)

    leaf_count_result  = leaf_count(result)
    leaf_count_optimal = leaf_count(optimal)

    #print("leaf_count_result=",leaf_count_result)
    #print("leaf_count_optimal=",leaf_count_optimal)

    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)

    if str(result).find("Integral") != -1:
        grade = "F"
        grade_annotation =""
    else:
        if expnType_result <= expnType_optimal:
            if result.has(I):
                if optimal.has(I): #both result and optimal complex
                    if leaf_count_result <= 2*leaf_count_optimal:
                        grade = "A"
                        grade_annotation =""
                    else:
                        grade = "B"
                        grade_annotation ="Both result and optimal contain complex but leaf count of result is larger
                else: #result contains complex but optimal is not
                    grade = "C"
                    grade_annotation ="Result contains complex when optimal does not."
            else: # result do not contain complex, this assumes optimal do not as well
                if leaf_count_result <= 2*leaf_count_optimal:
                    grade = "A"
                    grade_annotation =""
                else:
                    grade = "B"
                    grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(lea
        else:
            grade = "C"
            grade_annotation ="Result contains higher order function than in optimal. Order "+str(ExpnType_re
```

```
    #print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

    return grade, grade_annotation
```

### 4.2.4 SageMath grading function

```
#Dec 24, 2019. Nasser: Ported original Maple grading function by
#          Albert Rich to use with Sagemath. This is used to
#          grade Fricas, Giac and Maxima results.
#Dec 24, 2019. Nasser: Added 'exp_integral_e' and 'sng', 'sin_integral'
#          'arctan2','floor','abs','log_integral'
#June 4, 2022  Made default grade_annotation "none" instead of "" due
#          issue later when reading the file.
#July 14, 2022. Added ellipticF. This is until they fix sagemath, then remove it.

from sage.all import *
from sage.symbolic.operators import add_vararg, mul_vararg

debug=False;

def tree_size(expr):
    r"""
    Return the tree size of this expression.
    """
    #print("Enter tree_size, expr is ",expr)

    if expr not in SR:
        # deal with lists, tuples, vectors
        return 1 + sum(tree_size(a) for a in expr)
    expr = SR(expr)
    x, aa = expr.operator(), expr.operands()
    if x is None:
        return 1
    else:
        return 1 + sum(tree_size(a) for a in aa)

def is_sqrt(expr):
    if expr.operator() == operator.pow:   #isinstance(expr,Pow):
        if expr.operands()[1]==1/2: #expr.args[1] == Rational(1,2):
            if debug: print ("expr is sqrt")
            return True
        else:
            return False
    else:
        return False
```

```
def is_elementary_function(func):
   #debug=False
   m = func.name() in ['exp','log','ln',
         'sin','cos','tan','cot','sec','csc',
         'arcsin','arccos','arctan','arccot','arcsec','arccsc',
         'sinh','cosh','tanh','coth','sech','csch',
         'arcsinh','arccosh','arctanh','arccoth','arcsech','arccsch','sgn',
      'arctan2','floor','abs'
      ]
   if debug:
      if m:
         print ("func ", func , " is elementary_function")
      else:
         print ("func ", func , " is NOT elementary_function")


   return m

def is_special_function(func):
   #debug=False
   if debug:
      print ("type(func)=", type(func))

   m= func.name() in ['erf','erfc','erfi','fresnel_sin','fresnel_cos','Ei',
        'Ei','Li','Si','sin_integral','Ci','cos_integral','Shi','sinh_integral'
        'Chi','cosh_integral','gamma','log_gamma','psi,zeta',
        'polylog','lambert_w','elliptic_f','elliptic_e','ellipticF',
        'elliptic_pi','exp_integral_e','log_integral']

   if debug:
      print ("m=",m)
      if m:
         print ("func ", func ," is special_function")
      else:
         print ("func ", func ," is NOT special_function")


   return m


def is_hypergeometric_function(func):
   return func.name() in ['hypergeometric','hypergeometric_M','hypergeometric_U']

def is_appell_function(func):
   return func.name() in ['hypergeometric']   #[appellf1] can't find this in sagemath
```

```
def is_atom(expn):

    #debug=False
    if debug:
        print ("Enter is_atom, expn=",expn)

    if not hasattr(expn, 'parent'):
        return False


    #thanks to answer at https://ask.sagemath.org/question/49179/what-is-sagemath-equivalent-to-atomic-
    try:
        if expn.parent() is SR:
            return expn.operator() is None
        if expn.parent() in (ZZ, QQ, AA, QQbar):
            return expn in expn.parent() # Should always return True
        if hasattr(expn.parent(),"base_ring") and hasattr(expn.parent(),"gens"):
            return expn in expn.parent().base_ring() or expn in expn.parent().gens()

        return False

    except AttributeError as error:
        print("Exception,AttributeError in  is_atom")
        print ("cought exception" , type(error).__name__ )
        return False


def expnType(expn):

    if debug:
        print (">>>>>Enter expnType, expn=", expn)
        print (">>>>>is_atom(expn)=", is_atom(expn))

    if is_atom(expn):
        return 1
    elif type(expn)==list:   #isinstance(expn,list):
        return max(map(expnType, expn))    #apply(max,map(ExpnType,expn))
    elif  is_sqrt(expn):
        if  type(expn.operands()[0])==Rational: #type(isinstance(expn.args[0],Rational):
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))   #max(2,expnType(expn.args[0]))
    elif expn.operator() == operator.pow:   #isinstance(expn,Pow)
        if type(expn.operands()[1])==Integer:   #isinstance(expn.args[1],Integer)
            return expnType(expn.operands()[0])    #expnType(expn.args[0])
        elif type(expn.operands()[1])==Rational:   #isinstance(expn.args[1],Rational)
            if type(expn.operands()[0])==Rational: #isinstance(expn.args[0],Rational)
```

```
            return 1
        else:
            return max(2,expnType(expn.operands()[0]))  #max(2,expnType(expn.args[0]))
    else:
        return max(3,expnType(expn.operands()[0]),expnType(expn.operands()[1])) #max(3,expnType(expn.op
    elif expn.operator() == add_vararg or expn.operator() == mul_vararg: #isinstance(expn,Add) or isinstan
        m1 = expnType(expn.operands()[0]) #expnType(expn.args[0])
        m2 = expnType(expn.operands()[1:]) #expnType(list(expn.args[1:]))
        return max(m1,m2)  #max(ExpnType(op(1,expn)),max(ExpnType(rest(expn))))
    elif is_elementary_function(expn.operator()):  #is_elementary_function(expn.func)
        return max(3,expnType(expn.operands()[0]))
    elif is_special_function(expn.operator()): #is_special_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(4,m1)   #max(4,m1)
    elif is_hypergeometric_function(expn.operator()): #is_hypergeometric_function(expn.func)
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(5,m1)   #max(5,m1)
    elif is_appell_function(expn.operator()):
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(6,m1)   #max(6,m1)
    elif str(expn).find("Integral") != -1: #this will never happen, since it
            #is checked before calling the grading function that is passed.
            #but kept it here.
        m1 = max(map(expnType, expn.operands()))      #max(map(expnType, list(expn.args)))
        return max(8,m1)   #max(5,apply(max,map(ExpnType,[op(expn)])))
    else:
        return 9

#main function
def grade_antiderivative(result,optimal):


    if debug:
        print ("Enter grade_antiderivative for sagemath")
        print("Enter grade_antiderivative, result=",result)
        print("Enter grade_antiderivative, optimal=",optimal)
        print("type(anti)=",type(result))
        print("type(optimal)=",type(optimal))

    leaf_count_result  = tree_size(result) #leaf_count(result)
    leaf_count_optimal = tree_size(optimal) #leaf_count(optimal)

    #if debug: print ("leaf_count_result=", leaf_count_result, "leaf_count_optimal=",leaf_count_optimal)


    expnType_result  = expnType(result)
    expnType_optimal = expnType(optimal)
```

```
if debug: print ("expnType_result=", expnType_result, "expnType_optimal=",expnType_optimal)

if expnType_result <= expnType_optimal:
    if result.has(I):
        if optimal.has(I): #both result and optimal complex
            if leaf_count_result <= 2*leaf_count_optimal:
                grade = "A"
                grade_annotation ="none"
            else:
                grade = "B"
                grade_annotation ="Both result and optimal contain complex but leaf count of result is larger t
        else: #result contains complex but optimal is not
            grade = "C"
            grade_annotation ="Result contains complex when optimal does not."
    else: # result do not contain complex, this assumes optimal do not as well
        if leaf_count_result <= 2*leaf_count_optimal:
            grade = "A"
            grade_annotation ="none"
        else:
            grade = "B"
            grade_annotation ="Leaf count of result is larger than twice the leaf count of optimal. "+str(leaf_
else:
    grade = "C"
    grade_annotation ="Result contains higher order function than in optimal. Order "+str(expnType_resu


print("Before returning. grade=",grade, " grade_annotation=",grade_annotation)

return grade, grade_annotation
```